Merewether High School

2020 **TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6–11)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Given there are 365 days in a non-leap year, what is the minimum number of people needed to guarantee that at least two people have the same birthday?

A. 183
B. 365
C. 366
D. 367

2 What is the primitive function of $\dfrac{2}{4+x^2}$?

A. $\ln\left(4+x^2\right)+C$

B. $\sin^{-1}\dfrac{x}{2}+C$

C. $\cos^{-1}\dfrac{x}{2}+C$

D. $\tan^{-1}\dfrac{x}{2}+C$

3 The unit vector of vector $\begin{pmatrix} 5 \\ -2\sqrt{6} \end{pmatrix}$ could be written as:

A. $\dfrac{1}{7}\left(5\underset{\sim}{i}-2\sqrt{6}\underset{\sim}{j}\right)$

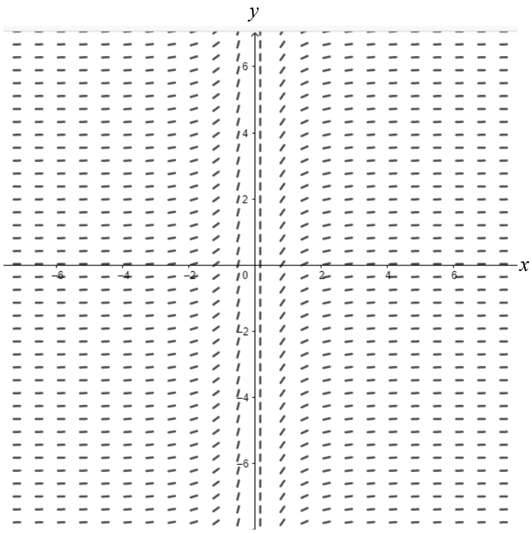
B. $\dfrac{1}{7}\left(-2\sqrt{6}\underset{\sim}{i}+5\underset{\sim}{j}\right)$

C. $\dfrac{1}{5\sqrt{2}}\left(5\underset{\sim}{i}-2\sqrt{6}\underset{\sim}{j}\right)$

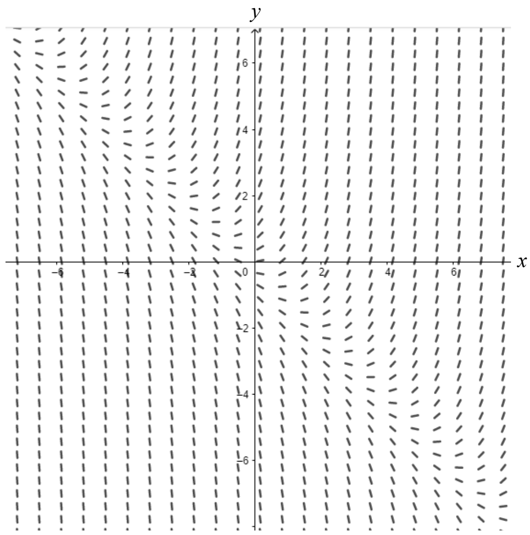
D. $\dfrac{1}{5\sqrt{2}}\left(-2\sqrt{6}\underset{\sim}{i}+5\underset{\sim}{j}\right)$

4 Which of the following slope field diagrams represents $y' = \dfrac{1}{x^2}$?

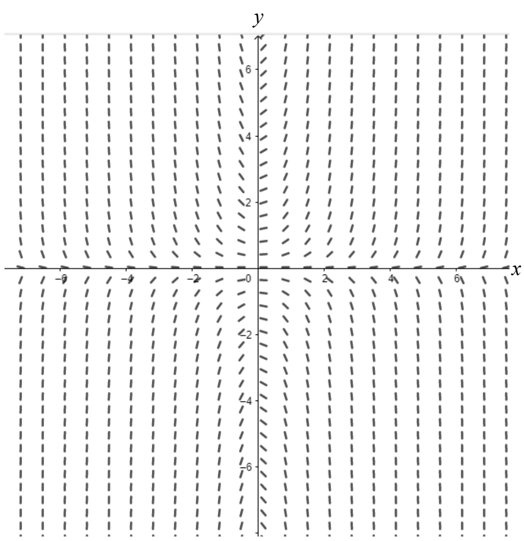
A.



