Mathematics Extension 1

HSC Marking Feedback 2020

Question 11

Part (a) (i)

Students should:

- apply their knowledge of function notation to substitute and simplify the expression, noting the value of the question to realise only one step was required.

In better responses, students were able to:

- obtain the result efficiently.

Areas for students to improve include:

- substituting into algebraic expressions
- demonstrating their knowledge or understanding of function notation.

Part (a) (ii)

Students should:

- use the result given in part (i) and their knowledge of the factor theorem to find the quadratic factor. This could be achieved by either long division, inspection or applying knowledge of the relationships between coefficients and roots of an equation.

In better responses, students were able to:

- recognise the linear factor and perform a long division to obtain the quadratic factor, or
- factorise the expression by inspection by using the given statement in part (i).

Areas for students to improve include:

- displaying their understanding of the factor theorem
- using their skills with long division of polynomials.

Part (b)

Students should:

- define, calculate and use the dot product of two vectors to examine the properties of perpendicular vectors. Alternatively, use the gradients of position vectors to determine properties of the two vectors.

In better responses, students were able to:

- state the property of perpendicular vectors
- apply the dot product formula from the Reference Sheet and solve the resultant quadratic equation
- state the relationship between the gradients of perpendicular position vectors and solve the resultant equation.

Areas for students to improve include:

- using their knowledge and understanding of vectors
- demonstrating the use and meaning of vector notation
- knowing how to apply the dot product or gradient formula to determine if vectors are perpendicular
- understanding methods of solving quadratic equations.

Part (c)

Students should:

- examine the relationship between $y = f(x)$ and $y = \frac{1}{f(x)}$, and recognise features to assist with graphing the functions.

In better responses, students were able to:

- initially recognise asymptotes and the y-intercept of $y = \frac{1}{f(x)}$
- use these features to graph the function $y = \frac{1}{f(x)}$.

Areas for students to improve include:

- understanding the relationship between functions and their reciprocal function
- avoiding the attempt to find the equation of the function and its reciprocal
- stating the important features of the function and hence its reciprocal
- knowing the difference between reciprocal and inverse functions.

Part (d)

Students should:

- convert an equation of the form $a\cos x + b\sin x$ to $A\sin(x+\alpha)$
- apply this to solving an equation of the form $a\cos x + b\sin x = c$.

In better responses, students were able to:

- efficiently find values for A and α in exact form
- obtain the solutions to the equation within the given domain.

Areas for students to improve include:

- using the Reference Sheet to expand $A\sin(x+\alpha)$ and finding exact values
- solving simultaneous equations

- developing their skills in solving trigonometric equations in a given domain.

Part (e)

Students should:

- solve a differential equation of the form $\frac{dy}{dx} = f(x)$.

In better responses, students were able to:

- manipulate the differential equation into a form that can be integrated
- integrate an exponential function
- realise there was no need to rewrite with y as the subject.

Areas for students to improve include:

- considering the mark value of the question and using it as a guide to the amount of working required
- rewriting a differential equation by separating the variables
- improving their knowledge of the difference between integration and differentiation of exponential functions.

Question 12

Part (a)

Students should:

- show their substitution when verifying the base case
- understand that only one term of the series is needed for $n = 1$
- take care when working with brackets
- show all steps in the proof.

In better responses, students were able to:

- deal with the LHS and RHS separately for base case $n = 1$ and for $n = k + 1$
- clearly articulate the inductive hypothesis for $n = k$
- state what they were required to prove when $n = k + 1$
- use factorisation rather than expansion when proving true for $n = k + 1$.

Areas for students to improve include:

- factorising algebraic expressions
- exercising care when working with brackets
- ensuring that each step of the proof is algebraically correct
- correctly use sigma notation, should they choose to use it
- setting out proofs in a logical sequence
- working from one side to the other when attempting proofs, and not manipulate both sides simultaneously.

Part (b)(i)

Students should:

- understand the parameters associated with binomial distributions, ie $X \sim \mathrm{Bin}(100, 0.6)$
- identify correct formula from Reference Sheet.

In better responses, students were able to:

- use the correct formula to arrive at the answer.

Areas for students to improve include:

- familiarising themselves with the Reference Sheet.

Part (b)(ii)

Students should:

- understand the relationship between variance and standard deviation
- calculate the variance and hence, the standard deviation.

In better responses, students were able to:

- substitute correctly in the formula for variance
- show that the standard deviation is $\sqrt{24}$.

Areas for students to improve include:

- focusing on using the Reference Sheet correctly.

Part (b)(iii)

Students should:

- link the probability to the standardised z-scores in normal distributions
- read-off the probability from the Reference Sheet.

In better responses, students were able to:

- use the empirical rule found in the Reference Sheet rather than attempting to calculate probabilities from z-scores.

