SYDNEY GRAMMAR SCHOOL

NAME

MATHS MASTER

CANDIDATE NUMBER

2026 Trial Examination

Form VI Mathematics Advanced

Wednesday 12th August, 2026

8:40am

General Instructions

- Reading time — 10 minutes
- Working time — 3 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.
- Carefully remove the central staple.

Thirty Five Questions — 100 Marks

Section I (10 marks) Questions 1 – 10

- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (90 marks) Questions 11 – 35

- Relevant mathematical reasoning and calculations are required.
- Answer the questions in this paper in the spaces provided.

Collection

- Write your name and candidate number on this page.
- Write your candidate number on each booklet and on the multiple choice sheet.
- Place everything inside this question booklet.

Checklist

- Reference sheet
- Multiple-choice answer sheet
- Candidature: 107 pupils

Writer: WJW

Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

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1. Which of the following is the y-intercept of the curve $y = e^x + e^{-x}$?

   (A) $(0, -1)$

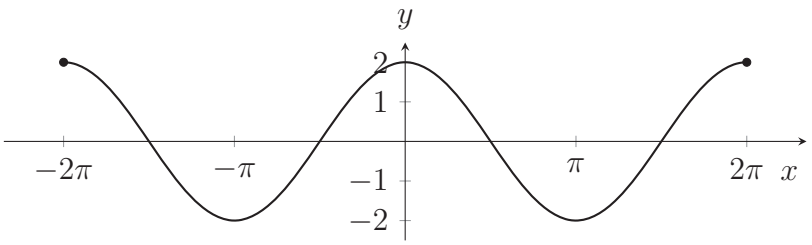
   (B) $(0, 0)$

   (C) $(0, 1)$

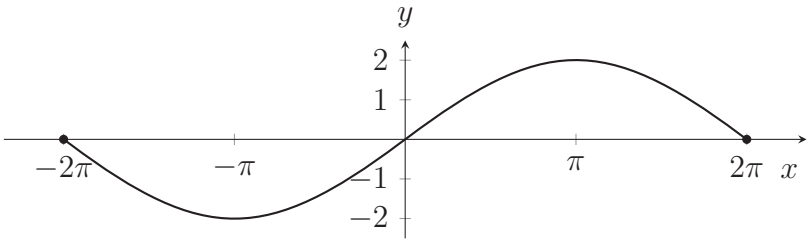
   (D) $(0, 2)$

2. Which of the following is the graph of $y = 2\cos x$ for $-2\pi \leq x \leq 2\pi$?

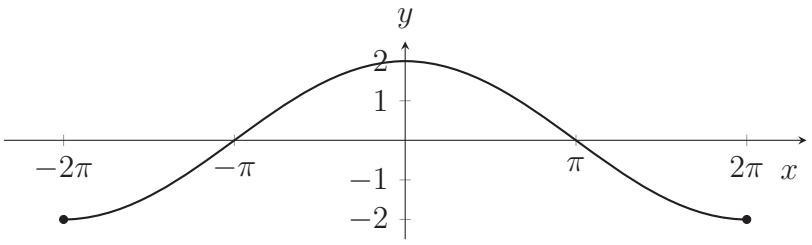
   (A)



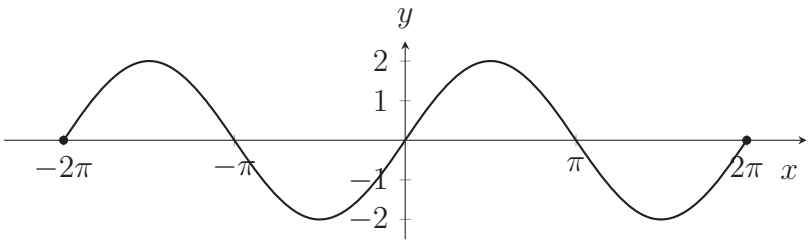
   (B)



   (C)



   (D)



3. Which of the following is a correct answer for $\int x^2 + 6x - 1\,dx$?

   (A) $2x + 6$

   (B) $\frac{x^3}{3} + 3x^2 - x$

   (C) $2x + 3x^2$

   (D) $\frac{x^2}{3} + 3x - x$

4. A table of future value interest factors is shown below.

Table of future value interest factors

Period	*Interest rate per period*					
	1%	2%	3%	4%	6%	8%
2	2.0100	2.0200	2.0300	2.0400	2.0600	2.0800
3	3.0301	3.0604	3.0909	3.1216	3.1836	3.2464
4	4.0604	4.1216	4.1836	4.2465	4.3746	4.5061
5	5.1010	5.2040	5.3091	5.4163	5.6371	5.8666
6	6.1520	6.3081	6.4684	6.6330	6.9753	7.3359
7	7.2135	7.4343	7.6625	7.8983	8.3938	8.9228
8	8.2857	8.5830	8.8923	9.2142	9.8975	10.6366
9	9.3685	9.7546	10.1591	10.5828	11.4913	12.4876
10	10.4622	10.9497	11.4639	12.0061	13.1808	14.4866
11	11.5668	12.1687	12.8078	13.4864	14.9716	16.6455
12	12.6825	13.4121	14.1920	15.0258	16.8699	18.9771

An investor wishes to accumulate \$50 000 over 2 years by making contributions at the end of each quarter. Interest is paid at 4% p.a., compounded quarterly.

Using the table of future value interest factors, what is the quarterly contributions required to accumulate \$50 000?

(A) \$5426

(B) \$6034

(C) \$24 510

(D) \$24 876

5. A fair coin is tossed four times. What is the probability of obtaining more heads than tails?

   (A) $\frac{1}{16}$

   (B) $\frac{1}{2}$

   (C) $\frac{5}{16}$

   (D) $\frac{3}{8}$

6. Which of the following statements is **TRUE**?

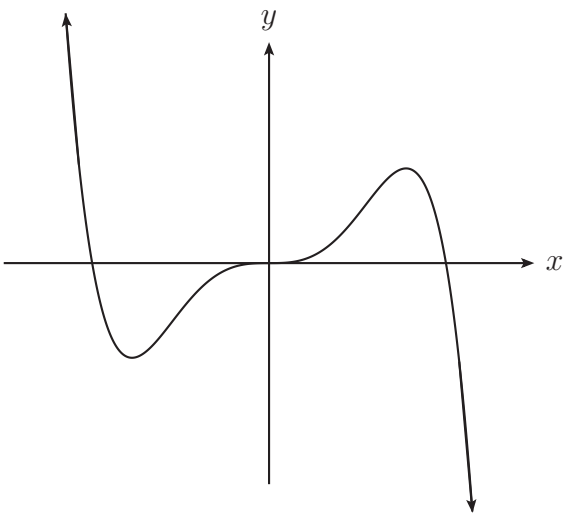
   (A) The sum of two even functions is odd.

   (B) The sum of two odd functions is even.

   (C) The product of two even functions is odd.

   (D) The product of two odd functions is even.

7. The diagram below shows the graph of $y = f(x)$.



   Which of the following statements must be false?

   (A) $f(x) = 0$ has three distinct solutions.

   (B) $f'(x) = 0$ has three distinct solutions.

   (C) $f''(x) = 0$ has three distinct solutions.

   (D) $f(x)f'(x) = 0$ has six distinct solutions.

8. Given $y = \sqrt{1+\sqrt{x}}$, which of the following is the correct expression for $\dfrac{dy}{dx}$?

(A) $\dfrac{1}{2\sqrt{1+\sqrt{x}}}$

(B) $\dfrac{x}{4\sqrt{1+\sqrt{x}}}$

(C) $\dfrac{1}{2\sqrt{x}\sqrt{1+\sqrt{x}}}$

(D) $\dfrac{1}{4\sqrt{x+x\sqrt{x}}}$

9. Let $f(x) = \sin 2x$ and $g(x) = |x-\pi|$, where $0 \le x \le 2\pi$.
Which of the following is the correct expression for $f \circ g(x)$?

(A) $f \circ g(x) = \begin{cases} -\sin 2x, & 0 \le x < \pi, \\ \sin 2x, & \pi \le x \le 2\pi. \end{cases}$

(B) $f \circ g(x) = \begin{cases} \sin 2x, & 0 \le x < \pi, \\ \sin 2x, & \pi \le x \le 2\pi. \end{cases}$

(C) $f \circ g(x) = \begin{cases} -\sin 2x, & 0 \le x < \pi, \\ -\sin 2x, & \pi \le x \le 2\pi. \end{cases}$

(D) $f \circ g(x) = \begin{cases} \sin 2x, & 0 \le x < \pi, \\ -\sin 2x, & \pi \le x \le 2\pi. \end{cases}$

10. The graph of $y = f(x)$ undergoes the following sequence of transformations in order.

    1. Translate the graph a units to the right, where $a > 0$.
    2. Dilate the graph horizontally by a factor of b, where $b > 1$.
    3. Translate the graph a units to the left.
    4. Dilate the graph horizontally by a factor of $\dfrac{1}{b}$.

    Which of the following describes the overall effect of these transformations?

    (A) A single translation of $a\left(1 - \dfrac{1}{b}\right)$ units to the right.

    (B) A single translation of $a\left(1 - \dfrac{1}{b}\right)$ units to the left.

    (C) A single translation of $a(b - 1)$ units to the right.

    (D) No change to the graph.

End of Section I

The paper continues in the next section

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B L A N K P A G E

CANDIDATE NUMBER

Section II

Part A

QUESTION ELEVEN (2 marks)

(a) Find the domain of $f(x) = \sqrt{x+4}$. **1**

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(b) Find the range of $f(x) = e^x - 3$. **1**

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QUESTION TWELVE (2 marks)

Find the derivative of:

(a) $y = \sin 3x$ **1**

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(b) $y = \ln(4 - x)$ **1**

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QUESTION THIRTEEN (4 marks)

Find:

(a) $\int (2x-5)^2 \, dx$ **2**

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(b) $\int_0^{\frac{\pi}{4}} \cos 2x \, dx$ **2**

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QUESTION FOURTEEN (2 marks)

A school library records the total number of books at the beginning of each year. The library contained 4200 books at the beginning of 2023 and 4380 books at the beginning of 2026.