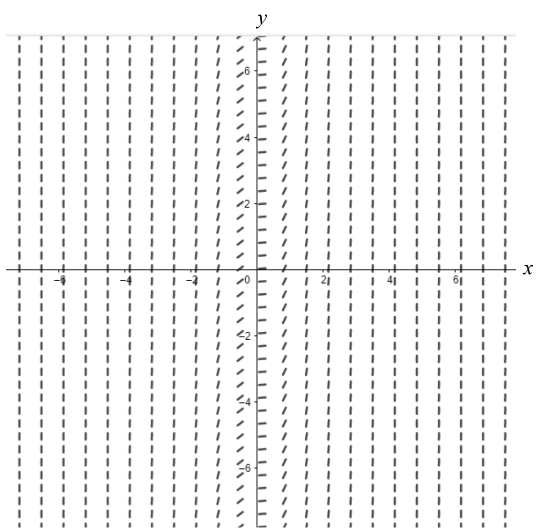
B.



C.



D.



5 The expression for $y = \cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$ is:

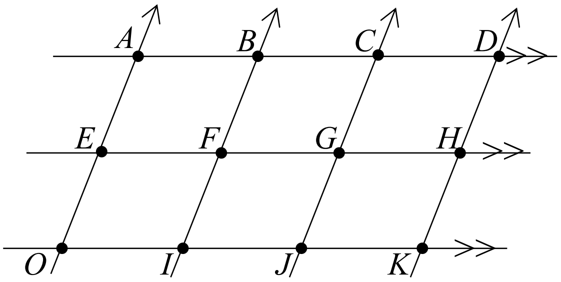
A. $2\cos\left(x+\frac{\pi}{6}\right)$

B. $\cos\left(x+\frac{\pi}{6}\right)$

C. $2\cos\left(x+\frac{\pi}{3}\right)$

D. $\cos\left(x+\frac{\pi}{3}\right)$

6 In the following diagram, all the short line segments are equal in length.



The vector $\overrightarrow{OA} = \underset{\sim}{a}$ and the vector $\overrightarrow{OJ} = \underset{\sim}{j}$. The expression $\frac{1}{2}\underset{\sim}{a} + \frac{3}{2}\underset{\sim}{j}$ represents:

A. $\overrightarrow{OE}$

B. $\overrightarrow{OF}$

C. $\overrightarrow{OG}$

D. $\overrightarrow{OH}$

7 A jury of 12 people is to be chosen from a group of 12 males and 18 females. How many possible jury combinations could be formed if the final jury had at least 9 male members?

A. 13 613 040

B. 189 835

C. 1287

D. 13 984 880

8 The expression $\cos\left[\sin^{-1}\frac{x}{2}+\sin^{-1}\frac{1}{y}\right]$ simplifies to:

A. $\dfrac{\sqrt{\left(4-x^2\right)\left(y^2-1\right)}-x}{2y}$

B. $\dfrac{\sqrt{\left(4-x^2\right)\left(y^2-1\right)}+2x}{y}$

C. $\dfrac{\sqrt{\left(x^2-4\right)\left(1-y^2\right)}-xy}{2}$

D. $\dfrac{\sqrt{\left(x^2-4\right)\left(1-y^2\right)}+2y}{x}$

9 A 150cm tall person is walking away from a 10m lamp post at 2m/s.
How fast is the person's shadow changing?

A. $\dfrac{3}{17}$ m/s

B. $\dfrac{6}{17}$ m/s

C. $\dfrac{40}{17}$ m/s

D. 2 m/s

10 The polynomial $x^3-3x^2+4x-1=0$ has roots α, β and δ. Find the value of $(\alpha+1)(\beta+1)(\delta+1)$.

A. 9
B. 8
C. 1
D. 0

Section II

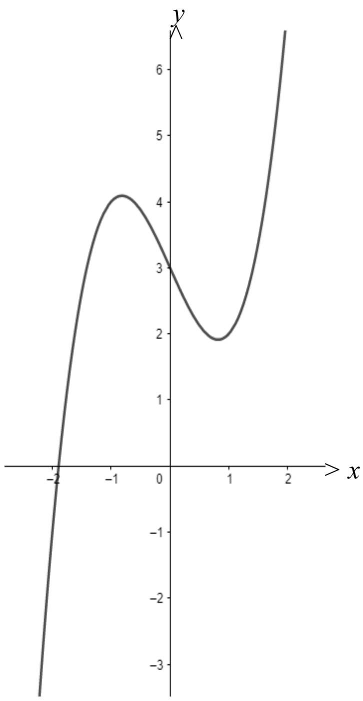
60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Question 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE Writing Booklet.

