Mathematics Extension 1

HSC Marking Feedback 2021

Question 11

Part (a)

Students should:

- add and subtract simple vectors.

In better responses, students were able to:

- demonstrate their knowledge of course content.

Areas for students to improve include:

- having the knowledge and application of adding and subtracting like vectors.

Part (b)

Students should:

- use and engage with the coefficients of the binomial expansion using Pascal's triangle
- square a binomial twice, in this instance $(2a - b)^2(2a - b)^2$, and collect like terms to arrive at the simplified expression for the expansion.

In better responses, students were able to:

- successfully use Pascal's Triangle to find the coefficients with the correct signs
- successfully expand the squares of the two binomials and arrive at the correct expansion.

Areas for students to improve include:

- using Pascal's Triangle effectively, especially considering the negative sign in $(2a - b)^4$
- expanding a simple binomial $(2a - b)^2(2a - b)^2$, by itself, ensuring the negative sign is considered.

Part (c)

Students should:

- use the substitution of u to arrive at $x = u - 1$ and $dx = du$
- obtain the integrand in terms of u
- obtain the correct primitive in terms of u
- replace u with x as their final solution.

In better responses, students were able to:

- obtain the correct substitution in terms of u
- integrate successfully to arrive at the correct primitive in terms of x
- resubstitute u for x in their final solution.

Areas for students to improve include:

- integrating expressions with fractional indices correctly
- realising the need to replace the substituted u by x in their final solution
- realising it is an indefinite integral and hence will need a constant in their final solution.

Part (d)

Students should:

- use combinatorics knowledge to arrive at the correct solution.

In better responses, students were able to:

- effectively demonstrate their knowledge in applying combinatorics to the problem.

Areas for students to improve include:

- recognising that when calculating the number of ways the committee can be formed, they need to use multiplication of the outcomes rather than addition.

Part (e)

Students should:

- recall the formula for the volume of a sphere
- find the derivative of the volume of the sphere with respect to r correctly
- obtain the correct chain rule for the expression $\frac{dv}{dt}$
- substitute correctly into the $\frac{dv}{dt}$ expression to arrive at the answer, correct to 1 decimal place.

In better responses, students were able to:

- obtain the correct $\frac{dv}{dt}$ expression
- substitute correctly to arrive at the answer correct to 1 decimal place.

Areas for students to improve include:

- recalling the formula for the volume of a sphere and correctly finding the derivative with respect to r
- obtaining and using the chain rule
- evaluating correct to 1 decimal place.

Part (f)

Students should:

- integrate to obtain the correct inverse trigonometric primitive

- evaluate the limits correctly to obtain the correct solution
- ensure their answer is in radians.

In better responses, students were able to:

- apply their knowledge of inverse trigonometric functions
- substitute the limits of integration correctly to arrive at $\frac{\pi}{3}$.

Areas for students to improve include:

- familiarising themselves with the use of the Reference Sheet, using it for integration leading to inverse trigonometric functions
- correctly evaluating limits to arrive at an answer in radians.

Part (g)

Students should:

- recognise a polynomial in terms of $\sin x$
- factorise the polynomial by removing the highest common factor
- solve $\sin^2 x = \frac{1}{2}$, ensuring they obtain $\sin x = \pm\frac{1}{\sqrt{2}}$
- find all the possible solutions in the required domain.

In better responses, students were able to:

- apply their knowledge of trigonometric functions and polynomials by fully factorising a trigonometric polynomial equation
- solve for all the angles, in radians, in the given domain.

Areas for students to improve include:

- recognising the highest common factor when factorising a polynomial
- recognising that the square root must have both the plus and minus signs
- avoiding dividing by a factor because this results in the elimination of possible solutions
- finding trigonometric solutions in radians, where required.

Part (h)

Students should:

- apply the relevant formulae relating to roots and coefficients of a quartic equation
- recognise the coefficients of b and c are 0 in this quartic equation
- write the four terms as a single fraction with a common denominator.

In better responses, students were able to:

- apply their knowledge of the relationships between the roots and coefficients of polynomial equations, finding the product and sum of roots taken three at a time correctly to obtain a simplified answer of $\frac{1}{2}$.

