BAULKHAM HILLS HIGH SCHOOL

YEAR 12 TASK 2 2023

Mathematics Extension 1

General Instructions
- Reading time – 10 minutes
- Working time – 1 hour
- Write using black pen
- Calculators approved by NESA may be used
- Show all necessary working in questions 6-8
- Marks may be deducted for careless or badly arranged work
- Reference sheet is provided

Total marks – 46
Exam consists of 6 pages.

This paper consists of TWO sections.

Section I – (5 marks) Pages (2-3)
Questions 1-5
- Attempt Questions 1-5
- Answer on answer sheet provided
- Allow about 8 minutes for this section

Section II – (41 marks) Pages (4-6)
- Attempt Questions 6-8
- Allow about 52 minutes
- Answer each question on the appropriate page of your answer booklet

<u>Section I</u> - 5 marks

Use the multiple-choice answer sheet for questions 1-5.

1. Which of the following is the derivative of $\log_e(\sin^{-1} x)$?

   A) $\dfrac{1}{\sqrt{1-x^2}}$

   B) $\dfrac{1}{\sqrt{1-x^2}\sin^{-1} x}$

   C) $\ln\left(\sqrt{1-x^2}\right)$

   D) $\dfrac{1}{\sqrt{1-x^2}\cos^{-1} x}$

2. Which of the following is an expression for $\int \cos^2 2x \, dx$?

   A) $x - \dfrac{1}{4}\sin 4x + c$

   B) $x + \dfrac{1}{4}\sin 4x + c$

   C) $\dfrac{x}{2} - \dfrac{1}{8}\sin 4x + c$

   D) $\dfrac{x}{2} + \dfrac{1}{8}\sin 4x + c$

3. Which of the following is an expression for $\int \dfrac{e^x}{1+e^{2x}} dx$?

   Use the substitution $u = e^x$.

   A) $e^x \tan^{-1} e^x + c$

   B) $e^x \tan^{-1} e^{2x} + c$

   C) $\tan^{-1} e^x + c$

   D) $\tan^{-1} e^{2x} + c$

4. Which of the following is the exact volume of the solid formed by rotating the curve $y = \frac{4}{x-2}$ about the $x-$axis, between $x = 3$ and $x = 6$?

A) $4\pi \ln 4$

B) $16\pi \ln 4$

C) 6π

D) 12π

5. What is the exact value of $\int_{\frac{\pi}{2}}^{\pi} \left(\sin^2 x + x\right) dx$?

A) $\frac{3\pi^2 + \pi + 2}{8}$

B) $\frac{3\pi^2 + \pi}{8}$

C) $\frac{3\pi^2 + 2\pi + 2}{8}$

D) $\frac{3\pi^2 + 2\pi}{8}$

End of Section I

Section II

41 marks
Attempt Questions 6-8
Allow about 52 minutes for this section

Answer each question on the appropriate page of your answer booklet. Extra paper is available.
Your responses should include relevant calculations and/or mathematical reasoning.

Question 6 (14 marks) **Marks**

a) i. Show that $\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)=\sqrt{2}\sin x$. **2**

ii. Hence, or otherwise, solve $\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)=1$ for $0\le x\le\pi$. **1**

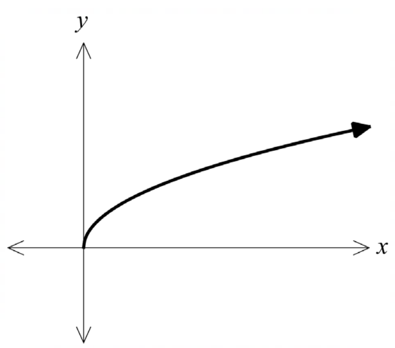
b) Use the substitution $u=\tan x$ to find $\int_0^{\frac{\pi}{4}}\tan^2 x\sec^2 x\,dx$. **3**

c) Find $\int_0^1 \frac{dx}{\sqrt{9-4x^2}}$, leaving your answer in exact form. **2**

d) Differentiate $e^{3x}\tan^{-1}\left(\frac{x}{2}\right)$. **2**

e) Evaluate $\int_0^{\frac{\pi}{8}}\sin^2 3x\ dx$. **3**

f) Sketch the solid formed when the region between the curve $y=\sqrt{x}$ and the $y-$axis from $y=0$ to $y=4$ is rotated about the $y-$axis. **1**