Areas for students to improve include:

- reading the question carefully to ascertain when and how normal distribution approximation should be used.

Part (c)

Students should:

- evaluate the correct expression for ${}^nC_r$
- identify and quantify the pigeons and pigeonholes

- be prepared to answer questions on the Year 11 content in the HSC.

In better responses, students were able to:

- work out that a minimum of 393 students are needed to ensure that at least eight students pass exactly the same three topics
- establish that 392 pigeons are needed to equally fill each pigeonhole with 7 pigeons and the 8 remaining pigeons could be added to any one or more of the pigeonholes.

Areas for students to improve include:

- clearly explaining the pigeonhole principle
- understanding the ceiling function (or why we round up) when using the pigeonhole principle
- solving the problem using logic and sound reasoning, instead of simply using a formula with the hope of getting the correct answer
- practicing interpreting worded problems
- thoroughly revising Year 11 content.

Part (d)

Students should:

- be familiar with trigonometric identities on the Reference Sheet
- know the rules of integration: the integral of products is not always equal to the product of the integrals
- check results when integrating trigonometric functions, paying particular attention to the signs and fractional coefficients involved.

In better responses, students were able to:

- use the appropriate transformation and provide the correct solution in concise manner.

Areas for students to improve include:

- recognising when to use trigonometric identities
- ensuring accuracy when working with negatives and fractions.

Part (e)

Students should:

- understand the need and process of separation of variables when dealing with differential equations
- not assume that the subject of the equation must be y, or that the function must be linear.

In better responses, students were able to:

- use definite integrals to arrive at the solution, thereby obviating the need to work with constants of integration
- use only one constant of integration
- realise that the solution does not have to be a function.

Areas for students to improve include:

- understanding that differential equations should not be integrated where there are multiple variables on one side
- avoiding placing constants of integration on both sides as this can cause confusion when evaluating the constant of the primitive
- focusing on algebraic skills when dealing with negative numbers, particularly when substituting and solving equations.

Question 13

Part (a) (i)

Students should:

- be able to differentiate $3\sin^3\theta$, using the chain rule.

In better responses, students were able to:

- correctly use the chain rule to differentiate and arrive at the correct answer.

Areas for students to improve include:

- enhancing their knowledge and application of the chain rule.

Part (a) (ii)

Students should:

- find the limits of integration in terms of θ
- substitute the trigonometric ratios correctly, obtaining a simplified integrand
- integrate the trigonometric expression, and substitute the limits of integration to obtain a simplified numerical solution
- follow the instruction in the question to substitute
- recognise the connection to part (i) of the question.

In better responses, students were able to:

- find the new limits of integration and substitute correctly
- successfully integrate and evaluate the limits of integration to arrive at the correct numerical solution
- simplify the resulting equation into a form which allowed the substitution of the result from part (i).

Areas for students to improve include:

- realising the need for finding the derivative of $\tan\theta$ to replace dx
- finding limits in terms of θ integrating trigonometric ratios correctly
- relating part (ii) to part (i) to obtain a simplified integrand
- substituting the limits of integration into the required trigonometric ratios to obtain a simplified numerical expression, ie correct algebraic manipulation.

Part (b)

Students should:

- find the point of intersection of the two functions
- write the required volume as a difference of the volumes of the two functions
- use the double angle formula of $\cos 2\theta$ correctly
- substitute the limits of integration into the required trigonometric ratios to obtain a simplified numerical expression.

In better responses, students were able to:

- find and chose the correct the point of intersection
- correctly use the double angle formula
- correctly integrate the functions, and correctly manipulate the limits of integration to obtain the correct numerical solution.

Areas for students to improve include:

- realising the regions of the functions and hence writing the correct order of the difference of the volumes
- using the Reference Sheet to determine the correct forms to use for both $sin^2(nx)$ and $cos^2(nx)$
- knowing that the volume of the solid of revolution is the difference of the squares of each function, rather than the difference of the functions all squared.

Part (c) (i)

Students should:

- differentiate $\tan(\cos^{-1}(x))$ using the chain rule, and $\frac{\sqrt{1-x^2}}{x}$ using the quotient rule
- either use a drawing of a right-angled triangle, or simplify $\sec^2(\cos^{-1} x)$, to arrive at $\frac{1}{x^2}$
- manipulate the quotient rule to create a simplified fraction involving a square root expression in the denominator.

In better responses, students were able to:

- find both derivatives correctly
- correctly manipulate the fraction involving surds
- find the derivative of $\tan(\cos^{-1}(x))$
- arrive at the correct derivatives and equate them.