(a) Assuming the same number of books was added each year, calculate the annual increase in the number of books. **1**

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(b) Hence determine the number of books in the library at the beginning of 2025. **1**

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QUESTION FIFTEEN (2 marks)

Consider the function $f(x) = \tan(\pi x) + 3$.

(a) State the period. **1**

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(b) State the equation of the vertical asymptote of $y = f(x)$ for $0 \leq x \leq 1$. **1**

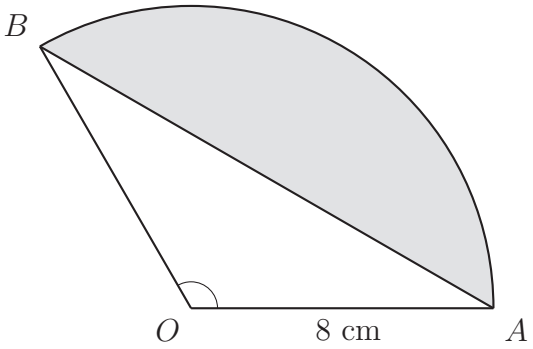
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QUESTION SIXTEEN (4 marks)

The diagram shows a sector OAB of a circle with centre O and radius 8 cm.



(a) Find the length of the chord AB, given that $\angle AOB = \dfrac{2\pi}{3}$. **2**

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(b) Find the exact area of the minor segment, shaded in the diagram above. **2**

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QUESTION SEVENTEEN (3 marks)

Find the equation of the tangent to $y = e^{2x}$ at the point where $x = \dfrac{1}{2}$. **3**

Extra writing space (Use this space only for questions in Part A)

If you use this space, clearly indicate which question you are answering.

Extra writing space (Use this space only for questions in Part A)

If you use this space, clearly indicate which question you are answering.

B L A N K P A G E

CANDIDATE NUMBER

Section II

Part B

QUESTION EIGHTEEN (4 marks)

The function $y = a\sin(bx) + c$ has maximum value 6, minimum value -2, and period $\dfrac{2\pi}{3}$ radians, where $a > 0$.

(a) Find the values of a, b and c. **3**

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(b) Hence, state the number of solutions to the equation **1**

$$a\sin(bx) + c = 4, \qquad \text{for } 0 \leq x \leq 2\pi.$$

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QUESTION NINETEEN (3 marks)

The point $P(0, \frac{1}{2})$ lies on the curve $y = \frac{1}{2}(\sin x + \cos x)^2$. Another point Q lies on the curve in the domain $0 \leq x \leq \pi$ such that the tangent to the curve at Q is perpendicular to the tangent at P. Find the coordinates of Q. **3**

QUESTION TWENTY (2 marks)

The table shows present value interest factors for an annuity of \$1.

$Rate(r)$ / $Period(n)$	0.0050	0.0055	0.0060	0.0065	0.0070
58	50.23911	49.54343	48.86109	48.19179	47.53525
59	50.98419	50.26696	49.56370	48.87411	48.19786
60	51.72556	50.98653	50.26213	49.55202	48.85587
61	52.46324	51.70217	50.95639	50.22555	49.50931

(a) Sophie plans to pay \$200 into an investment account at the end of each month for 58 months. Her investment will earn 0.55% interest per month. Use the information in the table to calculate the present value of this annuity. **1**

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(b) Daniel borrows \$22 600 at an interest rate of 7.8% per annum, compounded monthly. He can afford repayments of at most \$450 per month. Using the table, determine the minimum number of monthly repayments required. **1**

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QUESTION TWENTY ONE (4 marks)

The table shows the probability distribution of a discrete random variable.

x	5	10	15	20
$P(X = x)$	k^2	0.27	0.67	$0.1k$

(a) Find the value of k. **3**

(b) Find the expected value $E(X)$. **1**

QUESTION TWENTY TWO (2 marks)

Prove that **2**

$$\frac{(\cot\theta - \tan\theta)^2}{\cot^2\theta - \tan^2\theta} = \cos^2\theta - \sin^2\theta.$$

QUESTION TWENTY THREE (3 marks)

A survey of pupils at a school found that one-third of the pupils are members of the Chess Club, one-half are members of the Debating Club, and two-thirds are members of either the Chess Club or the Debating Club or both.

One pupil from the survey is selected at random. Let C be the event that the selected pupil is a member of the Chess Club, and let D be the event that the selected pupil is a member of the Debating Club.

(a) Find $P(C \cap D)$. **1**

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(b) Hence state whether C and D are mutually exclusive. Justify your answer. **1**

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(c) Determine whether C and D are independent. Justify your answer with calculations. **1**

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CANDIDATE NUMBER

Section II

Part C

QUESTION TWENTY FOUR (2 marks)

The table shows the velocity of a cyclist at different times during a ride. **2**

Time, t seconds	0	10	20	30	40
Velocity, v m/s	3.2	4.8	6.1	5.7	4.9

Use the trapezoidal rule with 4 subintervals to estimate the distance travelled by the cyclist during the first 40 seconds. Give your answer correct to one decimal place.

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QUESTION TWENTY FIVE (4 marks)

Consider the geometric series $\cos x + \cos^3 x + \cos^5 x + \cos^7 x + \cdots$, where $0 < x < \dfrac{\pi}{2}$.

(a) Show that the limiting sum of this series exists. **1**

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(b) Hence solve $\cos x + \cos^3 x + \cos^5 x + \cos^7 x + \cdots = \dfrac{2}{3}$ for $0 < x < \dfrac{\pi}{2}$. **3**

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QUESTION TWENTY SIX (5 marks)

A medical substance is injected into a patient's bloodstream. The amount of the substance remaining in the bloodstream after t hours is modelled by

$A = A_0 e^{-kt}$, where A_0 is the initial amount and k is a positive constant.

After 6 hours, the amount of substance remaining is 40% of the initial amount.

(a) Show that $k = \frac{1}{6}\ln\left(\frac{5}{2}\right)$. **1**

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(b) Find the rate, in terms of A_0, at which the amount of the substance is changing when the remaining amount is equal to half of its initial amount. **2**

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(c) Find the time taken for the amount of the substance to fall to 12% of its initial amount. Give your answer to the nearest hour. **2**

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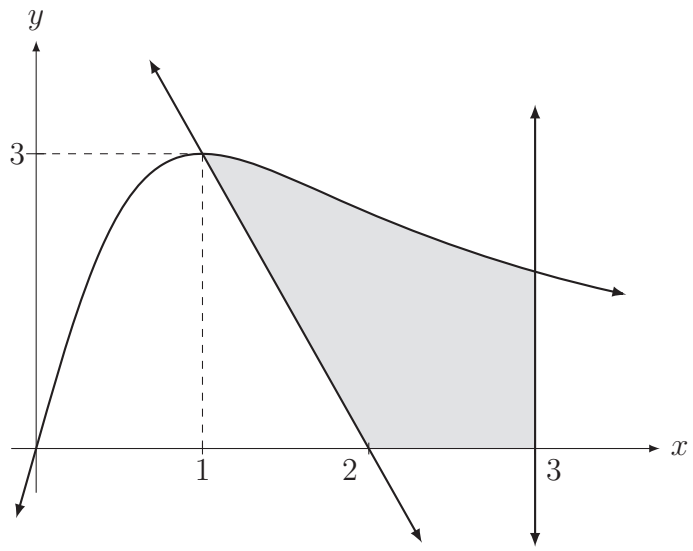
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QUESTION TWENTY SEVEN (3 marks)

The diagram shows the shaded region bounded by the curve $y = \dfrac{6x}{x^2+1}$, the line $y = -3x+6$, the line $x = 3$ and the x-axis. **3**



Find the exact area of the shaded region.

QUESTION TWENTY EIGHT (6 marks)

Naomi takes out a mortgage of \$650 000 to purchase a home. The mortgage is to be repaid over 30 years in equal monthly repayments, paid at the end of each month. Interest is charged at 4.8% per annum, compounded monthly. Let A_n be the amount owing on the mortgage immediately after the nth monthly repayment.