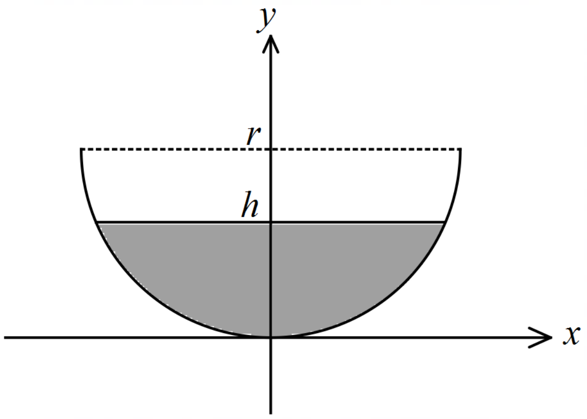
Question 7 (13 marks)

a) i. Express $7\cos\theta - \sin\theta$ in the form $R\cos(\theta+\alpha)$, where $R>0$ and $0° \le \alpha \le 90°$. **2**

ii. Hence solve $7\cos\theta - \sin\theta = 5$ for $0° \le \theta \le 360°$, giving answers to the nearest degree. **2**

b) If $\displaystyle\int_{-2}^{2\sqrt{3}} \frac{dx}{\sqrt{16-x^2}} = k\pi$, find the value of k. **3**

c) A hemi-spherical bowl of radius r contains water up to the height h, where $0<h<r$. By considering the hemi-spherical bowl as a solid generated when the semi-circle $y = r - \sqrt{r^2 - x^2}$ is rotated about the y-axis, prove that the volume of the water in the bowl is $\dfrac{\pi h^2(3r-h)}{3}$. **3**



d) Use the substitution $t = \tan x$ to solve the equation $\sin 2x - 3\cos 2x = 3$ for $0 \le x \le 2\pi$. Give your answer correct to two decimal places. **3**

Question 8 (14 marks) **Marks**

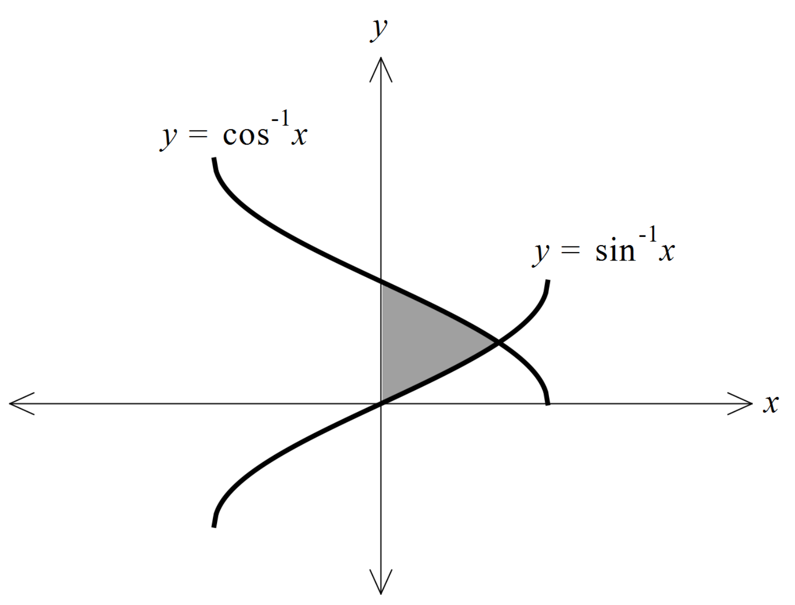
a) i. Prove that $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$. **3**

ii. Hence, or otherwise, find $\displaystyle\int \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\,dx$. **2**

b) By using $u = x - 1$, find $\displaystyle\int \frac{x+1}{\sqrt[3]{x-1}}\,dx$. **3**

c) Use polynomial division, or otherwise, to find the exact value of $\displaystyle\int_0^2 \frac{x^4 + x^2 - 4}{x^2 + 4}\,dx$. **3**

d) Find the exact area bounded by $y = \sin^{-1} x$, $y = \cos^{-1} x$ and the $y-$axis. **3**



End of Exam

Baulkham Hills High School
Task 2 Examination 2023

Marking Guideline- Yr 12 Mathematics Extension 1

Section I (5 marks)

Award 1 marks to each correct answer.