(a) The diagram below is a sketch of the graph $y = f(x)$.



(i) Sketch the graph of $y = f(|x|)$. **1**

(ii) On a separate number plane, sketch the graph of $y = |f(x)|$. **1**

Question 11 continues on page 7

(b) (i) Prove the trigonometric identity $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$. **3**

(ii) Hence, find expressions for the exact values of the solutions to the equation $3x - x^3 = 1 - 3x^2$. **4**

(c) A curved funnel has a shape formed by rotating part of the parabola $y = 2\sqrt{x}$ about the y-axis, where x and y are given in cm. The funnel is 4cm deep. Find the exact volume of liquid which the funnel will hold if it is sealed at the bottom. **3**

(d) The position of a particle at time t seconds is given by the parametric equations $x = 3\sin t$ and $y = 3\cos t$, where x and y are in metres.

(i) Find the velocity of the particle when $t = 0$. **1**

(ii) Determine the cartesian equation of the particle's path. **2**

End of Question 11

Question 12 begins on page 8

Question 12 (15 marks) Use a SEPARATE Writing Booklet.

(a) Evaluate $\int_{-3}^{0} \frac{2x}{\sqrt{1-x}}\,dx$ using the substitution $u = 1 - x$. **3**

(b) If $y = e^{kx}$ satisfies the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$,

(i) Show that $\frac{d^2y}{dx^2} = k^2 y$ **1**

(ii) Find the possible values of k. **2**

(c) For the function $f(x) = (x-1)^2 - 6$

(i) Sketch the function, clearly labelling the x and y intercepts and the coordinates of the vertex and explain why the domain of the function must be restricted if $f(x)$ is to have an inverse function. **3**

(ii) Give the equation for $f^{-1}(x)$ if the domain of $f(x)$ is restricted to $x \geq 1$. **1**

(iii) State the domain and range of $f^{-1}(x)$, given the restriction in part (ii) and sketch the curve $y = f^{-1}(x)$ on a separate number plane. **2**

(d) Given displacement vectors $\underset{\sim}{u} = -12\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{v} = 15\underset{\sim}{i} - 4\underset{\sim}{j}$, find the magnitude and direction to the nearest minute, of the resultant vector $\underset{\sim}{u} + \underset{\sim}{v}$. **3**

End of Question 12

Question 13 begins on page 9

Question 13 (15 marks) Use a SEPARATE Writing Booklet.

(a) For the differential equation $y' = \dfrac{5}{2x^2 + 7x + 3}$

(i) Show that $\dfrac{5}{2x^2 + 7x + 3} = \dfrac{2}{2x+1} - \dfrac{1}{x+3}$. **1**

(ii) Find the solution to the differential equation, given that when $x = \dfrac{1}{2}, y = 1$. **2**

(b) Given the expression $2\sin 4x \sin x$,

(i) Use a formula on the reference sheet to rewrite this in terms of $\cos 3x$ and $\cos 5x$. **1**

(ii) Hence, find an exact value for $\displaystyle\int_0^{\frac{\pi}{4}} \sin 4x \, \sin x \; dx$. **3**

(c) Write an equation of a trigonometric function that has every one of the following four features:

- Is continuous for all values of x.
- The range is $-1 \le y \le 3$.
- The period is $\dfrac{2\pi}{3}$.
- At the point $\left(\dfrac{\pi}{3}, -1\right)$, the gradient of the function is zero. **2**

(d) Find the two possible acute values of $\tan\dfrac{\theta}{2}$, given $2\sin\theta - \cos\theta - 2 = 0$. **2**

(e) Show that $P = 33 + 212e^{-0.05t}$ if $\dfrac{dP}{dt} = -0.05(P - 33)$ and initially $P = 245$. **4**

End of Question 13

Question 14 begins on page 10

Question 14 (15 marks) Use a SEPARATE Writing Booklet.

(a) Use mathematical induction to prove that $2+10+24+\cdots+n(3n-1)=n^2(n+1)$ for all integers $n \geq 1$. **3**

(b) Jacob's yacht is moored directly south of a sandy access point onshore. Jacob intends to give his dog some exercise. He will need to row his dinghy, from his yacht, across to this access point 100 metres directly north of his yacht. Jacob's rowing speed is 5 metres per second when the water is still. The current is flowing east at a rate of 4 metres per second. Jacob's dinghy will also be impacted by a wind blowing from the south-west, which will push the dinghy at 8 metres per second. He starts rowing the dinghy directly across to the sandy access point onshore so that he is perpendicular to his yacht.

