HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 1

Year 12

Assessment 2 2021

STUDENT NUMBER: ____________________

General Instructions

- Reading Time – 5 minutes
- Working Time – 1.5 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided
- Show all necessary working
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination

Total marks – 54

Section I Pages 2 – 3

5 marks

Attempt Questions 1 – 5

Answer on the Objective Response Answer Sheet provided

Section II Pages 4 – 7

49 marks

Attempt Questions 6 – 8. Start each question on a new page. Write your student number on page

At the end of the assessment:

- Order your solutions, starting with Objective Response answer sheet, then Questions 6-8
- Place your question paper on top
- Staple through

Question	***1-5***	***6***	***7***	***8***	***Total***
Total	*/5*	*/16*	*/17*	*/16*	*/54*

This assessment task constitutes 25% of the Higher School Certificate Course School Assessment

Section I

5 marks
Attempt Questions 1 – 5
Allow about 10 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 5

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1 If $\underset{\sim}{u} = 3\underset{\sim}{i} - 10\underset{\sim}{j}$, $\underset{\sim}{v} = -6\underset{\sim}{i} + 8\underset{\sim}{j}$ and $\underset{\sim}{w} = 5\underset{\sim}{i} - \underset{\sim}{j}$, which of the following represents $\frac{1}{2}\underset{\sim}{v} + 3\underset{\sim}{u} - \underset{\sim}{w}$?

(A) $\underset{\sim}{i} - 25\underset{\sim}{j}$

(B) $3\underset{\sim}{i} - 29\underset{\sim}{j}$

(C) $-3\underset{\sim}{i} - 29\underset{\sim}{j}$

(D) $\underset{\sim}{i} + 30\underset{\sim}{j}$

2 In how many ways can all six dogs and eight cats gather around two non-identical circular bowls if two particular dogs cannot eat from the same bowl at the same time and there is always the same number of animals at each bowl?

(A) $14! \times 7! \times 2$

(B) ${}^{14}C_6 \times {}^{8}C_7 \times 2$

(C) ${}^{12}C_6 \times 6! \times 6! \times 2$

(D) ${}^{14}C_7 \times {}^{7}C_5$

3 The angle between the vectors $\underset{\sim}{u} = 5\underset{\sim}{i} + 6\underset{\sim}{j}$ and $\underset{\sim}{v} = -3\underset{\sim}{i} + \underset{\sim}{j}$ is closest to:

(A) $112°22'$

(B) $111°22'$

(C) $68°38'$

(D) $68°37'$

4 A bag contains 10 red marbles, 10 white marbles and 10 blue marbles. What is the minimum number of marbles that have to be chosen randomly from the bag to ensure that 4 marbles of the same colour are chosen?

(A) 4

(B) 10

(C) 11

(D) 5

5 How many solutions does the equation $(\cos^2 x - 1)(\cot^2 x - 1) = 0$ have in the domain $\left[-2\pi, 2\pi\right]$?

(A) 13

(B) 4

(C) 6

(D) 8

End of Section I

Section II

49 marks
Attempt Questions 6 – 8
Allow about 80 minutes for this section

Begin each question on a new page, indicating the question number.
Extra writing paper is available.

Question 6 (16 marks) **Start a new page.**

(a) Find $\lim_{x \to 0} \frac{\sin 4x}{\frac{x}{8}}$ **2**

(b) In how many ways can 10 distinct beads be arranged around a bangle? **1**

(c) If $t = \tan\frac{\theta}{2}$,

(i) Show that $3\sin\theta - 4\cos\theta - 4 = \frac{6t-8}{1+t^2}$. **1**

(ii) Hence, solve $3\sin\theta - 4\cos\theta = 4$ for $0° \le \theta \le 360°$, expressing the solutions correct to the nearest minute, where applicable. **3**

(d) Differentiate $y = \cos^4 5x$ **2**

(e) Peter left a base camp three days ago on a journey into a jungle. The three days of his journey can be described by displacement vectors $\underset{\sim}{\delta}_1$, $\underset{\sim}{\delta}_2$ and $\underset{\sim}{\delta}_3$, where $\underset{\sim}{\delta}_1 = (7,8)$, $\underset{\sim}{\delta}_2 = (6,2)$ and $\underset{\sim}{\delta}_3 = (2,9)$.
All distances are in kilometres.