Answers: 1. B 2. D 3. C 4. D 5. D

Question	Suggested solutions	Answer
1	$\frac{d}{dx}\left(\log_e\left(\sin^{-1}x\right)\right)=\frac{1}{\sqrt{1-x^2}\times\sin^{-1}x}$	B
2	$\int\cos^2 2x\ dx$ $=\int\frac{1}{2}(1+\cos 4x)\ dx$ $=\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)+c$ $=\frac{x}{2}+\frac{1}{8}\sin 4x+c$	D
3	$\int\frac{e^x}{1+e^{2x}}dx$ $=\int\frac{du}{1+u^2}$ $\quad u=e^x$, $dx=\frac{du}{e^x}$ $=\tan^{-1}u+c$ $=\tan^{-1}e^x+c$	C
4	$V=\pi\int_3^6\left(\frac{4}{x-2}\right)^2 dx$ $=16\pi\int_3^6(x-2)^{-2}dx$ $=-16\pi\left[(x-2)^{-1}\right]_3^6$ $=-16\pi\left[(6-2)^{-1}-(3-2)^{-1}\right]$ $=12\pi$	D

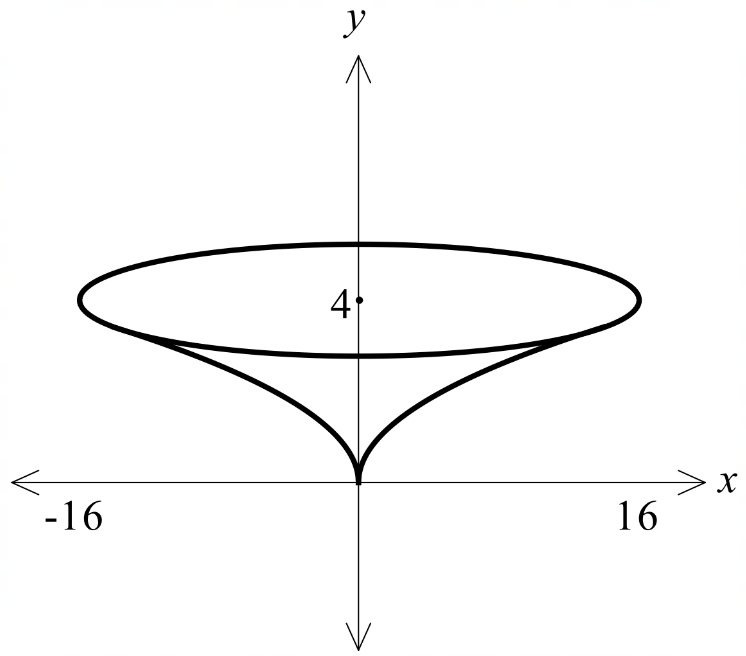
5	$\int_{\frac{\pi}{2}}^{\pi} (\sin^2 x + x^2)\ dx$ $= \int_{\frac{\pi}{2}}^{\pi} \left[\frac{1}{2}(1-\cos 2x) + x\right] dx$ $= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{x^2}{2}\right]_{\frac{\pi}{2}}^{\pi}$ $= \left(\frac{1}{2}\pi - \frac{1}{4}\sin 2x + \frac{\pi^2}{2}\right) - \left(\frac{1}{2}\times\frac{\pi}{2} - \frac{1}{4}\sin\pi + \frac{\pi^2}{8}\right)$ $= \frac{3\pi^2 + 2\pi}{8}$	D

Section II (40 marks)

In all questions, award full marks for correct answers with necessary working.

Use the suggested solutions in conjunction with the marking criteria.