(i) Show that the velocity of Jacob's dinghy can be expressed as the component vector $\left(4+4\sqrt{2}\right)\underset{\sim}{i}+\left(5+4\sqrt{2}\right)\underset{\sim}{j}$. **2**

(ii) Calculate the speed of the dinghy, correct to 2 decimal places. **1**

(iii) Determine how long it took Jacob to reach the shore, and the distance he rowed. **3**

Question 14 continues on page 11

(c) A golf ball is hit at a velocity of 90 ms^{-1} at an angle θ to the horizontal. The position vector $s(t)$, from the starting point, of the ball after t seconds is given by $s = 90t\cos\theta \underset{\sim}{i} + \left(90t\sin\theta - 3t^2\right)\underset{\sim}{j}$, where gravity is 9.8 ms^{-2}

(i) Show that the maximum horizontal range of the ball is $1350\sin 2\theta$ metres. **2**

(ii) To ensure that the ball lands on the green, it must travel between 400 and 450 metres. What values of θ, correct to the nearest minute, would allow this to happen? **1**

(iii) The golfer aims accurately and hits the ball directly towards the green. After 3.4 seconds of flight, at a point 9 metres above the ground, the ball hits a low flying TV drone. If it had not hit the drone or any other obstacles, would the ball have landed on the green? Justify your answer mathematically. **3**

End of paper

2020 Mathematics Extension 1 trial solutions.

Multiple choice answers

1. C
2. D
3. A
4. A
5. C
6. D
7. B
8. A
9. B
10. A

2020 Extension 1 Trial MHS

Solutions.

1. 365 pigeon-holes

$\therefore$ to guarantee at least 2 in one of them, we need 366 people.

C.

2. $\int \frac{2}{4+x^2} = \tan^{-1}\frac{x}{2} + C$ (using ref. sheet) D.

3. $\underset{\sim}{v} = 5\underset{\sim}{i} - 2\sqrt{6}\underset{\sim}{j}$

$|\underset{\sim}{v}| = \sqrt{25+24}$

$= 7$

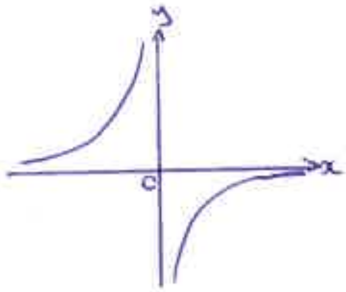
The unit vector $\hat{\underset{\sim}{v}} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|}$

$= \frac{1}{7}(5\underset{\sim}{i} - 2\sqrt{6}\underset{\sim}{j})$

A.

4. $y' = \frac{1}{x^2} = x^{-2}$

$y = -x^{-1} = -\frac{1}{x} + C$



A.

5. $a = 1$, $b = \sqrt{3}$

$R = \sqrt{1^2 + \sqrt{3})^2} = 2$

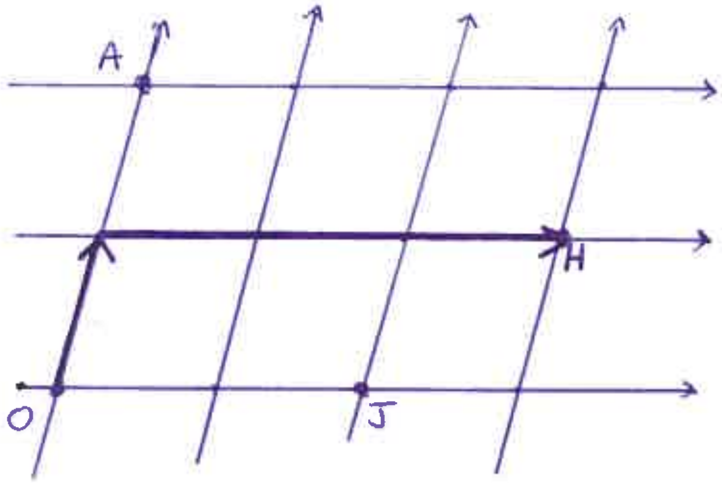
$\tan^{-1}\left(\frac{b}{a}\right) = \alpha$

$\therefore \alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$

C.