(i) Show that Peter's displacement vector, $\underset{\sim}{\delta}_T$, from base camp is given by $\underset{\sim}{\delta}_T = (15,19)$. **1**

(ii) Calculate the magnitude and direction of the displacement vector from the base camp. Express your answer correct to 2 decimal places and correct to the nearest degree. **2**

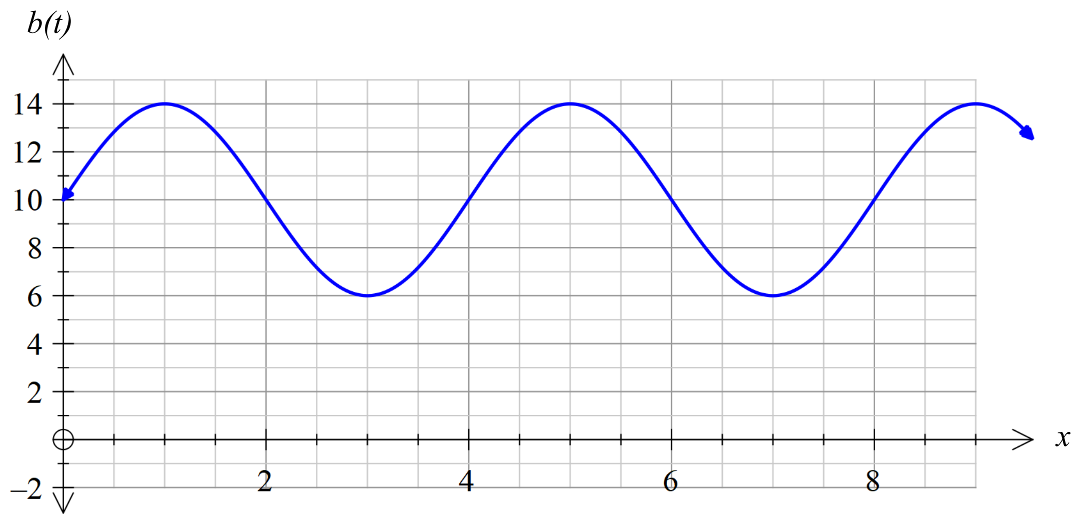
Question 6 continues on next page

Question 6 continued

(f) The population of bees on an island can be modelled by the function

$$b(t) = a\sin\left(\frac{\pi x}{2}\right) + 10$$

where x is the time in years and the population, $b(t)$, is measured in **thousands**. The graph of the bee population is shown below.



(i) What is the value of a? **1**

On the same island, an infestation of parasitic mites attacks the bee population. The population of these mites can be modelled by the function

$$m(t) = \cos\left(\frac{\pi(x-3)}{2}\right) + 2$$

where x is the time in years and the population, $m(t)$, is measured in **thousands**.

(ii) Neatly sketch *m(t)* in the domain [0,8], showing all important features. **2**

(iii) What is the population of the mites when the bee population reaches 14 000 for the **second** time**?** **1**

Question 7 (17 marks) **Start a new page.**

(a) A number plate is made up of two letters followed by two digits followed by two letters. How many arrangements are possible if:

(i) there are no restrictions. **1**

(ii) the letter **G** is not used. **1**

(iii) the letter **G** is used at least once. **1**

(b) Find the term independent of x in the expansion of $\left(x^2+\frac{1}{2x^2}\right)^{12}$. **3**

(c) (i) Express $6\sin 2\theta+8\cos 2\theta$ in the form $R\sin(2\theta+\beta)$, expressing β in radians correct to 2 decimal places and $R>0$. **3**

(ii) Hence, or otherwise, solve $6\sin 2\theta+8\cos 2\theta=5$ in the domain $\left[0,\pi\right]$. Express θ correct to two decimal places. **2**

(d) Consider the points $P(a,2a)$, $Q(-a,5a)$, $R(3a,4a)$ and $S(9a,12a)$, where a is a positive real number.