(a) Find a formula for the amount, A_n, and calculate Naomi's monthly repayment, correct to the nearest cent. **3**

(b) Instead of making the monthly repayments found in part (a), Naomi decides to make monthly repayments of \$4000 throughout the term of the loan. The interest rate remains unchanged. Calculate how long it will take her to repay the loan. **2**

(c) Hence, calculate how much interest Naomi will save over the life of the loan by increasing her monthly repayments to \$4000. **1**

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CANDIDATE NUMBER

Section II

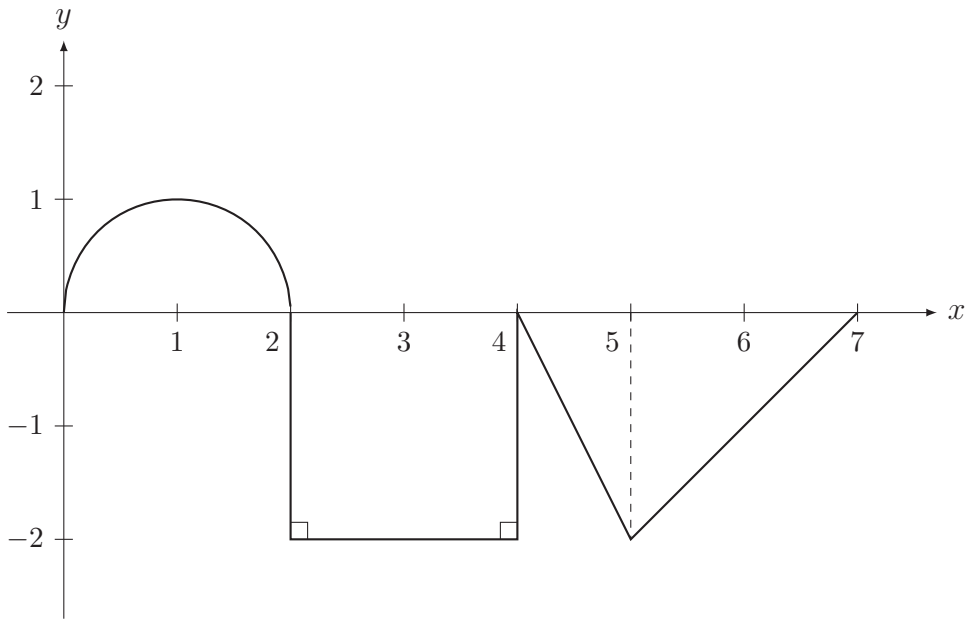
Part D

QUESTION TWENTY NINE (3 marks)

Let P be the point $(\cos 2\theta, \sqrt{3}\sin 2\theta)$, where $0 < \theta < 2\pi$. The point A has coordinates $(1, 0)$. The midpoint of AP lies on the line $y = x - \dfrac{1}{2}$. Find all possible values of θ. **3**

QUESTION THIRTY (3 marks)

The diagram below shows the graph of $y = f(x)$, formed by a semicircular arc and five line segments. **3**



Evaluate $\int_0^5 2(f(x)+1)\,dx$. Give your answer as an exact value.

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QUESTION THIRTY ONE (3 marks)

(a) Differentiate $\sqrt{e^{4x}-1}$ with respect to x. **1**

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(b) The gradient function of a curve is $\dfrac{dy}{dx} = \dfrac{e^{4x}}{\sqrt{e^{4x}-1}}$. The curve passes through the point $(\ln 3, 0)$. Find the equation of the curve, giving your answer in exact form. **2**

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QUESTION THIRTY TWO (8 marks)

Consider the function $f(x) = x \ln x - 2x$.

(a) State the domain of $f(x)$. **1**

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(b) Find all stationary points of the function and determine their nature. **3**

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(c) Find all x-intercepts of the function. **2**

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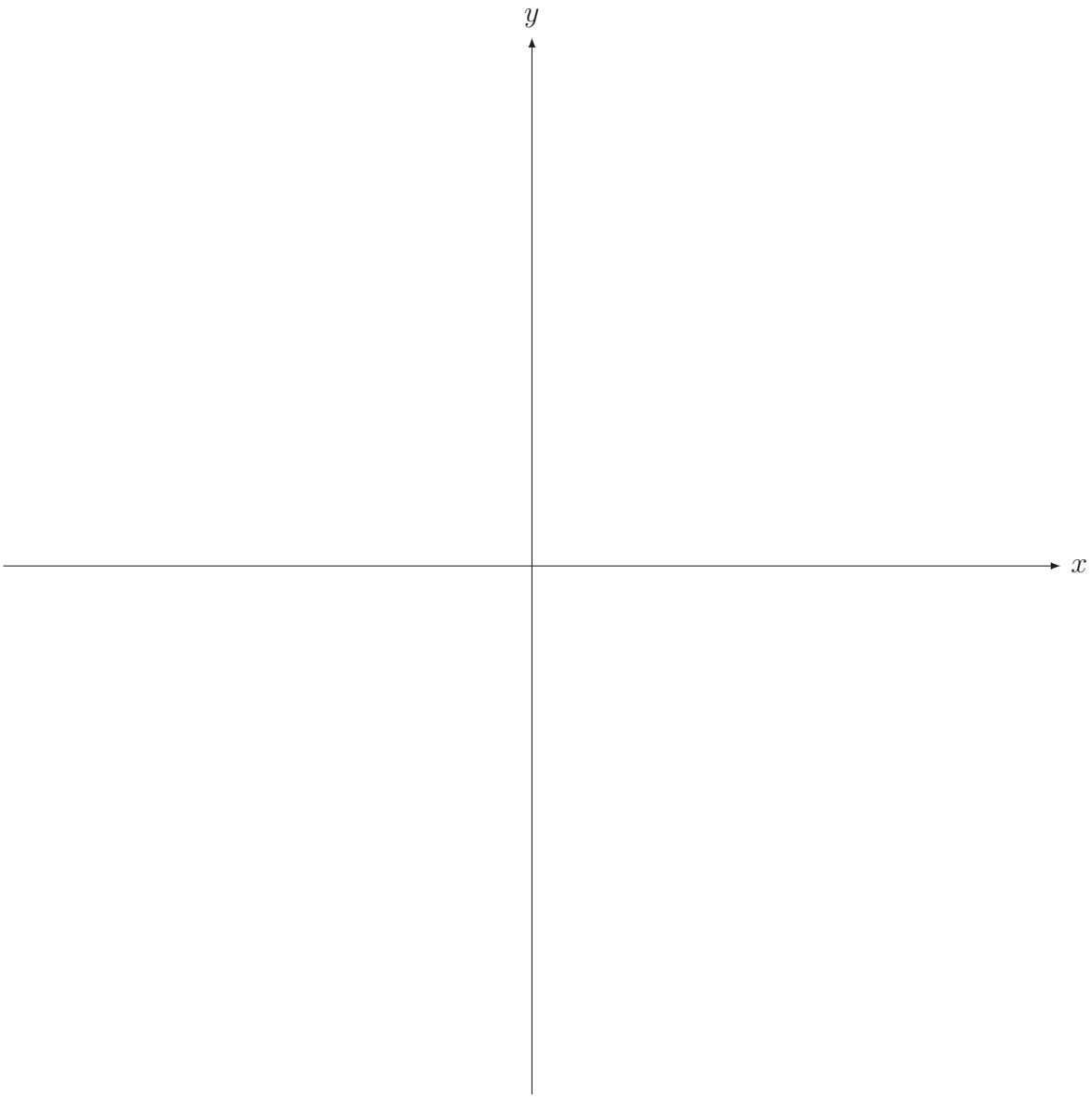
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(d) Hence, sketch the graph of $y = f(x)$. **2**



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B L A N K P A G E

CANDIDATE NUMBER

Section II

Part E

QUESTION THIRTY THREE (4 marks)

Let the concentration of hydrogen ions in a solution be H, measured in $\text{mol}\,\text{L}^{-1}$.

A scientist prepares a number of solutions whose concentrations of hydrogen ions form a geometric sequence. Let H_n be the concentration of the nth solution. The first solution has concentration $H_1 = 10^{-2}\ \text{mol}\,\text{L}^{-1}$ and each subsequent solution has half the concentration of the previous solution.

(a) Find the concentration of hydrogen ions in the nth solution. **1**

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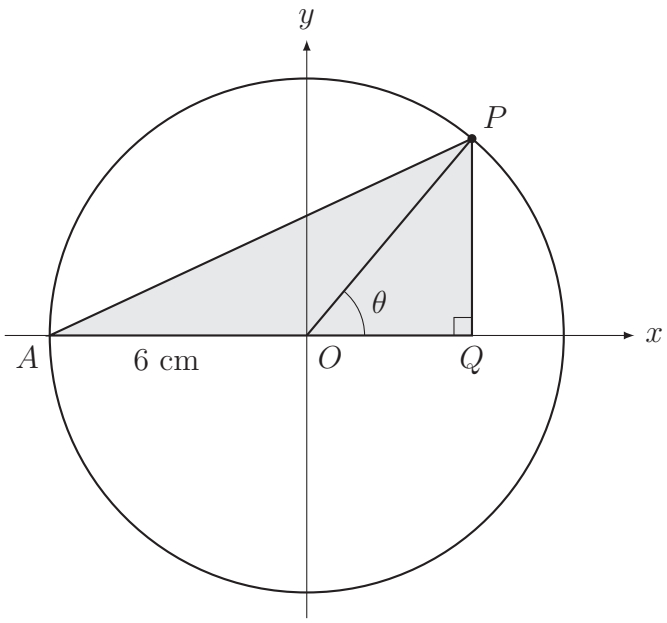
(b) The acidity of a solution, $A(H)$, is measured by its pH value, where $A(H) = -\log_{10}(H)$. Show that the acidity of the nth solution is given by $A_n = 2 + (n-1)\log_{10} 2$. **2**

(c) Determine the first solution in the sequence whose acidity exceeds 7. **1**

QUESTION THIRTY FOUR (6 marks)

The diagram shows a circle with centre O and radius 6 cm. The circle meets the negative x-axis at point A.

The point P lies on the circle in the first quadrant, and OP makes an angle θ radians with the positive x-axis. The point Q is the foot of the perpendicular from P to the x-axis.



(a) Show that the area of triangle APQ is $18\sin\theta(1+\cos\theta)$ cm^2. **2**

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(b) Use calculus to find the maximum possible area of triangle APQ. **4**

QUESTION THIRTY FIVE (6 marks)

The curve $y = 4x - x^2$ and the x-axis enclose a region.

(a) Show that the area of the region is $\frac{32}{3}$ square units. **2**

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(b) A horizontal line $y = 4 - k$, where $0 < k < 4$, cuts the curve at two points and divides the region into two equal areas. Find the exact value of k. **4**

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SYDNEY GRAMMAR SCHOOL

NAME

MATHS MASTER

CANDIDATE NUMBER

2026 Trial Examination

Form VI Mathematics Advanced

Wednesday 12th August, 2026

8:40am

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Writer: WJW

Section I

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1. Which of the following is the y-intercept of the curve $y = e^x + e^{-x}$?