Q	Suggested Solutions	Marking criteria
6ai	$\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)=\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}+\sin x\cos\frac{\pi}{4}-\cos x\sin\frac{\pi}{4}$ $=\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\cos x$ $=\frac{2}{\sqrt{2}}\sin x$ $=\sqrt{2}\sin x$	2- correct solution 1- expands using the compound angle formula or equivalent
6aii	$\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)=1$ $\sqrt{2}\sin x=1$ $\sin x=\frac{1}{\sqrt{2}}$ $x=\frac{\pi}{4},\frac{3\pi}{4}$	1- correct solution
6b	$\int_0^{\frac{\pi}{4}}\tan^2 x\sec^2 x\ dx$ $u=\tan x$ $\frac{du}{dx}=\sec^2 x$ $=\int_0^1 u^2\,du$ $du=\sec^2 x\,dx$ $=\left[\frac{u^3}{3}\right]_0^1$ $u=\tan\frac{\pi}{4}$, $u=1$ $u=\tan 0$, $u=0$ $=\frac{1}{3}$	3- correct solution 2- uses the substitution method correctly and correctly identifying the limits 1- correctly identifies the new limits or uses the substitution method correctly
6c	$\int_0^1\frac{dx}{\sqrt{9-4x^2}}$ $=\int_0^1\frac{dx}{\sqrt{3^2-2^2x^2}}$ $=\frac{1}{2}\left[s\text{in}^{-1}\left(\frac{2x}{3}\right)\right]_0^1$ $=\frac{1}{2}\sin^{-1}\frac{2}{3}$	2- correct solution 1- obtains the inverse $R\sin^{-1}\left(\frac{2x}{3}\right)$

6d	$\frac{d}{dx}\left(e^{3x}\tan^{-1}\left(\frac{x}{2}\right)\right)$ $v=\tan^{-1}\left(\frac{x}{2}\right)$ $u=e^{3x}$ $\quad v'=\frac{1}{2}\times\frac{1}{1+\left(\frac{x}{2}\right)^2}$ $u'=3e^{3x}$ $v'=\frac{2}{x^2+4}$ $u'v+v'u=3e^{3x}\tan^{-1}\left(\frac{x}{2}\right)+\frac{2e^{3x}}{x^2+4}$	2- correct solution 1- correct differentiates $\tan^{-1}\frac{x}{2}$
6e	$\int_0^{\frac{\pi}{8}}\sin^2 3x\,dx$ $=\int_0^{\frac{\pi}{8}}\left(\frac{1}{2}-\frac{1}{2}\cos(2\times 3x)\right)dx$ $=\int_0^{\frac{\pi}{8}}\left(\frac{1}{2}-\frac{1}{2}\cos 6x\right)dx$ $=\left[\frac{x}{2}-\frac{1}{12}\sin 6x\right]_0^{\frac{\pi}{8}}$ $=\left(\frac{\pi}{16}-\frac{1}{12}\sin\left(\frac{\pi}{8}\times 6\right)\right)-\left(0-\frac{1}{12}\sin 0\right)$ $=\frac{\pi}{16}-\frac{1}{12\sqrt{2}}$	3- correct solution 2- obtains the correct primitive 1- transforms the integrand
6f		1- correct graph

7ai	$R\cos(\theta+\alpha)=R\cos\theta\cos\alpha-R\sin\theta\sin\alpha$ $7\cos\theta-\sin\theta\equiv R\cos\theta\cos\alpha-R\sin\theta\sin\alpha$ $R\cos\alpha=7 \quad R\sin\alpha=1$ $\cos\alpha=\frac{7}{R} \quad \sin\alpha=\frac{1}{R}$ $R=\sqrt{1^2+7^2}$ $R=\sqrt{50}$ $\tan\alpha=\frac{1}{7}$ $\alpha=8°7'48.37''$ $\therefore 7\cos\theta-\sin\theta=5\sqrt{2}\cos(\theta+8°)$	2- correct solution 1- correctly finds R or α
7aii	$5\sqrt{2}\cos(\theta+8°)=5$ $\cos(\theta+8°)=\frac{1}{\sqrt{2}}$ $\theta+8°=45°,315° \quad 0°\le\theta\le360°$ $\therefore \theta=37°,307° \quad 8°\le\theta+8°\le368°$	2- correct solution 1- correctly finds 1 angle/solution
7b	$\int_{-2}^{2\sqrt{3}}\frac{dx}{\sqrt{16-x^2}}=k\pi$ $\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_{-2}^{2\sqrt{3}}$ $=\sin^{-1}\left(\frac{2\sqrt{3}}{4}\right)-\sin^{-1}\left(\frac{-2}{4}\right)$ $=\frac{\pi}{2}$ $\therefore k=\frac{1}{2}$	3- correct solution 2- correctly integrates for sin inverse and substitutes the values in 1- correctly integrates for sin inverse
7c	$y=r-\sqrt{r^2-x^2}$ $(y-r)^2=r^2-x^2$ $\therefore x^2=r^2-(y-r)^2$ $x^2=2ry-y^2$ $V=\pi\int_0^h(2ry-y^2)dy$ $=\pi\left[ry^2-\frac{y^3}{3}\right]_0^h$ $=\pi\left(rh^2-\frac{h^3}{3}\right)$ $=\pi h^2\left(r-\frac{h^3}{3}\right)$ $=\frac{\pi h^2(3r-h)}{3}\text{units}^3$	3- correctly obtains the answer 2- obtains the correct integrand to find the volume 1- finds the correct equation for x^2