(i) Express $\overrightarrow{PQ}$ in component form. **1**

(ii) Show that $\overrightarrow{RS}=6a\underset{\sim}{i}+8a\underset{\sim}{j}$. **1**

(iii) Given that the scalar projection of $\overrightarrow{PQ}$ onto $\overrightarrow{RS}$ is 12, find the value of a. **2**

(e) In how many ways can the letters of the word **AMARAKS** be chosen three at a time? **2**

Question 8 (16 marks) **Start a new page.**

(a) (i) Using the expansion for $\cos(A+B)$, show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

(ii) Hence, or otherwise, solve the equation $\cos 3\theta - \cos\theta = 0$ for $\left[0, 2\pi\right]$. **3**

(b) A function $f(x)$ is given by $f(x) = x^2 \ln\dfrac{1}{x}$, where $x > 0$.

(i) Show that $f'(x) = -x\left(2\ln x + 1\right)$. **2**

(ii) Show that the function has a maximum turning point at $\left(\dfrac{1}{\sqrt{e}}, \dfrac{1}{2e}\right)$. **3**

(iii) Find the point(s) of inflexion, expressing your answer as decimals. **2**

(iv) Sketch the function showing all of the above characteristics, asymptotes and x-intercepts. **3**

(v) Show that $\ln x \geq \dfrac{-1}{2ex^2}$ for all $x > 0$. **1**

End of paper

HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 1
Year 12
Assessment 2 2021

OBJECTIVE RESPONSE ANSWER SHEET

Student Number:_________________________

Questions 1 – 5

Completely fill the response oval representing the most correct answer.

1.	A ⬭	B ⬭	C ⬭	D ⬭
2.	A ⬭	B ⬭	C ⬭	D ⬭
3.	A ⬭	B ⬭	C ⬭	D ⬭
4.	A ⬭	B ⬭	C ⬭	D ⬭
5.	A ⬭	B ⬭	C ⬭	D ⬭

Hornsby Girls High School
Year 12 Mathematics Extension 1 HSC Assessment2 2021
Solutions

Solutions	Marker's Comments
Question 1 **A** $\frac{1}{2}\left(-6\underset{\sim}{i}+8\underset{\sim}{j}\right)+3\left(3\underset{\sim}{i}-10\underset{\sim}{j}\right)-\left(5\underset{\sim}{i}-\underset{\sim}{j}\right)$ $=-3\underset{\sim}{i}+4\underset{\sim}{j}+9\underset{\sim}{i}-30\underset{\sim}{j}-5\underset{\sim}{i}+\underset{\sim}{j}$ $=\underset{\sim}{i}-25\underset{\sim}{j}$	
Question 2 **C** ${}^{12}C_6\times 6!\ \times {}^{6}C_6\times 6!\ \times 2$	
Question 3 **B** $\underset{\sim}{u}\cdot\underset{\sim}{v}=-15+6$ $=-9$ $\lvert\underset{\sim}{u}\rvert=\sqrt{5^2+6^2}$ $=\sqrt{61}$ $\lvert\underset{\sim}{v}\rvert=\sqrt{(-3)^2+1^2}$ $=\sqrt{10}$ $\cos\theta=\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert}$ $\cos\theta=\frac{-9}{\sqrt{61}\times\sqrt{10}}$ $\theta=\cos^{-1}\left(\frac{-9}{\sqrt{61}\times\sqrt{10}}\right)$ $=111°22$	
Question 4 **B** $3+3+3+1=10$	
Question 5 **D** $\cos^2 x=1$ $\cos x=\pm 1$ $x=-2\pi,-\pi,0,\pi,2\pi$ $\cot^2 x=1\quad(\sin x\neq 0,\cos x\neq 1)$ $\tan x=\pm 1$ $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4},\frac{-\pi}{4},\frac{-3\pi}{4},\frac{-5\pi}{4},\frac{-7\pi}{4}$ $x\neq -2\pi,-\pi,0,\pi,2\pi$ $\therefore$ 8 solutions	