   (A) $(0, -1)$

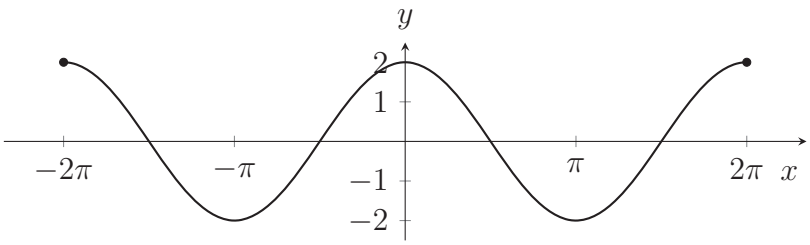
   (B) $(0, 0)$

   (C) $(0, 1)$

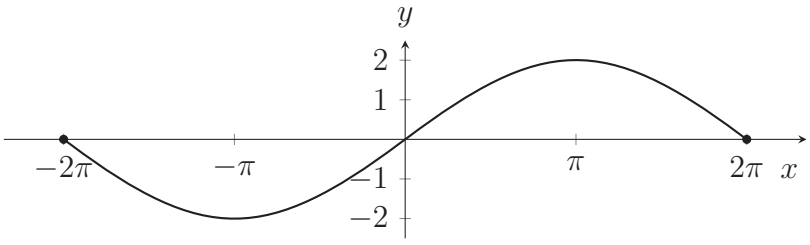
   (D) $(0, 2)$

2. Which of the following is the graph of $y = 2\cos x$ for $-2\pi \leq x \leq 2\pi$?

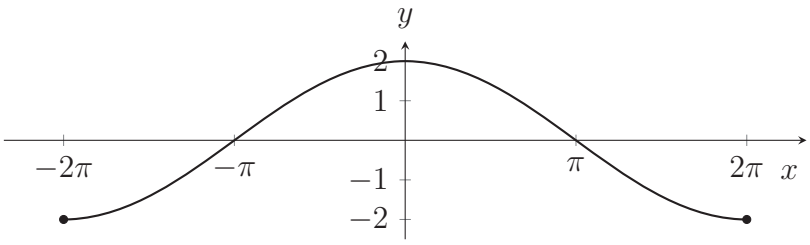
   (A)



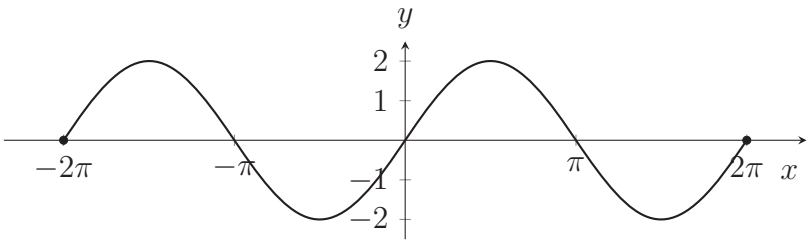
   (B)



   (C)



   (D)



3. Which of the following is a correct answer for $\int x^2 + 6x - 1\,dx$?

   (A) $2x + 6$

   (B) $\frac{x^3}{3} + 3x^2 - x$

   (C) $2x + 3x^2$

   (D) $\frac{x^2}{3} + 3x - x$

4. A table of future value interest factors is shown below.

Table of future value interest factors

Period	*Interest rate per period*					
	1%	2%	3%	4%	6%	8%
2	2.0100	2.0200	2.0300	2.0400	2.0600	2.0800
3	3.0301	3.0604	3.0909	3.1216	3.1836	3.2464
4	4.0604	4.1216	4.1836	4.2465	4.3746	4.5061
5	5.1010	5.2040	5.3091	5.4163	5.6371	5.8666
6	6.1520	6.3081	6.4684	6.6330	6.9753	7.3359
7	7.2135	7.4343	7.6625	7.8983	8.3938	8.9228
8	8.2857	8.5830	8.8923	9.2142	9.8975	10.6366
9	9.3685	9.7546	10.1591	10.5828	11.4913	12.4876
10	10.4622	10.9497	11.4639	12.0061	13.1808	14.4866
11	11.5668	12.1687	12.8078	13.4864	14.9716	16.6455
12	12.6825	13.4121	14.1920	15.0258	16.8699	18.9771

An investor wishes to accumulate \$50 000 over 2 years by making contributions at the end of each quarter. Interest is paid at 4% p.a., compounded quarterly.

Using the table of future value interest factors, what is the quarterly contributions required to accumulate \$50 000?

(A) \$5426

(B) \$6034

(C) \$24 510

(D) \$24 876

5. A fair coin is tossed four times. What is the probability of obtaining more heads than tails?

   (A) $\frac{1}{16}$

   (B) $\frac{1}{2}$

   (C) $\frac{5}{16}$

   (D) $\frac{3}{8}$

6. Which of the following statements is **TRUE**?

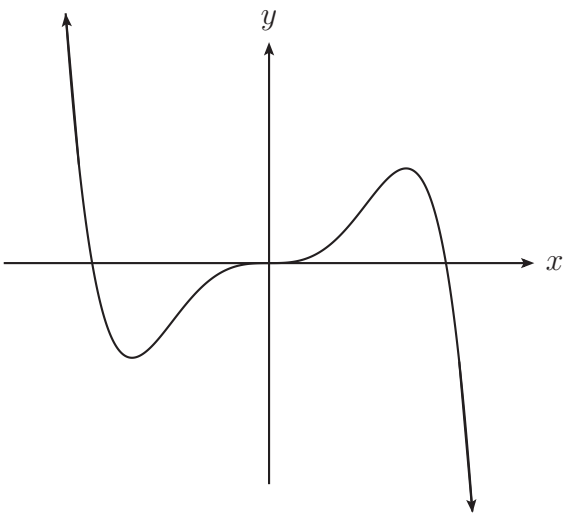
   (A) The sum of two even functions is odd.

   (B) The sum of two odd functions is even.

   (C) The product of two even functions is odd.

   (D) The product of two odd functions is even.

7. The diagram below shows the graph of $y = f(x)$.



   Which of the following statements must be false?

   (A) $f(x) = 0$ has three distinct solutions.

   (B) $f'(x) = 0$ has three distinct solutions.

   (C) $f''(x) = 0$ has three distinct solutions.

   (D) $f(x)f'(x) = 0$ has six distinct solutions.

8. Given $y = \sqrt{1+\sqrt{x}}$, which of the following is the correct expression for $\dfrac{dy}{dx}$?

(A) $\dfrac{1}{2\sqrt{1+\sqrt{x}}}$

(B) $\dfrac{x}{4\sqrt{1+\sqrt{x}}}$

(C) $\dfrac{1}{2\sqrt{x}\sqrt{1+\sqrt{x}}}$

(D) $\dfrac{1}{4\sqrt{x+x\sqrt{x}}}$

9. Let $f(x) = \sin 2x$ and $g(x) = |x-\pi|$, where $0 \le x \le 2\pi$.
Which of the following is the correct expression for $f \circ g(x)$?

(A) $f \circ g(x) = \begin{cases} -\sin 2x, & 0 \le x < \pi, \\ \sin 2x, & \pi \le x \le 2\pi. \end{cases}$

(B) $f \circ g(x) = \begin{cases} \sin 2x, & 0 \le x < \pi, \\ \sin 2x, & \pi \le x \le 2\pi. \end{cases}$

(C) $f \circ g(x) = \begin{cases} -\sin 2x, & 0 \le x < \pi, \\ -\sin 2x, & \pi \le x \le 2\pi. \end{cases}$

(D) $f \circ g(x) = \begin{cases} \sin 2x, & 0 \le x < \pi, \\ -\sin 2x, & \pi \le x \le 2\pi. \end{cases}$

10. The graph of $y = f(x)$ undergoes the following sequence of transformations in order.

    1. Translate the graph a units to the right, where $a > 0$.
    2. Dilate the graph horizontally by a factor of b, where $b > 1$.
    3. Translate the graph a units to the left.
    4. Dilate the graph horizontally by a factor of $\frac{1}{b}$.

    Which of the following describes the overall effect of these transformations?

    (A) A single translation of $a\left(1 - \frac{1}{b}\right)$ units to the right.

    (B) A single translation of $a\left(1 - \frac{1}{b}\right)$ units to the left.

    (C) A single translation of $a(b - 1)$ units to the right.

    (D) No change to the graph.

End of Section I

The paper continues in the next section

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Section II

Part A

QUESTION ELEVEN (2 marks)

(a) Find the domain of $f(x) = \sqrt{x+4}$. **1**

$x+4 \geq 0$

$x \geq -4$ or $[-4, \infty)$ (1)

(b) Find the range of $f(x) = e^x - 3$. **1**

$y > -3$ or $(-3, \infty)$ (1)

QUESTION TWELVE (2 marks)

Find the derivative of:

(a) $y = \sin 3x$ **1**

$y' = 3\cos 3x$ (1)

(b) $y = \ln(4 - x)$ **1**

$y' = \frac{-1}{4-x}$ or $\frac{1}{x-4}$ (1)

QUESTION THIRTEEN (4 marks)

Find:

(a) $\int (2x-5)^2\,dx$ **2**

$\frac{1}{2}\int 2(2x-5)^2\,dx$ or $\int 4x^2-20x+25\,dx$

$= \frac{4}{3}x^3 - 10x^2 + 25x + C$

$= \frac{1}{2}\times\frac{1}{3}(2x-5)^3 + C$

$= \frac{1}{6}(2x-5)^3 + C$ (1)

(1) First integration question, 1 mark for +C
Only pin a mark if boys did not put +C

(b) $\int_0^{\frac{\pi}{4}} \cos 2x\,dx$ **2**

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\cos 2x\,dx$

$= \frac{1}{2}\left[\sin 2x\right]_0^{\frac{\pi}{4}}$ (1)

$= \frac{1}{2}\times\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= \frac{1}{2}$ (1)

QUESTION FOURTEEN (2 marks)

A school library records the total number of books at the beginning of each year. The library contained 4200 books at the beginning of 2023 and 4380 books at the beginning of 2026.

