BAULKHAM HILLS
HIGH
SCHOOL

2020

YEAR 12
HALF YEARLY
ASSESSMENTS

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 1 hour and 30 minutes
- Write using black pen
- Write your NESA# and Teacher's name on your answer booklet
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for careless or badly arranged work

Total marks: 53

Section I – 5 marks (pages 2 – 3)
- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II – 48 marks (pages 4 – 7)
- Attempt Questions 6 – 8
- Allow about 1 hour and 22 minutes for this section

Section I

5 marks
Attempt Questions 1 – 5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 5

1 $\displaystyle\int \frac{dx}{4+9x^2} = \ ?$

(A) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{3x}{2}\right)+c$

(B) $\dfrac{1}{6}\tan^{-1}\left(\dfrac{3x}{2}\right)+c$

(C) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{2x}{3}\right)+c$

(D) $\dfrac{1}{6}\tan^{-1}\left(\dfrac{2x}{3}\right)+c$

2 The region bounded by the curve $y = x^4$, the y-axis and the line $y = 16$ is rotated about the y-axis.

Which of the following expressions is correct for the volume of the solid of revolution created by this rotation?

(A) $\pi\displaystyle\int_0^2 x^8\,dx$

(B) $\pi\displaystyle\int_0^{16} x^8\,dx$

(C) $\pi\displaystyle\int_0^2 \sqrt{y}\,dy$

(D) $\pi\displaystyle\int_0^{16} \sqrt{y}\,dy$

3 What is the maximum value of $6\cos\theta + 4\sin\theta$?

(A) 10

(B) $2\sqrt{13}$

(C) 6

(D) $2\sqrt{5}$

4 Which of the following is the primitive of $\sin^2 x$?

(A) $x - \frac{1}{2}\sin 2x + c$

(B) $x + \frac{1}{2}\sin 2x + c$

(C) $\frac{x}{2} - \frac{1}{4}\sin 2x + c$

(D) $\frac{x}{2} + \frac{1}{4}\sin 2x + c$

5 Using the substitution $u = 2x + 1$, $\int (2x-1)\sqrt{2x+1}\;dx$ can be expressed as

(A) $\frac{1}{2}\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

(B) $\frac{1}{2}\int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du$

(C) $2\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

(D) $2\int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du$

END OF SECTION I

Section II

48 marks
Attempt Questions 6 – 8
Allow about 1 hour 22 minutes for this section

Answer each question on the appropriate answer sheet. Each answer sheet must show your NESA#. Extra paper is available.

In Questions 6 to 8, your responses should include relevant mathematical reasoning and/or calculations.

Marks

<u>**Question 6**</u> ***(15 marks)*** Use a *separate* answer sheet

(a) Differentiate the following functions with respect to x

(i) $f(x) = [\sin^{-1}(3x)]^3$ *2*

(ii) $f(x) = \ln(\cos^{-1}x)$ *2*

(iii) $f(x) = (x^2 + 1)\tan^{-1}x$ *2*

(b) Evaluate

(i) $\displaystyle\int_{-2}^{2} \frac{dx}{x^2 + 4}$ *2*

(ii) $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{4 - 3x^2}}$ *3*

(c) Solve $2\sin\theta - \cos\theta = 1$, for $0° \leq \theta \leq 360°$, correct to the nearest minute, using the substitution $t = \tan\dfrac{\theta}{2}$ *4*

Marks

<u>**Question 7**</u> ***(20 marks)*** Use a *separate* answer sheet

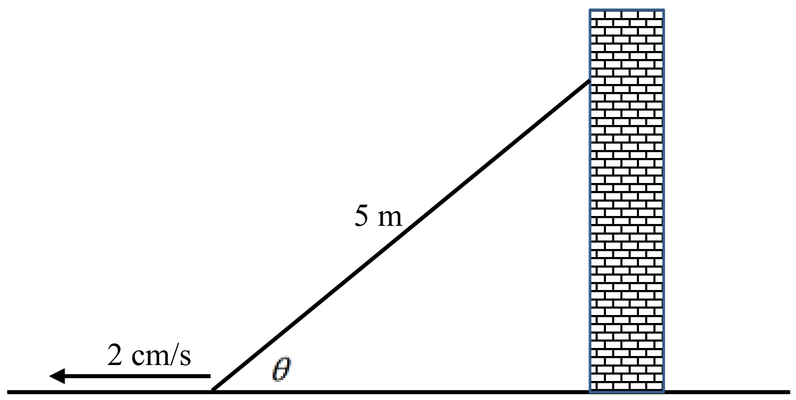
(a) Find the following integrals using the given substitution