6.



$\frac{1}{2}\underset{\sim}{a} + \frac{3}{2}\underset{\sim}{j} = \overrightarrow{OH}$

D.

7. 12 in the jury choosing from 12 males and 18 females with <u>at least</u> 9 males chosen:

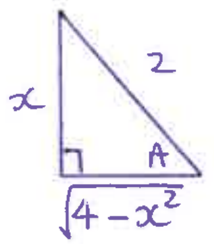
$^{12}C_9 \cdot {}^{18}C_3 + {}^{12}C_{10} \cdot {}^{18}C_2 + {}^{12}C_{11} \cdot {}^{18}C_1 + {}^{12}C_{12} \cdot {}^{18}C_0$

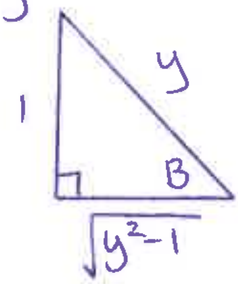
$= 189\,835$

B.

8. Let $\sin^{-1}\frac{x}{2} = A$ and $\sin^{-1}\frac{1}{y} = B$

$\frac{x}{2} = \sin A$ $\qquad$ $\frac{1}{y} = \sin B$





$$\cos\left[\sin^{-1}\frac{x}{2} + \sin^{-1}\frac{1}{y}\right] = \cos(A+B)$$

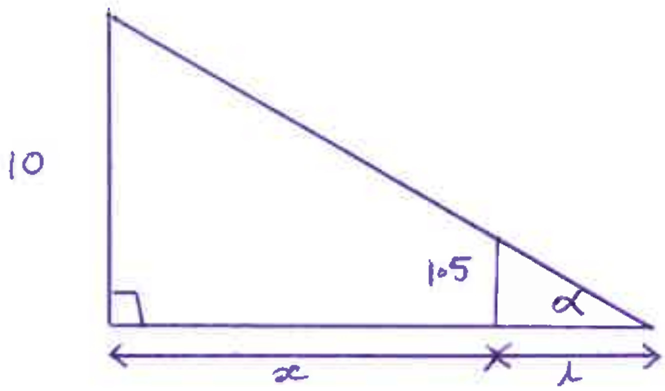
$$= \cos A\cos B - \sin A \sin B$$

$$= \frac{\sqrt{4-x^2}}{2}\cdot\frac{\sqrt{y^2-1}}{y} - \frac{x}{2}\cdot\frac{1}{y}$$

$$= \frac{\sqrt{(4-x^2)(y^2-1)} - x}{2y}$$

A.

9.



150 cm = 1.5 m

l is length of shadow.

$$\frac{dx}{dt} = 2 \text{ m/s}$$

$$\frac{dl}{dt} = \frac{dl}{dx} \cdot \frac{dx}{dt}$$

$$\tan\alpha = \frac{1.5}{l} = \frac{10}{x+l}$$

$$\frac{1.5}{l} = \frac{10}{x+l}$$

$$1.5x + 1.5l = 10l$$

$$1.5x = 8.5l$$

$$l = \frac{1.5x}{8.5} \qquad \therefore \frac{dl}{dx} = \frac{1.5}{8.5}$$

$$\text{so } \frac{dl}{dt} = \frac{1.5}{8.5} \cdot 2$$

$$= \frac{6}{17} \text{ m/s}$$

B.

10. $\alpha + \beta + \delta = -\frac{b}{a} = 3$ where $a = 1$, $b = -3$, $c = 4$, $d = -1$

$$\alpha\beta + \alpha\delta + \beta\delta = \frac{c}{a} = 4$$

$$\alpha\beta\delta = -\frac{d}{a} = 1$$

$$(\alpha+1)(\beta+1)(\delta+1) = (\alpha\beta + \alpha + \beta + 1)(\delta + 1)$$

$$= \alpha\beta\delta + \alpha\beta + \alpha\delta + \alpha + \beta\delta + \beta + \delta + 1$$

$$= \alpha\beta\delta + (\alpha\beta + \alpha\delta + \beta\delta) + (\alpha + \beta + \delta) + 1$$