(a) Assuming the same number of books was added each year, calculate the annual increase in the number of books. **1**

2023 2024 2025 2026

4200 4380

$T_n = a + (n-1)d$

$4380 = 4200 + 3d$

$3d = 180$

$d = 60$ (1)

(b) Hence determine the number of books in the library at the beginning of 2025. **1**

$4200 + 60 \times 2$

$= 4320$ (1)

QUESTION FIFTEEN (2 marks)

Consider the function $f(x) = \tan(\pi x) + 3$.

(a) State the period. **1**

Period $= \frac{\pi}{\pi}$

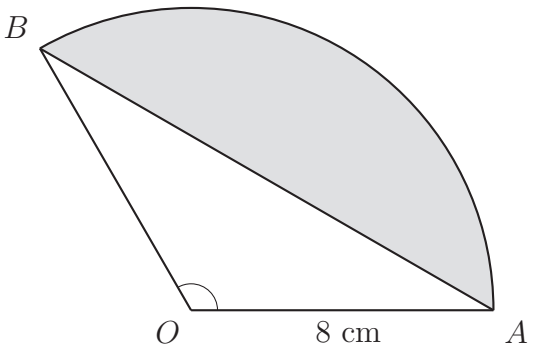
$= 1$ (1)

(b) State the equation of the vertical asymptote of $y = f(x)$ for $0 \leq x \leq 1$. **1**

$x = \frac{1}{2}$ (1)

QUESTION SIXTEEN (4 marks)

The diagram shows a sector OAB of a circle with centre O and radius 8 cm.



(a) Find the length of the chord AB, given that $\angle AOB = \dfrac{2\pi}{3}$. **2**

$AB^2 = 8^2 + 8^2 - 2\times 8\times 8\cos\frac{2\pi}{3}$ ①

$AB^2 = 64 + 64 - 128\times -\frac{1}{2}$

$= 192$

$AB = \sqrt{192}$

$= 8\sqrt{3}$ cm ①

or $\dfrac{AB}{\sin(\frac{2\pi}{3})} = \dfrac{8}{\sin(\frac{\pi}{6})}$

$AB = \dfrac{8\sin(\frac{2\pi}{3})}{\sin(\frac{\pi}{6})}$

$= 8\sqrt{3}$ cm.

(b) Find the exact area of the minor segment, shaded in the diagram above. **2**

$A = \frac{1}{2}\theta r^2 - \frac{1}{2}ab\sin\theta$

$= \frac{1}{2}\times\frac{2\pi}{3}\times 8^2 - \frac{1}{2}\times 8\times 8\sin\frac{2\pi}{3}$ ① either area of sector or area of triangle

$= \frac{\pi}{3}\times 64 - 32\times\frac{\sqrt{3}}{2}$

$= \left(\frac{64\pi}{3} - 16\sqrt{3}\right)$ cm^2 ①

QUESTION SEVENTEEN (3 marks)

Find the equation of the tangent to $y = e^{2x}$ at the point where $x = \frac{1}{2}$. **3**

$y = e^{2x}$

$\frac{dy}{dx} = 2e^{2x}$ ① Correct $\frac{dy}{dx}$

gradient of tangent at $x = \frac{1}{2}$

At $x = \frac{1}{2}$ $y = e^1$

$m_t = 2e^{2 \times \frac{1}{2}}$

$= 2e$ ① correct gradient

equation of tangent

$y - e = 2e(x - \frac{1}{2})$

$y = 2ex - e + e$

$y = 2ex$ ① correct equation

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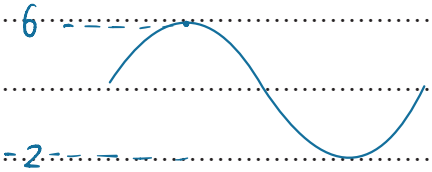
Section II

Part B

QUESTION EIGHTEEN (4 marks)

The function $y = a\sin(bx) + c$ has maximum value 6, minimum value -2, and period $\frac{2\pi}{3}$ radians, where $a > 0$.

(a) Find the values of a, b and c. **3**



$c = \frac{6+(-2)}{2}$

$= \frac{4}{2}$

$= 2$ (1)

$b = \frac{2\pi}{\frac{2\pi}{3}}$

$= 3$ (1)

$a = 6 - 2$

$= 4$ (1)

(b) Hence, state the number of solutions to the equation **1**

$$a\sin(bx) + c = 4, \qquad \text{for } 0 \leq x \leq 2\pi.$$

6 solutions (1)

QUESTION NINETEEN (3 marks)

The point $P(0, \frac{1}{2})$ lies on the curve $y = \frac{1}{2}(\sin x + \cos x)^2$. Another point Q lies on the curve in the domain $0 \leq x \leq \pi$ such that the tangent to the curve at Q is perpendicular to the tangent at P. Find the coordinates of Q. **3**

$y' = \frac{1}{2} \times 2(\sin x + \cos x) \times (\cos x - \sin x)$

$= \cos^2 x - \sin^2 x$ ① Differentiation

m_t at $P = (\cos 0)^2 - (\sin 0)^2$

$= 1$

m_t at $Q = -1$ ① Gradient of tangent at Q

Let $\cos^2 x - \sin^2 x = -1$ or $\cos 2x = -1$

$\cos^2 x - (-\cos^2 x) = -1$ $2x = \pi$

$2\cos^2 x - 1 = -1$ $x = \frac{\pi}{2}$

$\cos^2 x = 0$

$x = \frac{\pi}{2}$

$\therefore y = \frac{1}{2}(\sin \frac{\pi}{2} + \cos \frac{\pi}{2})^2$

$= \frac{1}{2}(1 + 0)^2$

$= \frac{1}{2}$

$\therefore Q(\frac{\pi}{2}, \frac{1}{2})$ ① correct coordinates of Q

QUESTION TWENTY (2 marks)

The table shows present value interest factors for an annuity of \$1.

Rate(r) / *Period*(n)	0.0050	0.0055	0.0060	0.0065	0.0070
58	50.23911	49.54343	48.86109	48.19179	47.53525
59	50.98419	50.26696	49.56370	48.87411	48.19786
60	51.72556	50.98653	50.26213	49.55202	48.85587
61	52.46324	51.70217	50.95639	50.22555	49.50931

(a) Sophie plans to pay \$200 into an investment account at the end of each month for 58 months. Her investment will earn 0.55% interest per month. Use the information in the table to calculate the present value of this annuity. **1**

\$200 × 49.54343
= \$9908.69 (1)

(b) Daniel borrows \$22 600 at an interest rate of 7.8% per annum, compounded monthly. He can afford repayments of at most \$450 per month. Using the table, determine the minimum number of monthly repayments required. **1**

\$22600 = \$450 × interest factor
x = 50.222

7.8% ÷ 12 = 0.0065

The minimum number of month is 61 months (1)

QUESTION TWENTY ONE (4 marks)

The table shows the probability distribution of a discrete random variable.

x	5	10	15	20
$P(X = x)$	k^2	0.27	0.67	$0.1k$

(a) Find the value of k. **3**

$k^2 + 0.27 + 0.67 + 0.1k = 1$

$k^2 + 0.1k - 0.06 = 0$ ① Sum of $xP(x)$

$100k^2 + 10k - 6 = 0$

$50k^2 + 5k - 3 = 0$ ① Progress towards

$(5k-1)(10k+3) = 0$

$k = \frac{1}{5}$ $k = -\frac{3}{10}$

(Not possible)

① Final correct k value

(b) Find the expected value $E(X)$. **1**

$E(X) = 5 \times \left(\frac{1}{5}\right)^2 + 10 \times 0.27 + 15 \times 0.67 + 20 \times 0.1 \times \frac{1}{5}$

$= 13.35$ ①

QUESTION TWENTY TWO (2 marks)

Prove that **2**

$$\frac{(\cot\theta - \tan\theta)^2}{\cot^2\theta - \tan^2\theta} = \cos^2\theta - \sin^2\theta.$$

$LHS = \dfrac{(\cot\theta - \tan\theta)^2}{(\cot\theta + \tan\theta)(\cot\theta - \tan\theta)}$

① Progress towards

$= \dfrac{\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}} \times \sin\theta\cos\theta$ $\times \sin\theta\cos\theta$

$= \dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$ ①

$= \cos^2\theta - \sin^2\theta$

$= RHS$

QUESTION TWENTY THREE (3 marks)

A survey of pupils at a school found that one-third of the pupils are members of the Chess Club, one-half are members of the Debating Club, and two-thirds are members of either the Chess Club or the Debating Club or both.

One pupil from the survey is selected at random. Let C be the event that the selected pupil is a member of the Chess Club, and let D be the event that the selected pupil is a member of the Debating Club.

(a) Find $P(C \cap D)$. **1**

$P(C) = \frac{1}{3}$ $P(D) = \frac{1}{2}$ $P(C \cup D) = \frac{2}{3}$

$P(C \cap D) = P(C) + P(D) - P(C \cup D)$

$= \frac{1}{3} + \frac{1}{2} - \frac{2}{3}$

$= \frac{1}{6}$ (1)

(b) Hence state whether C and D are mutually exclusive. Justify your answer. **1**

$P(C \cap D) = \frac{1}{6} \neq 0$ (1) Must provide reason for non mutually exclusive

$\therefore$ They are non mutually exclusive

(c) Determine whether C and D are independent. Justify your answer with calculations. **1**

$P(C) \times P(D) = \frac{1}{3} \times \frac{1}{2}$

$= \frac{1}{6}$

$P(C \cap D) = \frac{1}{6}$

$\therefore P(C) \times P(D) = P(C \cap D)$ (1) Must show $P(C) \times P(D) = P(C \cap D)$

$\therefore$ They are independent

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Section II

Part C

QUESTION TWENTY FOUR (2 marks)

The table shows the velocity of a cyclist at different times during a ride. **2**

Time, t seconds	0	10	20	30	40
Velocity, v m/s	3.2	4.8	6.1	5.7	4.9

Use the trapezoidal rule with 4 subintervals to estimate the distance travelled by the cyclist during the first 40 seconds. Give your answer correct to one decimal place.