(i) $\int x(1-x^2)^5 dx$ using the substitution $u = 1 - x^2$ *2*

(ii) $\int \frac{dx}{2x\ln x}$ using the substitution $u = \ln x$ *3*

(iii) $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}} dx$ using the substitution $u = 1 + \tan x$ *3*

(b) A 5 metre ladder is leaning against a wall on level ground. The base of the ladder is sliding away from the wall at a rate of 2 cm/s. *3*



Find the rate at which the angle of inclination the ladder makes with the ground, θ , is changing when the foot of the ladder is 3 metres from the base of the wall.

Question 7 continues on page 6

Marks

<u>Question 7</u> ***(continued)***

(c) (i) Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x + \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ *2*

(ii) Hence, or otherwise, solve; *3*

$$\cos x - \sqrt{3}\sin x = 1 \quad \text{for} \quad 0 \leq x \leq 2\pi$$

(d) (i) Show that $\frac{d}{dx}(\tan^3 x) = 3(\sec^4 x - \sec^2 x)$ *2*

(ii) Hence find $\int \sec^4 x \, dx$ *2*

End of Question 7

Marks

Question 8 ***(13 marks)*** Use a *separate* answer sheet

(a) Solve; *4*

$$\sin 3x = \sin 2x \cos x \quad \text{for} \quad 0 \leq x \leq \frac{\pi}{2}$$

(b) Find the volume of the solid generated when the curve $y = \cos x + \sec x$ is rotated about the x-axis between $x = 0$ and $x = \frac{\pi}{3}$ *4*

(c) (i) Show that; *2*

$$\frac{d}{dx}\left(\tan^{-1}\frac{1-x}{1+x}\right) = -\frac{1}{1+x^2}$$

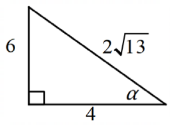
(ii) Prove that for $x > -1$, *2*

$$\tan^{-1}x + \tan^{-1}\frac{1-x}{1+x} = \frac{\pi}{4}$$

(iii) Hence calculate $\tan^{-1}(\sqrt{2} - 1)$ *1*

End of paper

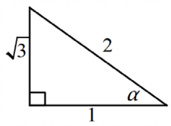
BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION HALF YEARLY 2020 SOLUTIONS

Solution	Marks	Comments
SECTION I		
1. B - $\displaystyle\int \frac{dx}{4+9x^2} = \frac{1}{3}\int \frac{3dx}{2^2+(3x)^2}$ $\displaystyle = \frac{1}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{3x}{2}\right)+c$ $\displaystyle = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)+c$	1	
2. D – $y = x^4 \Rightarrow x = y^{\frac{1}{4}}$ $\displaystyle V = \pi\int_0^{16}\left(y^{\frac{1}{4}}\right)^2 dy$ $\displaystyle = \pi\int_0^{16} y^{\frac{1}{2}}dy$ $\displaystyle = \pi\int_0^{16}\sqrt{y}\,dy$	1	
3. B – $6\cos\theta + 4\sin\theta$ $= 2\sqrt{13}\sin(\theta+\alpha)$ $\therefore$ maximum value is $2\sqrt{13}$  	1	
4. C – $f'(x) = \sin^2 x$ $\displaystyle = \frac{1}{2}(1-\cos 2x)$ $\displaystyle f(x) = \frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)+c$ $\displaystyle = \frac{x}{2}-\frac{1}{4}\sin 2x + c$	1	
5. B - $\displaystyle\int (2x-1)\sqrt{2x+1}du$ $\quad u = 2x+1$, $du = 2dx$ $\displaystyle = \frac{1}{2}\int (u-2)\sqrt{u}du$ $\displaystyle = \frac{1}{2}\int\left(u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right)du$	1	