Areas for students to improve include:

- learning the formulae relating roots and coefficients of quartic equations
- obtaining a single algebraic fraction from four algebraic terms
- evaluating algebraic fractions correctly.

Question 12

Part (a)

Students should:

- use the slope or direction fields to sketch a possible solution through a particular point.

In better responses, students were able to:

- apply their knowledge of slope or direction fields
- follow the tangent lines to sketch a possible solution passing through a given point.

Areas for students to improve include:

- demonstrating their understanding of direction fields and their use
- following the tangent lines to produce a sketch.

Part (b) (i)

Students should:

- solve a differential equation of the form $\frac{dT}{dt} = k(T - 25)$ resulting in a logarithmic function
- use initial conditions to find the value of the integration constant
- use given values to find the value of an unknown
- rearrange a logarithmic function to an exponential function
- substitute a given value and hence solve the equation to find the required value.

In better responses, students were able to:

- successfully use integration to solve the differential equation
- use given values to find the particular solution $T = 25 - 20e^{kt}$ where $k = \frac{1}{8} ln\left(\frac{3}{4}\right)$ or equivalent
- complete the question by finding the desired time correct to the nearest minute.

Areas for students to improve include:

- knowing how to solve differential equations
- using the Reference Sheet, in particular the integral resulting in a logarithmic function
- changing a logarithmic function to an exponential function
- using initial conditions to find the integration constant
- using the calculator correctly when calculating expressions that involve one or more logarithms
- using logarithmic laws correctly, for example, $\ln a - \ln b = \ln \frac{a}{b}$

- using the degree and minutes function correctly on a calculator, for example, $38.55..$ minutes $\neq\ 38°33'$.

Part (b) (ii)

Students should:

- graph logarithmic and exponential functions
- recognise important features such as intercepts and asymptotes
- rearrange functions, for example, rearrange $t = f(T)$ to $T = F(t)$.

In better responses, students were able to:

- graph $T = F(t)$ showing both the y-intercept and the asymptote.

Areas for students to improve include:

- finding intercepts and asymptotes of a given function
- realising that time cannot be negative, hence the graph should commence at $t = 0$.

Part (c)

Students should:

- work through the steps of an induction proof
- show the substitution when verifying $n = 1$ case
- clearly articulate the inductive hypothesis for $n = k$
- state what they are required to prove for $n = k + 1$
- perform algebraic manipulation to prove the left-hand side is equivalent to the right-hand side for the $n = k + 1$ case.

In better responses, students were able to:

- produce a proof by mathematical induction showing all necessary steps
- clearly showed the inductive step using the correct assumption in the $n = k + 1$ case
- simplify algebraic fractions
- display sufficient working to indicate a correct proof and obtain full marks.

Areas for students to improve include:

- clearly setting out the proof in a logical sequence
- explaining the use of the assumption for $n = k$, not just stating $T(k)$ and $T(k + 1)$
- ensuring that each step of the proof is algebraically correct, especially the inductive step
- working with algebraic fractions
- working from one side to the other when attempting proofs, and not manipulating both sides simultaneously.

Part (d) (i)

Students should:

- recognise the function as a concave down parabola
- sketch a parabola with a restricted domain
- show all necessary features on the graphs, in particular the intercepts in this case
- consider the vertex of the parabola to assist with sketching the parabola.

In better responses, students were able to:

- correctly sketch the correct left-hand side of the concave down parabola for the restricted domain
- as requested, indicate the x- and y-intercepts of the function
- indicate the end point of the function for the restricted domain, that is, where $x = 2$.

Areas for students to improve include:

- recognising the function as the left-hand side of a concave down parabola given the restricted domain
- sketching the parabola $y = 4 - \left(1 - \frac{x}{2}\right)^2$ and then indicating the required part of the parabola
- finding x- and y-intercepts and displaying them as requested on the graph.

Part (d) (ii)

Students should:

- find the inverse function either by interchanging x and y and then making y the subject, or by rearranging and making x the subject and then expressing this as the inverse function
- state the domain and range of the original function $y = 4 - \left(1 - \frac{x}{2}\right)^2$
- state the domain and range of the inverse function $y = 2 - 2\sqrt{4 - x}$.