Distance $= \frac{10}{2}(3.2 + 4.9 + 2\times(4.8+6.1+5.7))$ (1)

$= 5 \times 41.3$

$= 206.5\text{ m}$ (1)

QUESTION TWENTY FIVE (4 marks)

Consider the geometric series $\cos x + \cos^3 x + \cos^5 x + \cos^7 x + \cdots$, where $0 < x < \frac{\pi}{2}$.

(a) Show that the limiting sum of this series exists. **1**

$r = \frac{\cos^3 x}{\cos x}$

$= \cos^2 x$

$0 < \cos^2 x < 1$ for $0 < x < \frac{\pi}{2}$

$\therefore$ the limiting sum exists

(b) Hence solve $\cos x + \cos^3 x + \cos^5 x + \cos^7 x + \cdots = \frac{2}{3}$ for $0 < x < \frac{\pi}{2}$. **3**

$S_\infty = \frac{a}{1-r}$

$= \frac{\cos x}{1-\cos^2 x}$ ① Limiting sum of GP

let $\frac{\cos x}{1-\cos^2 x} = \frac{2}{3}$

$2 - 2\cos^2 x = 3\cos x$

$2\cos^2 x + 3\cos x - 2 = 0$ ① Equation of cosine

$(\cos x + 2)(2\cos x - 1) = 0$

$\cos x = -2$ (reject) $\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ ① correct solution

(if showed other solutions outside of domain, a mark was deducted)

QUESTION TWENTY SIX (5 marks)

A medical substance is injected into a patient's bloodstream. The amount of the substance remaining in the bloodstream after t hours is modelled by

$A = A_0 e^{-kt}$, where A_0 is the initial amount and k is a positive constant.

After 6 hours, the amount of substance remaining is 40% of the initial amount.

(a) Show that $k = \frac{1}{6}\ln\left(\frac{5}{2}\right)$. **1**

$A = A_0 e^{-kt}$ $t = 6$ $A = 40\% A_0$

$= \frac{2}{5} A_0$

$\frac{2}{5} A_0 = A_0 e^{-6k}$

$\frac{2}{5} = e^{-6k}$

$\frac{5}{2} = e^{6k}$

$\ln \frac{5}{2} = 6k$

$k = \frac{1}{6} \ln \frac{5}{2}$ ①

(b) Find the rate, in terms of A_0, at which the amount of the substance is changing when the remaining amount is equal to half of its initial amount. **2**

$\frac{dA}{dt} = -kA_0 e^{-kt}$ ① Differentiate correctly

$= -kA_0 \times \frac{1}{2}$

$= -\frac{1}{6} \times \frac{1}{2} \ln\left(\frac{5}{2}\right) A_0$ ①

$= -\frac{1}{12} \ln\left(\frac{5}{2}\right) A_0$

or

$\frac{1}{12} \ln\left(\frac{2}{5}\right) A_0$

Let $A = \frac{1}{2} A_0$

$\frac{1}{2} A_0 = A_0 e^{-kt}$

$\frac{1}{2} = e^{-kt}$

(c) Find the time taken for the amount of the substance to fall to 12% of its initial amount. Give your answer to the nearest hour. **2**

$A = A_0 e^{-\frac{1}{6}\ln(\frac{5}{2})t}$

Let $A = 12\% A_0$

$= \frac{3}{25} A_0$

$\frac{3}{25} A_0 = A_0 e^{-\frac{1}{6}\ln(\frac{5}{2})t}$

$\frac{3}{25} = e^{-\frac{1}{6}\ln(\frac{5}{2})t}$ ① progress towards

$\ln(\frac{25}{3}) = \frac{1}{6}\ln(\frac{5}{2})\, t$

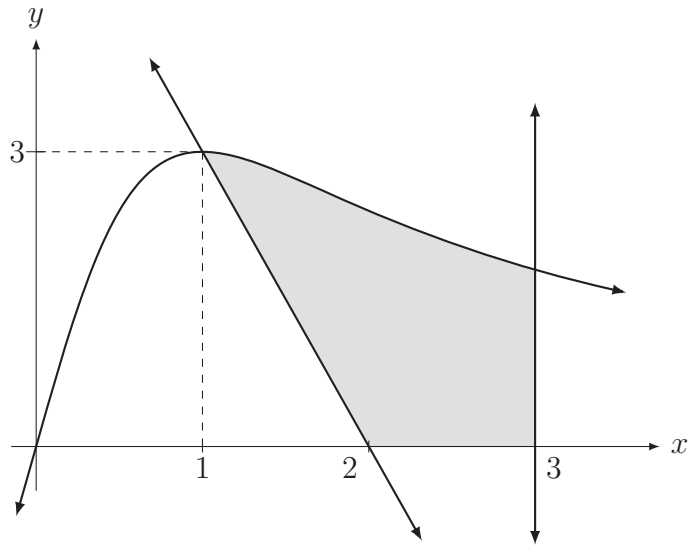
$t = \ln(\frac{25}{3}) \div \frac{1}{6}\ln(\frac{5}{2})$

$= 13.88$

$\doteq$ 14 hours ①

QUESTION TWENTY SEVEN (3 marks)

The diagram shows the shaded region bounded by the curve $y = \dfrac{6x}{x^2+1}$, the line $y = -3x+6$, the line $x = 3$ and the x-axis. **3**



Find the exact area of the shaded region.

Shaded area $= \int_1^3 \dfrac{6x}{x^2+1}\,dx$ − area of triangle

$= \left[3\ln|x^2+1|\right]_1^3 - \frac{1}{2}\times 1\times 3$

$= 3\ln|10| - 3\ln|2| - \frac{3}{2}$

$= \left(3\ln 5 - \frac{3}{2}\right)$ u²

① Finding area under the curve $\frac{6x}{x^2+1}$ from 1 to 3

① Finding the area of the triangle

① Final correct answer

QUESTION TWENTY EIGHT (6 marks)

Naomi takes out a mortgage of \$650 000 to purchase a home. The mortgage is to be repaid over 30 years in equal monthly repayments, paid at the end of each month. Interest is charged at 4.8% per annum, compounded monthly. Let A_n be the amount owing on the mortgage immediately after the nth monthly repayment.

(a) Find a formula for the amount, A_n, and calculate Naomi's monthly repayment, correct to the nearest cent. **3**

$R = 4.8\%$ p.a $\quad n = 30$ years

$= 0.4\%$ pm $\quad = 360$ months

$= 0.004$

① Develop the formula. Need to have A_1, A_2, A_3 or A_n

Let the monthly repayment be m

$A_1 = 650000 \times (1+0.004)^1 - m$

$A_2 = (650000 \times (1.004)^1 - m)(1.004) - m$

$= 650000 \times (1.004)^2 - m(1.004) - m$

$A_3 = 650000 \times (1.004)^3 - m(1.004)^2 - m(1.004)^1 - m$

$A_n = 650000 \times (1.004)^n - m(1.004)^{n-1} - m(1.004)^{n-2} - \cdots - m(1.004)^1 - m$

$A_{360} = 650000 \times (1.004)^{360} - (m + m(1.004) + m(1.004)^2 + \cdots + m(1.004)^{359})$

$= 650000 \times (1.004)^{360} - \left(\frac{m((1.004)^{360} - 1)}{0.004}\right)$ ① Correct sum of GP.

Let $A_n = 0$

$650000 \times (1.004)^{360} - \frac{m(1.004^{360} - 1)}{0.004} = 0$

$m = 650000 (1.004)^{360} \div \left(\frac{1.004^{360} - 1}{0.004}\right)$ ① Correct solution

$\doteq 3410.32$

(b) Instead of making the monthly repayments found in part (a), Naomi decides to make monthly repayments of \$4000 throughout the term of the loan. The interest rate remains unchanged. Calculate how long it will take her to repay the loan. **2**

$A_n = 650000 \times (1.004)^n - \frac{m(1.004^n - 1)}{0.004}$

Let $m = 4000$ find n

$0 = 650000 \times (1.004)^n - \frac{4000(1.004^n - 1)}{0.004}$ ① progress towards

$0 = 650000 \times (1.004)^n - 1000000(1.004^n - 1)$

$0 = 650000 \times (1.004)^n - 1000000 \times 1.004^n + 1000000$

$0 = -350000 \times 1.004^n + 1000000$

$350000 \times 1.004^n = 1000000$

$1.004^n = 2.85714$

$n = \log_{1.004} 2.85714$

$= 262.98$

$\doteq 263$ ①

(c) Hence, calculate how much interest Naomi will save over the life of the loan by increasing her monthly repayments to \$4000. **1**

\$3410.32 × 360 − \$4000 × 263

= \$175715.20

CANDIDATE NUMBER

Section II

Part D

QUESTION TWENTY NINE (3 marks)

Let P be the point $(\cos 2\theta, \sqrt{3}\sin 2\theta)$, where $0 < \theta < 2\pi$. The point A has coordinates $(1, 0)$. The midpoint of AP lies on the line $y = x - \frac{1}{2}$. Find all possible values of θ. **3**

midpoint AP $\left(\frac{1+\cos 2\theta}{2}, \frac{0+\sqrt{3}\sin 2\theta}{2}\right)$ is on the line