Solutions	Marker's Comments
Question 6 (a) $$\lim_{x\to 0}\frac{\sin 4x}{\frac{x}{8}}=\lim_{x\to 0}\frac{8\sin 4x}{x}$$ $$=\lim_{x\to 0}\frac{4\times 8\sin 4x}{4x}$$ $$=32\lim_{x\to 0}\frac{\sin 4x}{4x}$$ $$=32$$	For those who new the fundamental limit and how to manipulate it, this was an easy question to do. $$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$ There were a lot of students who did this poorly.
(b) $$10\text{ beads}=\frac{9!}{2}$$	A lot of students did not divide by 2 as this was a bangle/bracelet which could be worn the other way around, therefore clockwise and anticlockwise of one sequence of beads are the same permutation as each other.
(c) i) $$LHS=3\left(\frac{2t}{1+t^2}\right)-4\left(\frac{1-t^2}{1+t^2}\right)-4$$ $$=\frac{6t-4(1-t^2)-4(1+t^2)}{1+t^2}$$ $$=\frac{6t-4+4t^2-4-4t^2}{1+t^2}$$ $$=\frac{6t-8}{1+t^2}$$	Generally, well done
(c) ii) $3\sin\theta-4\cos\theta=4$ $3\sin\theta-4\cos\theta-4=0$ $\frac{6t-8}{1+t^2}=0$ $6t-8=0$ $t=\frac{4}{3}$ $\tan\frac{\theta}{2}=\frac{4}{3},\quad 0\le\frac{\theta}{2}\le\pi$ $\frac{\theta}{2}=\tan^{-1}\left(\frac{4}{3}\right)$ $=53°7'48.37''$ $\theta=106°16'$ check $\theta=180°$ $\therefore\theta=106°16',\ 180°$	Most did this well, but there were a lot who messed up using (i) with this. Also a lot of students did not check 180° as a possible solution, which it is. Students also need to check the domain properly

<table>
<tr><th>Solutions</th><th>Marker’s Comments</th></tr>
<tr><td>

Question 6

(d)

$y=(\cos 5x)^4$

$y'=4(\cos 5x)^3\times -5\sin 5x$

$\quad =-20\sin 5x.\cos^3 5x$

</td><td>Mostly well done. Some students need to learn how to apply the chain rule correctly.</td></tr>
<tr><td>

(e) i)

$\text{Horizontal component}: 7+6+2=15$

$\text{Vertical component}: 8+2+9=19$

$\therefore \delta_T=(15,19)$

</td><td>Generally, well done.</td></tr>
<tr><td>

(e) ii)

$|\delta_T|=\sqrt{15^2+19^2}$

$\quad =24.21 \text{ km}$

$\tan\theta=\frac{19}{15}$

$\theta=51°42'35.41''$

$\therefore$ The maginitude is 24.21 km. The direction is N38°17E, or 038°17T.