In better responses, students were able to:

- correctly find the inverse relation of $y = 4 - \left(1 - \frac{x}{2}\right)^2$ as $y = 2 \pm 2\sqrt{4 - x}$
- use the domain to correctly recognise the inverse function as $y = 2 - 2\sqrt{4 - x}$
- correctly state the domain as $(-\infty, 4]$.

Areas for students to improve include:

- understanding that finding $f^{-1}(x)$ requires more work than just interchanging x and y
- understanding the relationship between the domain and range of $y = f(x)$ and the range and domain of its inverse
- moving algebraically from $x = 4 - \left(1 - \frac{y}{2}\right)^2$ to $y = 2 \pm 2\sqrt{4 - x}$ then determining $y = f^{-1}(x)$ by considering the restriction on the domain.

Part (d) (iii)

Students should:

- sketch $y = f^{-1}(x)$ from the equation or by reflecting $y = f(x)$ through $y = x$, taking into consideration the restricted domain.

In better responses, students were able to:

- demonstrate their knowledge of inverse functions and restricted domain by accurately sketching $y = 2 - 2\sqrt{4 - x}$ for the restricted domain $(-\infty, 4]$.

Areas for students to improve include:

- understanding the definition of a function – for each x there is only one y
- displaying important features such as intercepts and endpoints
- understanding how an inverse can be sketched by reflecting the curve through $y = x$
- understanding how a graph is affected by restricted domains.

Question 13

Part (a)

Students should:

- recognise the need to evaluate volumes of revolution about the y-axis.

In better responses, students were able to:

- determine that the required volume was the result of the difference between an outer and an inner volume
- correctly write an expression for a volume which is created by rotating about the y-axis
- choose the correct limits of integration for each of the different volumes.

Areas for students to improve include:

- recognising which of the two curves created the outer and the inner volumes
- drawing a simple diagram which could help to choose correct limits of integration
- taking note of their evaluations and questioning a result in which the inner volume was calculated to be zero.

Part (b)

Students should:

- use their knowledge of projectile motion to show the required results.

In better responses, students were able to:

- find the time taken to reach maximum height and use this to show that the maximum height was less than 3 metres
- find either the time of flight for the ball to reach a horizontal displacement of 10 metres and show that the ball was still above ground level at this time or find the time of flight for maximum unrestricted range and show that this range is greater than 10 metres.

Areas for students to improve include:

- reading the question carefully to gain a full understanding of the situation, in particular, the relevance of the ball leaving the hand at 1 metre to the h term in the supplied equation
- taking care to ensure that the correct value for time is used with the relevant vertical or horizontal displacement expression
- understanding that, as the ball had left the hand at a height of 1 metre, the range would not be found by doubling the time to maximum height
- taking note of their results and acting on contradictory statements. For example, finding a height at 10 metres displacement which was higher than their previously stated maximum height
- showing all necessary working as opposed to making a statement which had not been fully proven
- using a few words to explain the relevance of a particular calculation
- noting that there was no need to derive equations from acceleration in this question.

Part (c)

Students should:

- use integration and, possibly, area formulae to evaluate the shaded area.

In better responses, students were able to:

- identify that the area bounded by the straight lines would be best evaluated as a triangle
- explain that, by symmetry or even functions, the areas on either side of the y-axis are equal
- understand that the area below the x-axis would be evaluated as a negative value and take the necessary actions to deal with this
- recognise that $|4 - 2\pi| = 2\pi - 4$, or similar
- correctly integrate the $\frac{8}{4+x^2}$ term and then evaluate the resulting substitution in radians.

Areas for students to improve include:

- understanding how to approach integrals involving absolute values
- recognising that their workload can be simplified through use of symmetry
- identifying the axis along which the limits of integration should be chosen
- developing methods to deal with areas which lie below the x-axis
- checking whether terms have positive or negative values and understanding how, and when, absolute value signs can be removed
- using the Reference Sheet for integrations which result in inverse trigonometric functions
- understanding that inverse trigonometric functions should only be evaluated in radians

Part (d) (i)

Students should:

- use trigonometric identities to show a relationship.

In better responses, students were able to:

- follow the hints in the question to substitute for A and C
- use the trigonometric identities on the Reference Sheet to simplify the expression which resulted from the substitution
- simplify with correct algebraic working.