① Finding the midpoint or any good progress towards

$\therefore \frac{\sqrt{3}\sin 2\theta}{2} = \frac{1+\cos 2\theta}{2} - \frac{1}{2}$

$\sqrt{3}\sin 2\theta = 1 + \cos 2\theta - 1$

$\sqrt{3}\sin 2\theta = \cos 2\theta$

$\sqrt{3}\tan 2\theta = 1$

$\tan 2\theta = \frac{1}{\sqrt{3}}$ $\quad 0 \le 2\theta \le 4\pi$

$2\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

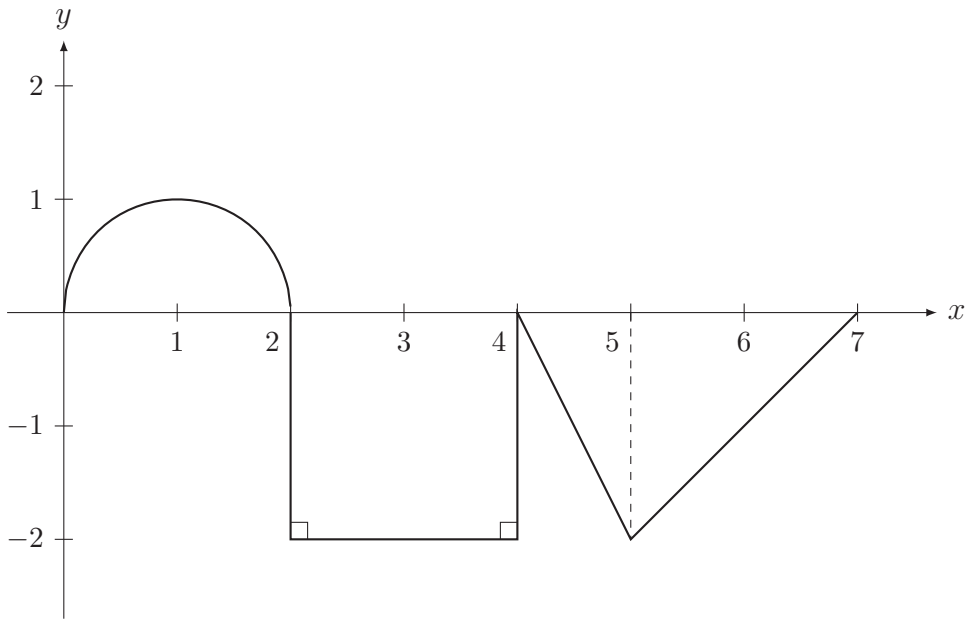
$= \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$

$\theta = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$

① ①

QUESTION THIRTY (3 marks)

The diagram below shows the graph of $y = f(x)$, formed by a semicircular arc and five line segments. **3**



Evaluate $\int_0^5 2(f(x)+1)\,dx$. Give your answer as an exact value.

$\int_0^5 f(x)\,dx$ = Area of semi-circle − Area of square − Area of triangle from 4-5

$= \frac{1}{2}\pi \times 1^2 - 2\times 2 - \frac{1}{2}\times 1\times 2$

$= \frac{1}{2}\pi - 4 - 1$

$= \frac{1}{2}\pi - 5$ ① consider area of semi-circle or area of square + area of triangle with negative sign.

$= 2\int_0^5 f(x)+1\,dx$

$= 2\int_0^5 f(x)\,dx + 2\int_0^5 1\,dx$ ① Find the value of $2\int_1^5 1\,dx$

$= 2(\frac{1}{2}\pi - 5) + 2[x]_0^5$

$= \pi - 10 + 2\times 5$

$= \pi$ ①

QUESTION THIRTY ONE (3 marks)

(a) Differentiate $\sqrt{e^{4x}-1}$ with respect to x. **1**

$\frac{d}{dx}\left(e^{4x}-1\right)^{\frac{1}{2}}$

$= \frac{1}{2}\left(e^{4x}-1\right)^{-\frac{1}{2}} \times 4e^{4x}$

$= 2e^{4x}\left(e^{4x}-1\right)^{-\frac{1}{2}}$ ①

(b) The gradient function of a curve is $\frac{dy}{dx} = \frac{e^{4x}}{\sqrt{e^{4x}-1}}$. The curve passes through the point $(\ln 3, 0)$. Find the equation of the curve, giving your answer in exact form. **2**

From part a)

$\int \frac{2e^{4x}}{\sqrt{e^{4x}-1}}\,dx = \sqrt{e^{4x}-1} + C_1$

$\therefore \int \frac{e^{4x}}{\sqrt{e^{4x}-1}}\,dx = \frac{1}{2}\sqrt{e^{4x}-1} + C_2$

$\therefore f(x) = \frac{1}{2}\sqrt{e^{4x}-1} + C$ ①

$f(\ln 3) = 0$

$\therefore 0 = \frac{1}{2}\sqrt{e^{4\ln 3}-1} + C$

$0 = \sqrt{3^4-1} + C$

$0 = \frac{1}{2}\sqrt{80} + C$

$C = -\frac{1}{2}\sqrt{80}$

$= -2\sqrt{5}$ ① Find the correct value of C, must simplify the value of C

$\therefore f(x) = \frac{1}{2}\sqrt{e^{4x}-1} - 2\sqrt{5}$

QUESTION THIRTY TWO (8 marks)

Consider the function $f(x) = x \ln x - 2x$.

(a) State the domain of $f(x)$. **1**

$x > 0$ ①

(b) Find all stationary points of the function and determine their nature. **3**

$f'(x) = \ln x + \frac{1}{x} \times x - 2$

$= \ln x + 1 - 2$

$= \ln x - 1$ ①

$f''(x) = \frac{1}{x}$

Let $f'(x) = 0$

$\ln x - 1 = 0$

$\ln x = 1$

$x = e$ ① Must find the y-value of the stationary point.

$f(e) = e \ln e - 2e$

$= e - 2e$

$= -e$

$\therefore (e, -e)$ is the stationary point

check the nature

$f''(e) = \frac{1}{e} > 0 \cup$ ① If check using the table, must show the values.

min

$\therefore (e, -e)$ is a min turning point

(c) Find all x-intercepts of the function. **2**

$x \ln x - 2x = 0$

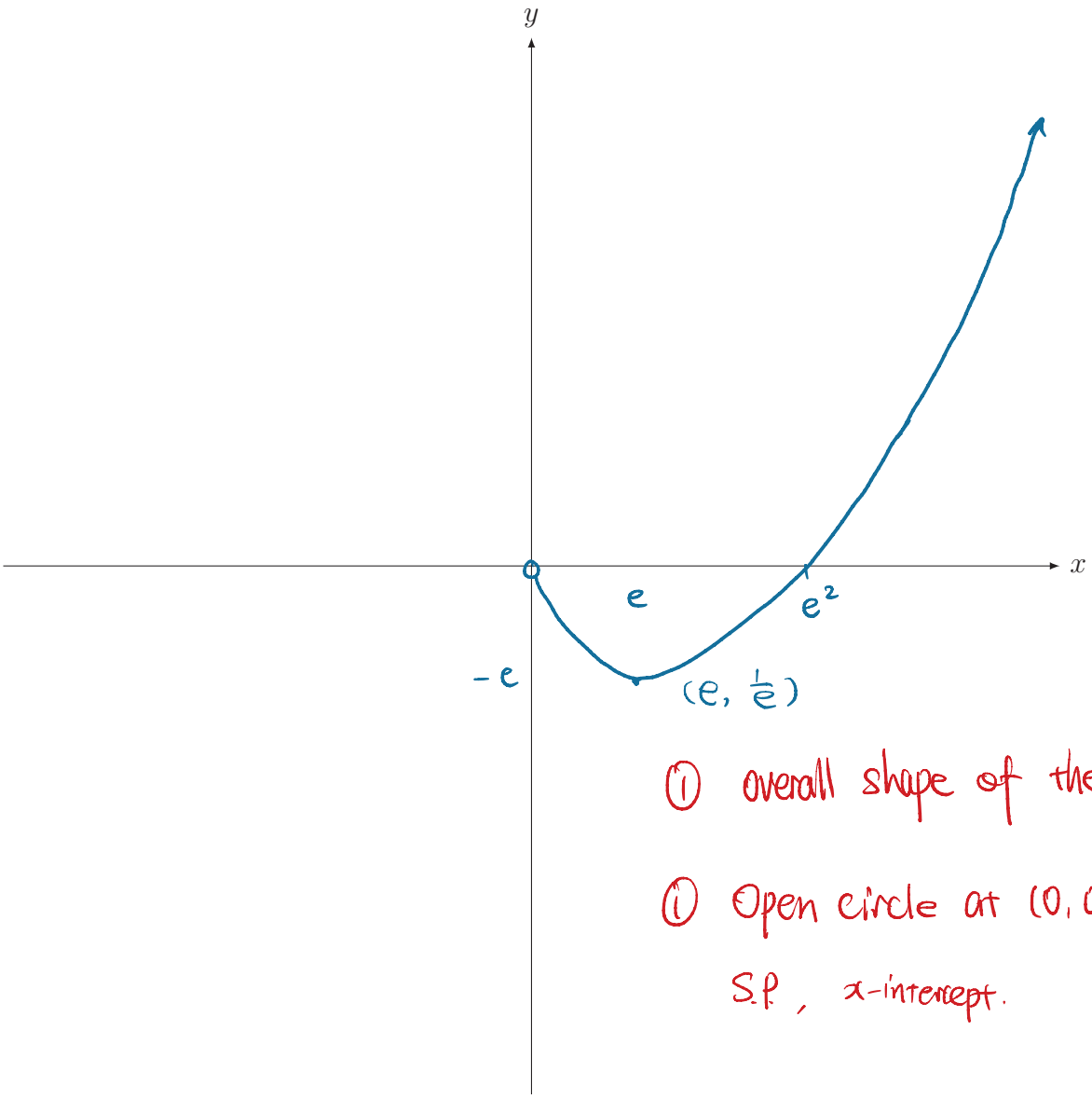
$x(\ln x - 2) = 0$

$x = 0$ (reject) ①

$\ln x = 2$

$x = e^2$ ①

(d) Hence, sketch the graph of $y = f(x)$. **2**



① overall shape of the curve

① Open circle at (0, 0)

S.P., x-intercept.