SECTION II

Solution	Marks	Comments
QUESTION 6		
6(a) (i) $f(x) = [\sin^{-1}(3x)]^3$ $f'(x) = 3[\sin^{-1}(3x)]^2 \times \dfrac{3}{\sqrt{1-9x^2}}$ $= \dfrac{9[\sin^{-1}(3x)]^2}{\sqrt{1-9x^2}}$	**2**	**2 marks** • Correct solution **1 mark** • Correctly differentiates $\sin^{-1}(3x)$ • Uses the chain rule with a valid attempt to differentiate $\sin^{-1}(3x)$
6(a) (ii) $f(x) = \ln(\cos^{-1}x)$ $f'(x) = \dfrac{\left(-\dfrac{1}{\sqrt{1-x^2}}\right)}{\cos^{-1}x}$ $= -\dfrac{1}{\sqrt{1-x^2}\cos^{-1}x}$	**2**	**2 marks** • Correct solution **1 mark** • Correctly differentiates $\cos^{-1}x$ • Differentiates the log function with a valid attempt to differentiate $\cos^{-1}x$
6(a) (iii) $f(x) = (x^2+1)\tan^{-1}x$ $f'(x) = (x^2+1)\left(\dfrac{1}{1+x^2}\right) + (\tan^{-1}x)(2x)$ $= 1 + 2x\tan^{-1}x$	**2**	**2 marks** • Correct solution **1 mark** • Correctly differentiates $\tan^{-1}x$ • Uses the product rule with a valid attempt to differentiate $\tan^{-1}x$
6(b) (i) $\displaystyle\int_{-2}^{2} \frac{dx}{x^2+4} = 2 \times \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$ $= \tan^{-1}1 - \tan^{-1}0$ $= \dfrac{\pi}{4} - 0$ $= \dfrac{\pi}{4}$	**2**	**2 marks** • Correct solution **1 mark** • Finds the correct primitive
6(b) (ii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-3x^2}}dx = \frac{1}{\sqrt{3}}\int_0^1 \frac{\sqrt{3}\,dx}{\sqrt{4-3x^2}}$ $= \dfrac{1}{\sqrt{3}}\left[\sin^{-1}\dfrac{\sqrt{3}\,x}{2}\right]_0^1$ $= \dfrac{1}{\sqrt{3}}\left(\sin^{-1}\dfrac{\sqrt{3}}{2} - \sin^{-1}0\right)$ $= \dfrac{1}{\sqrt{3}}\left(\dfrac{\pi}{3} - 0\right)$ $= \dfrac{\pi}{3\sqrt{3}}$	**3**	**3 marks** • Correct solution **2 marks** • Finds the correct primitive • Correctly evaluates definite integral using a primitive in the form of $k\sin^{-1}\dfrac{\sqrt{3}x}{2}$ **1 mark** • Establishes the primitive as $k\sin^{-1}\dfrac{\sqrt{3}x}{2}$

Solution	Marks	Comments
6(c) $2\sin\theta - \cos\theta = 1$ $2\left(\frac{2t}{1+t^2}\right) - \left(\frac{1-t^2}{1+t^2}\right) = 1$ $4t - 1 + t^2 = 1 + t^2$ $4t - 2 = 0$ $t = \frac{1}{2}$ $\tan\frac{\theta}{2} = \frac{1}{2}$ $\frac{\theta}{2} = 26°34'$ $\theta = 53°8'$ test $\theta = 180°$ $2\sin 180° - \cos 180° = 2 - (-1)$ $= 1$ $\therefore\ \theta = 180°$ is a solution $\theta = 53°8'\ ,\ 180°$	4	**4 marks** • Correct solution **3 marks** • Correctly finds the solution $\theta = 53°8'$ • Transforms the trig equation into a polynomial equation and establishes $\theta = 180°$ **2 marks** • Transforms the trig equation into a polynomial equation **1 mark** • Substitutes the correct t-results for $\sin\theta$ and $\cos\theta$ • establishes $\theta = 180°$ as a solution
QUESTION 7		
7(a) (i) $\int x(1-x)^5 dx = -\frac{1}{2}\int u^5 du$ $= -\frac{1}{12}u^6 + c$ $= -\frac{1}{12}(1-x^2)^6 + c$ $u = 1 - x^2$ $du = -2x dx$	2	**2 marks** • Correct solution **1 mark** • Rewrites integrand in terms of u
7(a) (ii) $\int \frac{dx}{2x\ln x} = \frac{1}{2}\int\frac{du}{u}$ $= \frac{1}{2}\ln\lvert u\rvert + c$ $= \frac{1}{2}\ln\lvert\ln x\rvert + c$ $u = \ln x$ $du = \frac{dx}{x}$	3	**3 marks** • Correct solution **2 marks** • Finds the primitive in terms of u **1 mark** • Rewrites integrand in terms of u
7(a) (iii) $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}} dx = \int_1^2 \frac{du}{\sqrt{u}}$ $= 2\left[\sqrt{u}\right]_1^2$ $= 2(\sqrt{2} - 1)$ $u = 1 + \tan x$ $du = \sec^2 x\ dx$ when $x = 0$, $u = 1$ when $x = \frac{\pi}{4}$, $u = 2$	3	**3 marks** • Correct solution **2 marks** • Finds the primitive in terms of u • Changes both the integrand and the limits in terms of u **1 mark** • Rewrites integrand in terms of u • Expresses limits in terms of u