$$= 1 + 4 + 3 + 1$$

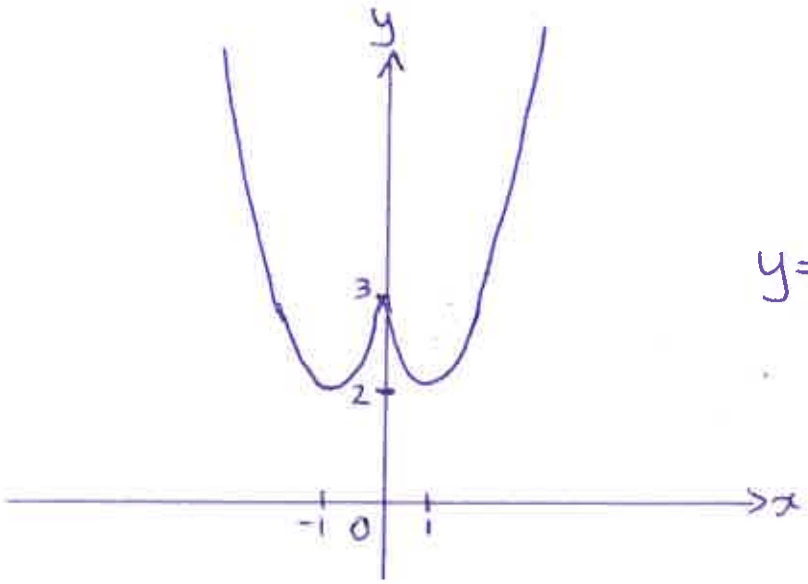
$$= 9$$

A.

Section II

Question 11

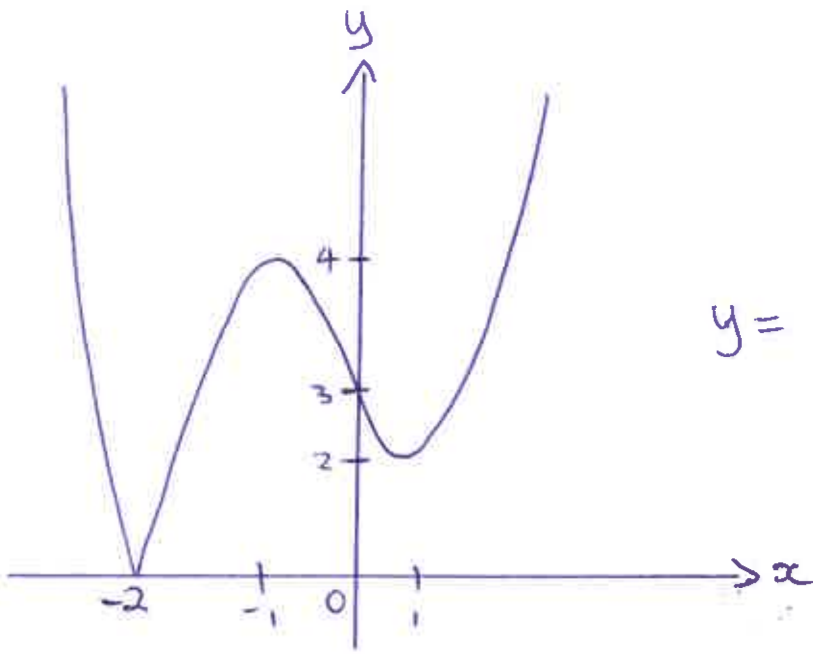
a) i)



(1)

$y = f(|x|)$

ii)



(1)

$y = |f(x)|$

11.) b) i.) Prove $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$$\tan 3\theta = \tan(\theta + 2\theta)$$

$$= \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta} \qquad (1)$$

$$= \frac{\tan\theta + \left[\frac{2\tan\theta}{1-\tan^2\theta}\right]}{1 - \tan\theta\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]} \qquad (1)$$

$$= \frac{\tan\theta - \tan^3\theta + 2\tan\theta}{1 - \tan^2\theta - 2\tan^2\theta} \qquad (1)$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