</td><td>Distance was well done, direction was poorly done.
Direction needs to be more than an angle. It is a compass direction or true bearing. Without any source of reference for the angle, it is incorrect.</td></tr>
<tr><td>

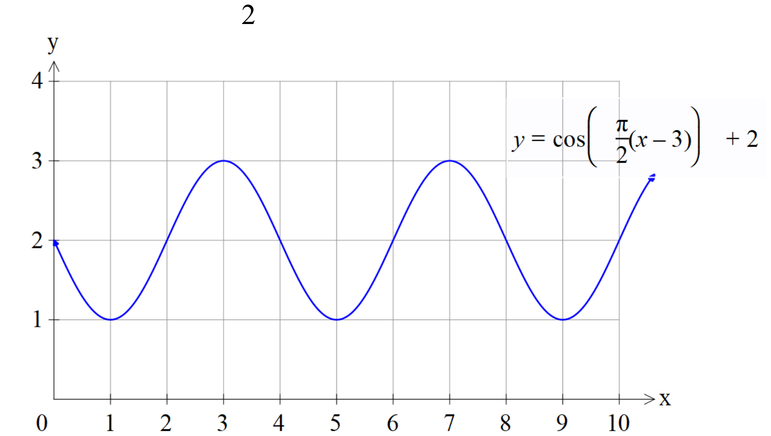
(f) i)

$a=4$

</td><td>$a=4000$ is incorrect. The function is in 1000’s so if you put 4000, it makes the functions values millions.</td></tr>
<tr><td>

(f) (ii)

$a=1,\ \text{period}=\frac{2\pi}{\frac{\pi}{2}}=4,\quad \text{average}=2$



</td><td>This was generally well done by most, however there were a number who need to work on their transformations.</td></tr>
<tr><td>

(f) (iii)

Substitude $x=5$ into $m(t)$,

$m(5)=\cos\left[\frac{\pi(5-3)}{2}\right]+2$

$\quad =\cos\pi+2$

$\quad =-1+2$

$\quad =1$

$\therefore$ The mites are 1000 when the bees reaches 14000.

</td><td>This was generally well done. Carried forward errors from graph were allowed. However, errors from incorrect substitution into formula or miscalculation were not.</td></tr>
</table>

<table>
<tr><th>Solutions</th><th>Marker's Comments</th></tr>
<tr><td>Question 7
(a) i)
$26\times26\times10\times10\times26\times26=26^4\times10^2$</td><td>Generally, well done.</td></tr>
<tr><td>(a) ii)
$25\times25\times10\times10\times25\times25=25^4\times10^2$</td><td>Generally, well done.</td></tr>
<tr><td>(a) iii)
$26^4\times10^2-25^4\times10^2=(26^4-25^4)\times10^2$</td><td>Generally, well done.</td></tr>
<tr><td>(b)
$T_{k+1}={}^{12}C_k\left(x^2\right)^{12-k}\left(\frac{1}{2x^2}\right)^k$
$={}^{12}C_k x^{24-2k}\left(\frac{1}{2}\right)^k x^{-2k}$
$={}^{12}C_k\left(\frac{1}{2}\right)^k x^{24-2k-2k}$
$={}^{12}C_k\left(\frac{1}{2}\right)^k x^{24-4k}$
For T_{k+1} to be constant, $24-4k=0$
$k=6$
$\therefore$ The term independent of x is: ${}^{12}C_6\left(\frac{1}{2}\right)^6$, which is $\frac{231}{16}$</td><td>Generally, well done. The general term should be used rather than finding individual constant terms.</td></tr>
<tr><td>(c) i)
$R\sin(2\theta+\beta)=R\sin2\theta\cos\beta+\cos2\theta\sin\beta$
$R=\sqrt{6^2+8^2}=10\quad(R>0)$
$R\cos\beta=6\qquad R\sin\beta=8$
$\cos\beta=\frac{6}{R}\qquad\sin\beta=\frac{8}{R}$
$\tan\beta=\frac{8}{6},\quad\beta\approx0.93$
$\therefore 6\sin2\theta+8\cos2\theta=10\sin(2\theta+0.93)$</td><td>Generally, well done. Sufficient workings need to be shown.