	Solution	Marks	Comments
7(b)	$\frac{dx}{dt} = \frac{1}{50}$ m/s $\cos\theta = \frac{x}{5}$ $\theta = \cos^{-1}\frac{x}{5}$ $\frac{d\theta}{dx} = \frac{-1}{\sqrt{25 - x^2}}$ $\frac{d\theta}{dt} = \frac{dx}{dt} \times \frac{d\theta}{dx}$ $= \frac{1}{50} \times \frac{-1}{\sqrt{25 - x^2}}$ $= \frac{-1}{50\sqrt{25 - x^2}}$ when $x = 3$; $\frac{d\theta}{dt} = \frac{-1}{50\sqrt{25 - 9}}$ $= -\frac{1}{200}$ rad/s ∴ the angle of inclination is decreasing at a rate of $\frac{1}{200}$ radians/second	**3**	**3 marks** • Correct solution, including correct units **2 marks** • Finds a correct expression for $\frac{d\theta}{dt}$ **1 mark** • Finds a correct expression for $\frac{d\theta}{dx}$ • Uses the chain rule to find an expression for $\frac{d\theta}{dt}$
7 (c) (i)	$\cos x - \sqrt{3}\sin x$ $= 2\cos\left(x + \frac{\pi}{3}\right)$  $\tan\alpha = \sqrt{3}$ $\alpha = \frac{\pi}{3}$	**2**	**2 marks** • Correct solution **1 mark** • Finds R or α
7(c) (ii)	$\cos x - \sqrt{3}\sin x = 1$ $2\cos\left(x + \frac{\pi}{3}\right) = 1$ $\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$ $x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}$ $x = 0, \frac{4\pi}{3}, 2\pi$	**3**	**3 marks** • Finds all three solutions **2 marks** • Finds two values of x **1 mark** • Finds two answers for $x + \frac{\pi}{3}$ • Finds one value of x
7(d) (i)	$\frac{d}{dx}(\tan^3 x) = 3(\tan^2 x)(\sec^2 x)$ $= 3(\sec^2 x - 1)(\sec^2 x)$ $= 3(\sec^4 x - \sec^2 x)$	**2**	**2 marks** • Correct solution **1 mark** • Uses chain rule
7(d) (ii)	$3\int (\sec^4 x - \sec^2 x)\, dx = \tan^3 x$ $3\int \sec^4 x\, dx = \tan^3 x + 3\int \sec^2 x\, dx$ $= \tan^3 x + 3\tan x + c$ $\int \sec^3 x\, dx = \frac{1}{3}\tan^3 x + \tan x + c$	**2**	**2 marks** • Correct solution **1 mark** • Uses part (i) to write an expression for $\int \sec^4 x\, dx$