[3 marks]

ii) Let $x = \tan\theta$ so $3x - x^3 = 1 - 3x^2$ becomes

$$3\tan\theta - \tan^3\theta = 1 - 3\tan^2\theta \qquad (1)$$

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 1$$

$\therefore \tan 3\theta = 1$ from part (i) (1)

$$3\theta = \frac{\pi}{4} \pm n\pi$$

$$\theta = \frac{\pi}{12} \pm n\frac{\pi}{3} = \frac{\pi \pm 4n\pi}{12} \qquad (1)$$

$$\therefore x = \tan\frac{\pi}{12},\ \tan\frac{5\pi}{12},\ \tan\frac{9\pi}{12} \qquad (1)$$

$$= \tan\frac{\pi}{12},\ \tan\frac{5\pi}{12},\ \tan\frac{3\pi}{4}$$

[4 marks]

b) ii) alternative

$$3x - x^3 = 1 - 3x^2$$

$$0 = x^3 - 3x^2 - 3x + 1$$

$P(1) = 1 - 3 - 3 + 1 \neq 0$

$P(-1) = -1 - 3 + 3 + 1 = 0$

$(x+1)$ is a factor of $P(x) = x^3 - 3x^2 - 3x + 1$

$$\begin{array}{r l}
 & x^2 - 4x + 1 \\
\cline{2-2}
x+1 \big) & x^3 - 3x^2 - 3x + 1 \\
- & x^3 + x^2 \\
\cline{2-2}
 & -4x^2 - 3x \\
- & -4x^2 - 4x \\
\cline{2-2}
 & x + 1
\end{array}$$

$$P(x) = (x+1)(x^2 - 4x + 1) = 0$$

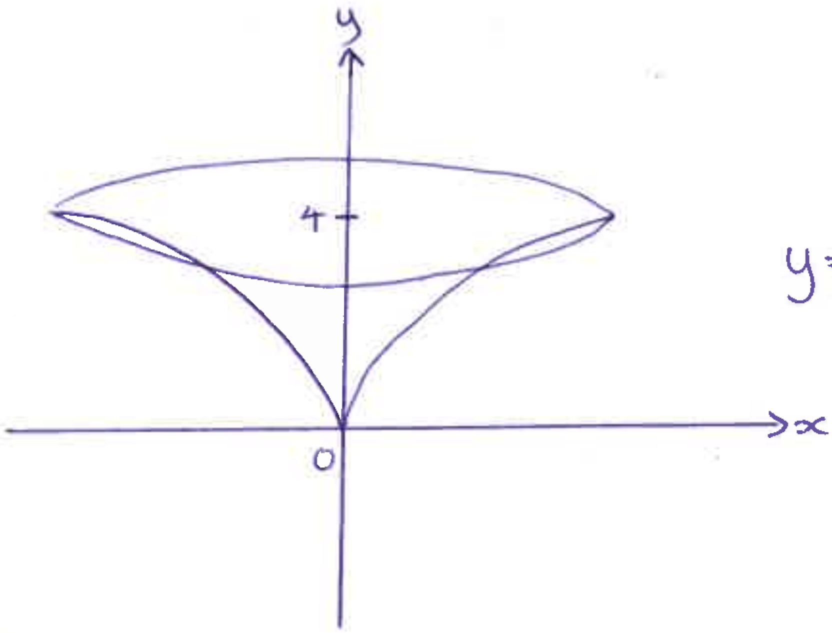
$x = -1 \qquad x = \dfrac{4 \pm \sqrt{16-4}}{2}$

$x = \dfrac{4 + \sqrt{12}}{2} \qquad x = \dfrac{4 - \sqrt{12}}{2}$

$= \dfrac{4 + 2\sqrt{3}}{2} \qquad = \dfrac{4 - 2\sqrt{3}}{2}$

$= 2 + \sqrt{3} \qquad = 2 - \sqrt{3}$

11) c)



$y = 2\sqrt{x}$

$y = 2\sqrt{x}$

$y^2 = 4x$

$x = \frac{y^2}{4}$

$x^2 = \frac{y^4}{16}$

$V = \pi \int_0^4 x^2 \, dy$

$= \frac{\pi}{16} \int_0^4 y^4 \, dy$ (1)

$= \frac{\pi}{16} \left[\frac{y^5}{5} \right]_0^4$ (1)

$= \frac{\pi}{16} \left[\frac{4^5}{5} - 0 \right]$

$= \frac{64\pi}{5}$ cubic cm (1)

$= \frac{64\pi}{5}$ ml.