Areas for students to improve include:

- reading the question carefully to understand that they had to equate the derivatives of both functions by manipulating and further simplifying the derived functions
- taking time to build the answer step by step
- manipulating surds as fractions
- taking care with algebraic processes to ensure that the expressions from one line to the next remain equivalent.

Part (c) (ii)

Students should:

- recognise the need to use part (i) to find the functions of each derivative
- observe that the difference of both functions will be a constant, through the process of integration
- recognise that the difference of the derivatives should be 0
- recognise there is a restricted domain
- test a point on each side of the domain of both functions to arrive at the same answer.

In better responses, students were able to:

- recognise the integrals of the two functions are equal to a constant
- equate the derivative of both functions
- acknowledge there is a domain restriction to both functions
- test on both sides of the domain to prove the functions are equal.

Areas for students to improve include:

- focusing on the setting out of formal proofs
- recognising and considering the natural domain restrictions of trigonometric as well as rational functions
- testing both sides of the domain for each function to enable them to arrive at a correct conclusion
- recognising the constant of integration.

Question 14

Part (a) (i)

Students should:

- recognise that the coefficients of terms in each of these expansions will be equal
- note that the term on the LHS of the required result is the coefficient of x^n in the expansion of $(1+x)^{2n}$.

In better responses, students were able to:

- clearly state that the coefficient of x^n in expansion of $(1+x)^n(1+x)^n$ is
$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{k}\binom{n}{n-k}+\cdots+\binom{n}{n}\binom{n}{0}$$
- explain why $\binom{n}{n}=\binom{n}{0}$ etc, using $\binom{n}{k}=\binom{n}{n-k}$.

Areas for students to improve include:

- demonstrating knowledge of the binomial expansions of these expressions.
- using the Pascal triangle relationship represented by $\binom{n}{k}=\binom{n}{n-k}$.

Part (a) (ii)

Students should:

- ensure that they fully justify how they are deriving the (supplied) answer.

In better responses, students were able to:

- explain, using a pattern, how each case could be generated
- add these cases together to derive the required result.

Areas for students to improve include:

- reading the question carefully to gain a full understanding of the situation
- taking time to build the answer step by step
- knowing when cases should be multiplied and added.

Part (a) (iii)

Students should:

- ensure that they fully justify how they are deriving the (supplied) answer.

In better responses, students were able to:

- explain, using a pattern, how each case could be generated
- clearly differentiate between the leader and the group formations
- add these cases together to derive the required result.

Areas for students to improve include:

- reading the question carefully to gain a full understanding of the situation
- taking time to build the answer step by step
- knowing when cases should be multiplied and added.

Part (a) (iv)

Students should:

- note that the choices for the leadership group will be the same in all cases
- recognise that the number of men and women from which to choose the groups has now been reduced by one
- recognise that the reversed method will, ultimately, generate an equivalent number of choices as the method in part (iii).

In better responses, students were able to:

- express the choices for leaders as $\binom{n}{1}^2 = n^2$
- express the choices for the members of the groups in the form $\binom{n-1}{k}^2$
- multiply these parts to generate an expression for each individual case
- add these cases together to derive a correct result

- use the result from part (i) to simplify this summation.

Areas for students to improve include:

- reading the question carefully to gain a full understanding of the situation
- taking time to build the answer step by step
- knowing when cases should be multiplied and added
- realising that parts of questions are often connected, finding these connections and using them.

Part (b) (i)

Students should:

- recognise the need to apply trigonometric identities.

In better responses, students were able to:

- apply relevant trigonometric identities correctly in a manner such that the expression is simplified
- outline their proof with clear and correct algebraic steps.

Areas for students to improve include:

- focusing on the setting out of formal proofs
- taking care with algebraic processes to ensure that the expressions from one line to the next remain equivalent.

Part (b) (ii)

Students should:

- follow the instruction in the question to substitute
- recognise the connection to part (i) of the question.

In better responses, students were able to:

- correctly make the suggested substitution
- simplify the resulting equation into a form which allows the substitution of the result from part (i)
- correctly simplify this equation to obtain the required result.

Areas for students to improve include:

- demonstrating skills in substitution
- demonstrating skills in algebraic simplification.

Part (b) (iii)

Students should:

- recognise a question involving sum and product of roots results
- recognise connections to previous parts of questions.

In better responses, students were able to:

- correctly solve the $\sin(3\theta) = \frac{1}{2}$ equation from part (ii)
- use their solutions to generate roots of the cubic equation in the form $x = 4\sin\theta$
- correctly justify why the three (supplied) roots are distinct
- apply the sum and product of roots results to achieve the required result.

Areas for students to improve include:

- recognising that the first step of this part is to continue from the result in part (ii) of the question
- understanding how to determine which roots are repeated and which are distinct
- showing the ability to generate the $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ result.