7d	let $t=\tan x$ $\left(\dfrac{2t}{1+t^2}\right)-3\left(\dfrac{1-t^2}{1+t^2}\right)=3$ $2t-3+3t^2=3+3t^2$ $t=3$ $\tan x=3$ $x=1.25,4.39$ test for $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$ $\therefore x=1.25,4.39,\dfrac{\pi}{2},\dfrac{3\pi}{2}$	3- correct solution 2- finds 2 out of the 4 solutions 1- solves for t
8ai	$\tan 3x=\tan(2x+x)$ $=\dfrac{\tan 2x+\tan x}{1-\tan 2x\tan x}$ $=\left(\dfrac{2\tan x}{1-\tan^2 x}+\tan x\right)\div\left(1-\dfrac{2\tan^2 x}{1-\tan^2 x}\right)$ $=\dfrac{2\tan x+\tan x-\tan^3 x}{1-\tan^2 x-2\tan^2 x}$ $=\dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$	3- uses correct methods 2- uses compound angle formula and double angle formula correctly 1- correctly uses the compound angle formula or equivalent
8aii	$\displaystyle\int\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}dx=\int\tan 3x\,dx$ $\displaystyle\int\tan 3x\ dx=\int\frac{\sin 3x}{\cos 3x}$ $=-\dfrac{1}{3}\ln\lvert\cos 3x\rvert+c$	2- correct solution 1- correctly transforms the integrand to $\dfrac{\sin 3x}{\cos 3x}$
8b	$\displaystyle\int\frac{x+1}{\sqrt[3]{x-1}}\ dx$ $\displaystyle=\int\frac{2+u}{\sqrt[3]{u}}\ du$ $\displaystyle=\int\left(2u^{-\frac{1}{3}}+u^{\frac{2}{3}}\right)du$ $=3u^{\frac{2}{3}}+\dfrac{3u^{\frac{5}{3}}}{5}+c$ $=3(x-1)^{\frac{2}{3}}+\dfrac{3(x-1)^{\frac{5}{3}}}{5}+c$ $u=x-1\quad x=u+1$ $\dfrac{du}{dx}=1$ $du=dx$	3- correct solution 2- correct primitive 1- obtains the correct substitution for u and du

8c	Using polynomial division: $\frac{x^4+x^2-4}{x^2+4}=x^2-3+\frac{8}{x^2+4}$ $\int_0^2 x^2-3+\frac{8}{x^2+4}dx$ $=\left[\frac{x^3}{3}-3x+\frac{8}{2}\tan^{-1}\frac{x}{2}\right]_0^2$ $=\frac{2^3}{3}-3(2)+4\tan^{-1}\left(\frac{2}{2}\right)$ $=\pi-\frac{10}{3}$	3- correct solution 2- correct primitive 1- transforms integrand into partial fraction
8d	$y=\cos^{-1}x \quad y=\sin^{-1}x$ $x=\cos y \quad x=\sin y$ $\cos y=\sin y$ $\tan y=1$ $y=\frac{\pi}{4}$ $\int_0^{\frac{\pi}{4}}\sin y\ dy+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos y\ dy$ $=[-\cos y]_0^{\frac{\pi}{4}}+[\sin y]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $=-\cos\frac{\pi}{4}+\cos+\sin\frac{\pi}{2}-\sin\frac{\pi}{4}$ $=-\frac{\sqrt{2}}{2}+1+1-\frac{\sqrt{2}}{2}$ $=2-\sqrt{2}$	3- correct solution 2-forms the correct integrand 1-finds $y=\frac{\pi}{4}$ or $x=\frac{1}{\sqrt{2}}$