Solution		Marks	Comments
QUESTION 8			
8(a)	$\sin 3x = \sin 2x\cos x$ $= \frac{1}{2}(\sin 3x + \sin x)$ $2\sin 3x = \sin 3x + \sin x$ $\sin 3x = \sin x$ **Q1 , Q2** $3x = x \;,\; \pi - x$ $2x = 0 \quad \text{or} \quad 4x = \pi$ $x = 0 \qquad x = \frac{\pi}{4}$	4	**4 marks** • Correct solution **3 marks** • Correctly finds one solution for x **2 marks** • establishes $\sin 3x = \sin x$ • states one correct solution for x **1 mark** • Uses "products to sum" formula to simplify $\sin 3x + \sin x$
8(b)	$V = \pi\int_0^{\frac{\pi}{3}} (\cos x + \sec x)^2 dx$ $= \pi\int_0^{\frac{\pi}{3}} (\cos^2 x + 2 + \sec^2 x)\, dx$ $= \pi\int_0^{\frac{\pi}{3}} \left\{\frac{1}{2}(1 + \cos 2x) + 2 + \sec^2 x\right\} dx$ $= \pi\int_0^{\frac{\pi}{3}} \left(\frac{5}{2} + \frac{1}{2}\cos 2x + \sec^2 x\right) dx$ $= \pi\left[\frac{5x}{2} + \frac{1}{4}\sin 2x + \tan x\right]_0^{\frac{\pi}{3}}$ $= \pi\left(\frac{5\pi}{6} + \frac{\sqrt{3}}{8} + \sqrt{3} - 0\right)$ $= \pi\left(\frac{5\pi}{6} + \frac{9\sqrt{3}}{8}\right)$ units^3	4	**4 marks** • Correct solution **3 marks** • Finds the primitive function • Successfully evaluates primitive that has been obtained by correct integration of a trig function found via a relevant process **2 marks** • Uses the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to rewrite integrand into an easily handled form • Finds a primitive that has been obtained by correct integration of a trig function found via a relevant process **1 mark** • finds the primitive of $\sec^2 x$ *Note: decimal approximation of 14.34624319 is acceptable*
8(c) (i)	$\frac{d}{dx}\left(\tan^{-1}\frac{1-x}{1+x}\right) = \dfrac{\frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}}{1 + \left(\frac{1-x}{1+x}\right)^2}$ $= \frac{-2}{(1+x)^2 + (1-x)^2}$ $= \frac{-2}{2 + 2x^2}$ $= -\frac{1}{1 + x^2}$	2	**2 marks** • Correct solution **1 mark** • Correctly uses the differentiation rule for $\tan^{-1} f(x)$

Solution	Marks	Comments
QUESTION 8		
8(c) (ii) $\frac{d}{dx}\left(\tan^{-1}x + \tan^{-1}\frac{1-x}{1+x}\right) = \frac{1}{1+x^2} - \frac{1}{1+x^2}$ $= 0$ $\therefore$ as both functions are continuous for $x > -1$, $\tan^{-1}x + \tan^{-1}\frac{1-x}{1+x} = c$ let $x = 0$; $\tan^{-1}x + \tan^{-1}\frac{1-x}{1+x} = \tan^{-1}0 + \tan^{-1}1$ $= \frac{\pi}{4}$ thus $\tan^{-1}x + \tan^{-1}\frac{1-x}{1+x} = \frac{\pi}{4}$, for $x > -1$ ***OR*** $\tan\left(\tan^{-1}x + \tan^{-1}\frac{1-x}{1+x}\right) = \frac{x + \frac{1-x}{1+x}}{1 - x\left(\frac{1-x}{1+x}\right)}$ $= \frac{x(1+x) + (1-x)}{(1+x) - x(1-x)}$ $= \frac{x^2+1}{1+x^2}$ $= 1$ $\therefore \tan^{-1}x + \tan^{-1}\frac{1-x}{1+x} = \tan^{-1}1$ $= \frac{\pi}{4}$	**2**	**2 marks** • Correct proves result **1 mark** • Explains why the function is a constant • Shows that $\tan\left(\tan^{-1}x + \tan^{-1}\frac{1-x}{1+x}\right) = 1$
8(c) (iii) let $x = \sqrt{2} - 1$; $\tan^{-1}(\sqrt{2}-1) + \tan^{-1}\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right) = \frac{\pi}{4}$ $\tan^{-1}(\sqrt{2}-1) + \tan^{-1}(\sqrt{2}-1) = \frac{\pi}{4}$ $2\tan^{-1}(\sqrt{2}-1) = \frac{\pi}{4}$ $\tan^{-1}(\sqrt{2}-1) = \frac{\pi}{8}$	**1**	**1 mark** • Correct exact value