Areas for students to improve include:

- taking the trigonometric identities from the Reference Sheet
- taking care algebraically to simplify correctly
- ensuring that sufficient working is shown to provide a convincing argument that the supplied result has been achieved.

Part (d) (ii)

Students should:

- use simultaneous equations to find an expression for B
- solve the resulting trigonometric equation for θ within the given domain.

In better responses, students were able to:

- recognise the connection to part (i) and let $A = \frac{5\theta}{7}$ and $C = \frac{6\theta}{7}$
- solve the resulting simultaneous equations to obtain $B = \frac{11\theta}{14}$
- solve the resulting trigonometric equation to obtain the two relevant values of θ.

Areas for students to improve include:

- identifying the connections between parts of questions
- recognising that the equation to solve had no B term and so the equation was to be solved for θ
- understanding that many trigonometric equations have more than one solution and ensuring that all relevant solutions are found
- understanding that, for trigonometric equations, the domain given indicates the form in which the answer should be given.

Question 14

Part (a)

Students should:

- read the worded question carefully and attempt to interpret it appropriately
- draw a neat diagram clearly labelling all the given information
- use appropriate trigonometric formulae to find the values of the required angles.

In better responses, students were able to:

- draw a clear and correct diagram, use the sine rule and their knowledge of bearings to arrive at the correct answer
- alternatively, use appropriate vector components and trigonometric ratios to find the correct bearing.

Areas for students to improve include:

- constructing and clearly labelling diagrams from the given information
- using the vector resolution method to solve questions involving motion.

Part (b)

Students should:

- separate variables in a differential equation and correctly integrate terms that lead to logarithmic expressions
- evaluate the constant of integration using the initial condition.

In better responses, students were able to:

- correctly solve the definite integral thereby removing the need for the constant of integration.

Areas for students to improve include:

- making effective use of the logarithmic and exponential laws to solve equations
- rearranging and manipulating terms to simultaneously solve two logarithmic equations
- solving differential equations.

Part (c) (i)

Students should:

- recognise that the angle between parallel vectors is zero
- use the correct notation for vectors and magnitude of vectors.

In better responses, students were able to:

- prove the given statement by using the dot product and either the fact that $\cos 0 = 1$ in this case or representing the two vectors in component form.

Areas for students to improve include:

- completing problems involving proofs
- showing all the steps in a proof
- not using specific values for variables when completing a proof.

Part (c) (ii)

Students should:

- be able to express a vector in terms of the sum of two other vectors
- effectively use the information given in the question, in this case, the fact that the magnitudes of the diagonals are equal

- use the result from part (i) as an indication to square the magnitude of each diagonal.

In better responses, students were able to:

- recognise the link between the result shown in part (i) and used it effectively as a key to the completing the proof for this part.

Areas for students to improve include:

- recognising the importance of using the information given in the question
- carefully considering the implications of the interconnectivity of parts within a question
- distinguishing between vectors and the magnitude of vectors by using the correct notation for each
- practising vector problems involving proofs of geometrical properties of polygons.

Part (d)

Students should:

- identify the relevant statistical descriptors from the information given
- understand the importance of finding the z-score of the standardised normal distribution.

In better responses, students were able to:

- find the correct z-score using the empirical rule
- show a clear understanding of the relationship between a score, the mean, the standard deviation and the z-score.

Areas for students to improve include:

- understanding the difference between variance and standard deviation and where each is used
- using the empirical rule to determine how many standard deviations is the score away from the mean
- analysing statistical data that exhibit binomial distribution.

Part (e)

Students should:

- use their knowledge of the product rule of differentiation to find expression for $f'(x)$
- realise that if $g(1) = 3$ then $g^{-1}(3) = 1$
- apply their knowledge of inverse functions to determine the gradient of the tangent at the given point.

In better responses, students were able to:

- correctly find the value of $\frac{d}{dx}[g^{-1}(3)]$ and used the product rule to arrive at the correct solution
- use implicit differentiation to correctly solve the question.

Areas for students to improve include:

- reviewing the application of the product rule of differentiation

- understanding the fundamentals of inverse functions
- not confusing gradients of a tangent and normal with gradients of a function and its inverse
- understanding the relationship between the derivative of a function at point (a, b) and the derivative of its inverse function at the point (b, a).