CANDIDATE NUMBER

Section II

Part E

QUESTION THIRTY THREE (4 marks)

Let the concentration of hydrogen ions in a solution be H, measured in mol L^{-1}.

A scientist prepares a number of solutions whose concentrations of hydrogen ions form a geometric sequence. Let H_n be the concentration of the nth solution. The first solution has concentration $H_1 = 10^{-2}$ mol L^{-1} and each subsequent solution has half the concentration of the previous solution.

(a) Find the concentration of hydrogen ions in the nth solution. **1**

$H_1 = 10^{-2}$ $H_2 = \frac{1}{2} \times 10^{-2}$ $H_3 = \frac{1}{2}^2 \times 10^{-2}$

common ratio $= \frac{1}{2}$

$H_n = ar^{n-1}$

$= 10^{-2}\left(\frac{1}{2}\right)^{n-1}$ ①

(b) The acidity of a solution, $A(H)$, is measured by its pH value, where $A(H) = -\log_{10}(H)$. Show that the acidity of the nth solution is given by $A_n = 2 + (n-1)\log_{10} 2$. **2**

$H_n = 10^{-2}\left(\frac{1}{2}\right)^{n-1}$

$A_n = -\log_{10}\left(10^{-2}\left(\frac{1}{2}\right)^{n-1}\right)$

$= -\left(\log_{10} 10^{-2} + \log_{10}\left(\frac{1}{2}\right)^{n-1}\right)$ ① progress towards

$= -\left(-2\log_{10} 10 + (n-1)\log_{10}\frac{1}{2}\right)$

$= 2 - (n-1)\log_{10}\frac{1}{2}$

$= 2 + (n-1)\log_{10} 2$ ①

(c) Determine the first solution in the sequence whose acidity exceeds 7. **1**

$2 + (n-1)\log_{10} 2 = 7$

$(n-1)\log_{10} 2 = 5$

$n - 1 = \frac{5}{\log_{10} 2}$

$n = \frac{5}{\log_{10} 2} + 1$

$= 17.6$

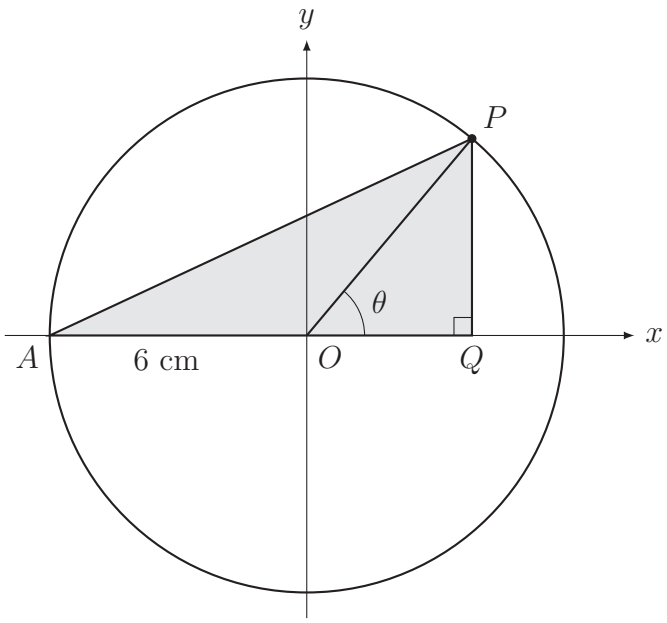
The first solution is when $n = 18$ (18th solution)

①

QUESTION THIRTY FOUR (6 marks)

The diagram shows a circle with centre O and radius 6 cm. The circle meets the negative x-axis at point A.

The point P lies on the circle in the first quadrant, and OP makes an angle θ radians with the positive x-axis. The point Q is the foot of the perpendicular from P to the x-axis.



(a) Show that the area of triangle APQ is $18\sin\theta(1+\cos\theta)$ cm^2. **2**

In $\triangle OPQ$

$\sin\theta = \frac{PQ}{6}$ $\quad \cos\theta = \frac{OQ}{6}$

$PQ = 6\sin\theta$ $\quad OQ = 6\cos\theta$ (1)

Area of $\triangle APQ = \frac{1}{2} \times AQ \times PQ$

$= \frac{1}{2} \times (6 + 6\cos\theta) \times 6\sin\theta$

$= 18\sin\theta(1+\cos\theta)$ (1)

(b) Use calculus to find the maximum possible area of triangle APQ. **4**

$A = 18\sin\theta(1+\cos\theta)$

$A' = 18\cos\theta(1+\cos\theta) - 18\sin^2\theta$

$= 18\cos\theta + 18\cos^2\theta - 18\sin^2\theta$

$= 18\cos\theta + 18\cos^2\theta - 18(1-\cos^2\theta)$

$= 18\cos\theta + 18\cos^2\theta - 18 + 18\cos^2\theta$

$= 36\cos^2\theta + 18\cos\theta - 18$ (1)

$A'' = -72\cos\theta\sin\theta - 18\sin\theta$

Let $A' = 0$ $\quad 36\cos^2\theta + 18\cos\theta - 18 = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$ $\quad \frac{1}{2}\ \ -1$

$(\cos\theta+1)(2\cos\theta-1) = 0$

$\cos\theta = -1 \quad \cos\theta = \frac{1}{2}$

(Not possible) $\quad \theta = \frac{\pi}{3}$ (1)

check $\theta = \frac{\pi}{3}$

$A'' = -72\cos\frac{\pi}{3}\sin\frac{\pi}{3} - 18\sin\frac{\pi}{3}$

$= -46.76 < 0$ $\cap$ max turning point (1)

Max Area $= 18\sin\frac{\pi}{3}(1+\cos\frac{\pi}{3})$

$= 18\times\frac{\sqrt{3}}{2}(1+\frac{1}{2})$

$= 9\sqrt{3}\times\frac{3}{2}$

$= \frac{27\sqrt{3}}{2}\ u^2$ (1)

QUESTION THIRTY FIVE (6 marks)

The curve $y = 4x - x^2$ and the x-axis enclose a region.

(a) Show that the area of the region is $\frac{32}{3}$ square units. **2**

$\int_0^4 4x - x^2 \, dx$

$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4$ (1)

$= \left(2\times 16 - \frac{1}{3}\times 64\right) - 0$

$= \frac{32}{3}\ u^2$ (1)

(b) A horizontal line $y = 4 - k$, where $0 < k < 4$, cuts the curve at two points and divides the region into two equal areas. Find the exact value of k. **4**

Since the line $y = 4-k$ divides the region into two equal areas

$A = \frac{1}{2}\times\frac{32}{3} = \frac{16}{3}$

$y = 4x - x^2$

$= -[x^2 - 4x + 4 - 4)$

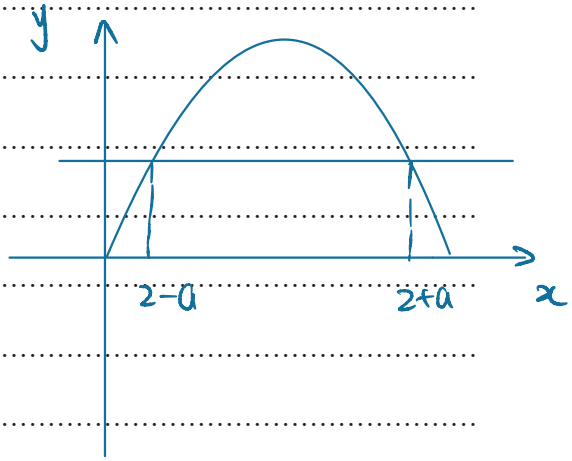
$= -(x-2)^2 + 4$

let $-(x-2)^2 + 4 = 4 - k$

$-(x-2)^2 = -k$

$x - 2 = \pm\sqrt{k}$

$x = 2 \pm \sqrt{k}$



let $a = \sqrt{k}$

$\therefore x = 2 \pm a$

$\int_{2-a}^{2+a} 4-(x-2)^2-4+k \; dx = \frac{16}{3}$ ① Find the interception points

$\int_{2-a}^{2+a} k-(x-2)^2 \; dx = \frac{16}{3}$ ① $\int$ top − bottom $= \frac{16}{3}$

$\int_{2-a}^{2+a} a^2-(x-2)^2 \; dx = \frac{16}{3}$

$\left[a^2x - \frac{1}{3}(x-2)^3\right]_{2-a}^{2+a} = \frac{16}{3}$ ① working

$\left(a^2(2+a) - \frac{1}{3}(a)^3\right) - \left(a^2(2-a) - \frac{1}{3}(-a)^3\right) = \frac{16}{3}$

$2a^2 + a^3 - \frac{1}{3}a^3 - 2a^2 + a^3 - \frac{1}{3}a^3 = \frac{16}{3}$

$2a^3 - \frac{2}{3}a^3 = \frac{16}{3}$

$\frac{4a^3}{3} = \frac{16}{3}$

$a^3 = 4 \quad a = \sqrt{k}$

$(\sqrt{k})^3 = 4$

$k^{\frac{3}{2}} = 4$

$k = 4^{\frac{2}{3}}$

$= \sqrt[3]{16}$ ①

END OF PAPER