The answer should be expressed in radian as required by the question.</td></tr>
<tr><td>(c) ii)
$10\sin(2\theta+0.93)=5$
$\sin(2\theta+0.93)=\frac{1}{2}$
$2\theta+0.93=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6}$
$\theta\approx0.84,2.94\qquad(0\le\theta\le\pi)$
Or use general solution: $2\theta+0.93=2n\pi+(-1)^n\left(\frac{\pi}{6}\right)$
$\theta=n\pi+(-1)^n\left(\frac{\pi}{12}\right)-\frac{0.93}{2}$
$n=1,\quad\theta\approx0.84$
$n=2,\quad\theta\approx2.94$</td><td>Some students missed the second solution or failed to rule out the negative solution. Remember to check the domain.</td></tr>
</table>

Solutions	Marker's Comments
Question 7 (d) i) $\overrightarrow{OP}+\overrightarrow{PQ}=\overrightarrow{PQ}$ $\overrightarrow{PQ}=\begin{pmatrix}-a\\5a\end{pmatrix}-\begin{pmatrix}a\\2a\end{pmatrix}$ $=\begin{pmatrix}-2a\\3a\end{pmatrix}$	Generally, well done.
(d) ii) $\overrightarrow{RS}=\begin{pmatrix}9a\\12a\end{pmatrix}-\begin{pmatrix}3a\\4a\end{pmatrix}$ $=\begin{pmatrix}6a\\8a\end{pmatrix}$ $=6a\underset{\sim}{i}+8a\underset{\sim}{j}$	Generally, well done.
(d) iii) $\lvert \text{Proj}_{\overrightarrow{RS}}\overrightarrow{PQ}\rvert=12$ $\dfrac{\overrightarrow{PQ}\cdot\overrightarrow{RS}}{\lvert\overrightarrow{RS}\rvert}=12$ $\dfrac{\begin{pmatrix}-2a\\3a\end{pmatrix}\cdot\begin{pmatrix}6a\\8a\end{pmatrix}}{\sqrt{(6a)^2+(8a)^2}}=12$ $\dfrac{-12a^2+24a^2}{\sqrt{100a^2}}=12$ $\dfrac{12a^2}{10a}=12$ $12a^2-120a=0$ $12a(a-10)=0$ $a=10\ \ (a>0)$	Generally, well done.
(e) $3A_s\to 1$ way $2A_s\to {}^4C_1=4$ way $1A_s\to {}^4C_2=6$ way no $A_s\to {}^4C_3=4$ way $1+4+6+4=15$ $\therefore$ Total number of ways are 15.	Generally, well done.

Solutions	Marker's Comments
Question 8 (a) i) $\cos(A+B)=\cos(2\theta+\theta)$ $=\cos 2\theta\cos\theta+\sin 2\theta\sin\theta$ $=(2\cos^2\theta-1)\cos\theta-2\sin^2\theta\cos\theta$ $=2\cos^3\theta-\cos\theta-2(1-\cos^2\theta)\cos\theta$ $=2\cos^3\theta-\cos\theta-2\cos\theta+2\cos^3\theta$ $=4\cos^3\theta-3\cos\theta$	Generally done well
(a) ii) $\cos 3\theta-\cos\theta=0$ $4\cos^3\theta-3\cos\theta-\cos\theta=0$ $4\cos^3\theta-4\cos\theta=0$ $4\cos\theta(\cos^2\theta-1)=0$ $\cos\theta=0,\quad \cos^2\theta=1$ $\theta=\frac{\pi}{2},\frac{3\pi}{2},0,\pi,2\pi$	Generally done well
(b) i) $f(x)=x^2\ln\frac{1}{x}$ $=-x^2\ln x$ $f'(x)=-(2x\ln x+x^2\times\frac{1}{x})$ $=-x(2\ln x+1)$	Generally done well
(b) ii) $-x(2\ln x+1)=0$ for stationary point $\therefore x=0,\quad 2\ln x+1=0$ $\ln x=-\frac{1}{2}\quad (x>0)$ $x=e^{-\frac{1}{2}}$ $f\left(e^{-\frac{1}{2}}\right)=-\left(e^{-\frac{1}{2}}\right)^2\ln\left(e^{-\frac{1}{2}}\right)$ $=-e^{-1}(-\frac{1}{2})$ $=\frac{1}{2e}$ $\therefore$ The stationary point is $\left(\frac{1}{\sqrt{e}},\frac{1}{2e}\right)$. Test the nature: <table><tr><td>x</td><td>0.1</td><td>$\frac{1}{\sqrt{e}}$</td><td>1</td></tr><tr><td>*f'(x)*</td><td>0.361</td><td>0</td><td>-1</td></tr><tr><td>*slope*</td><td>/</td><td>—</td><td>\</td></tr></table> $\therefore$ The maximum point is $\left(\frac{1}{\sqrt{e}},\frac{1}{2e}\right)$.	Generally done well Many students didn't show the substitution for the *y* value Generally done well

(b) iii)

$$f''(x) = -(2\ln x + 1)(-1) - x \times \frac{2}{x}$$
$$= -2\ln x - 3$$
$$-2\ln x - 3 = 0$$
$$\ln x = -\frac{3}{2}$$
$$x = e^{-\frac{3}{2}}$$
$$\approx 0.22$$
$$f\left(e^{-\frac{3}{2}}\right) = -\left(e^{-\frac{3}{2}}\right)\ln e^{-\frac{3}{2}}$$
$$= \frac{3}{2e^{\frac{3}{2}}}$$
$$\approx 0.075$$

Test the nature:

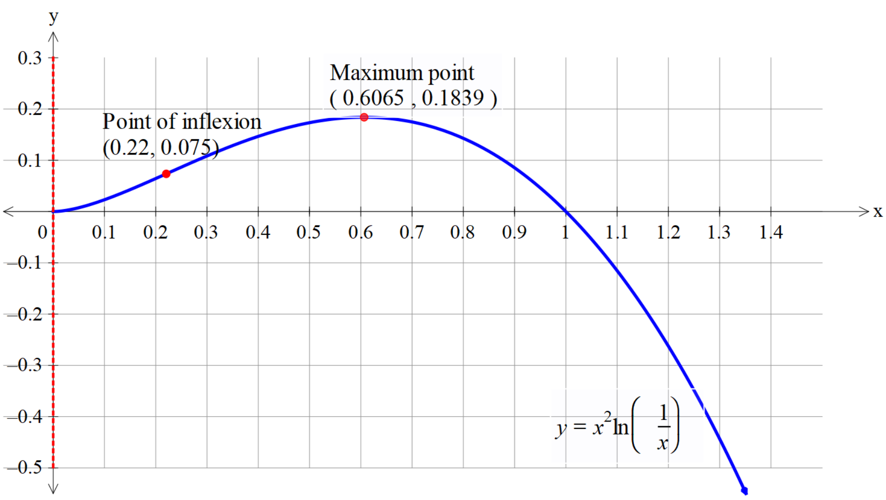
x	0.1	0.22	0.3
$f''(x)$	1.61	0	-0.59
concavity	$\cup$		$\cap$

$\therefore$ There is a point of inflexion at $\left(\frac{1}{e^{\frac{3}{2}}}, \frac{3}{2e^{\frac{3}{2}}}\right)$.

Generally done well

Many students didn't test for change in concavity

(b) iv)



Poorly done. Many curves were far too small to include all necessary information.

(b) v)

Since $y = \frac{1}{2e}$ is the maximum value,

$-x^2 \ln x \le \frac{1}{2e}$ as $y = -x^2 \ln x$

$\therefore \ln x \ge \frac{-1}{2ex^2}$

Poorly done. Very few students saw the connection with the maximum value.

End of Paper