[3marks]

11) d)

$x = 3\sin t \qquad y = 3\cos t$

i) $v = \dfrac{dx}{dt}$

$\dfrac{d}{dt}(3\sin t) = 3\cos t$

at $t = 0 \qquad v = 3\cos 0$

$= 3 \text{ ms}^{-1}$ (1) [1mark]

ii) Cartesian equation of path:

$x^2 + y^2 = 9\sin^2 t + 9\cos^2 t$ (1)

$x^2 + y^2 = 9(\sin^2 t + \cos^2 t)$

$x^2 + y^2 = 9$ (1) [2 marks]

Question 12

a) $\int_{-3}^{0} \frac{2x}{\sqrt{1-x}}\,dx$

$u = 1-x$, $x = 1-u$ $\quad \frac{du}{dx} = -1 \quad \therefore du = -dx$ or $dx = -du$

when $x = -3$ $\quad u = 1--3 = 4$

when $x = 0$ $\quad u = 1-0 = 1$

$$\int_{-3}^{0} \frac{2x}{\sqrt{1-x}}\,dx = -\int_{4}^{1} \frac{2(1-u)}{\sqrt{u}}\,du \qquad (1)$$

$$= +2\int_{1}^{4} \frac{1}{\sqrt{u}} - \frac{u}{\sqrt{u}}\,du$$

$$= 2\int_{1}^{4} u^{-\frac{1}{2}} - u^{\frac{1}{2}}\,du$$

$$= 2\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{1}^{4} \qquad (1)$$

$$= 2\left[2\sqrt{u} - \frac{2}{3}\sqrt{u^3}\right]_{1}^{4}$$

$$= 2\left[\left(2\sqrt{4} - \frac{2}{3}\sqrt{64}\right) - \left(2\sqrt{1} - \frac{2}{3}\sqrt{1}\right)\right]$$

$$= 2\left[\left(4 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right)\right] \qquad (1)$$

$$= -\frac{16}{3}$$

[3 marks]

12.) b.) i.)

$$y = e^{kx}$$

$$\frac{dy}{dx} = ke^{kx} = ky$$

$$\frac{d^2y}{dx^2} = k^2e^{kx} = k^2y \qquad (1) \qquad [1 \text{ mark}]$$

ii.) $$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

$$\therefore \quad k^2y + ky - 6y = 0 \qquad (1)$$

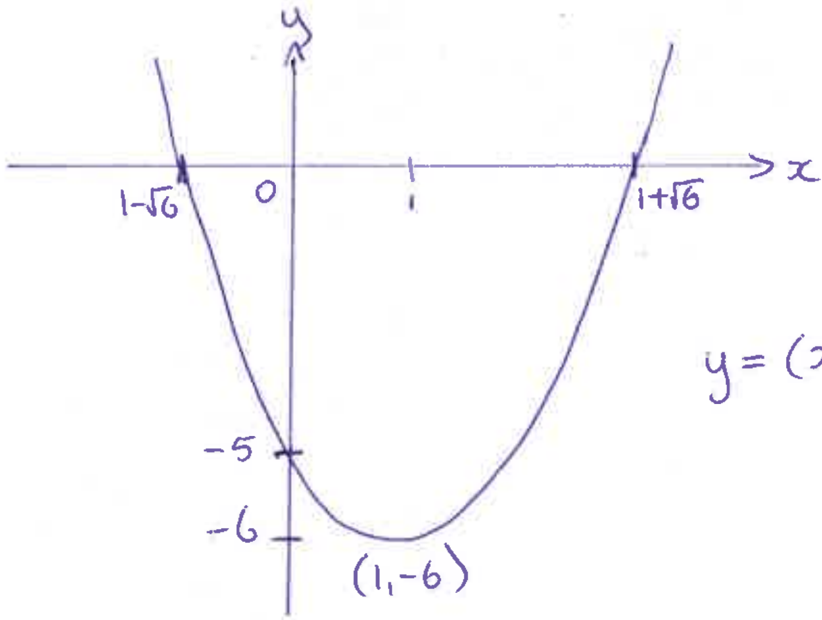
$$y(k^2 + k - 6) = 0$$

$$k^2 + k - 6 = 0$$

$$(k+3)(k-2) = 0$$

$$\therefore k = -3, 2. \qquad (1) \qquad [2 \text{ marks}]$$

12.) c) i)



(1 for x and y intercepts)

(1 for correct shape and coordinates of vertex shown)

$y=(x-1)^2-6$

The domain of $f(x)$ must be restricted in order to have an inverse function so that the function passes the horizontal line test and it will be a one-to-one function either monotonic increasing or monotonic decreasing. (1)

[3 marks]

ii)
$$x=(y-1)^2-6$$
$$x+6=(y-1)^2$$
$$\pm\sqrt{x+6}=y-1$$
$$y=1\pm\sqrt{x+6}$$

since $x \geq 1$ for $f(x)$

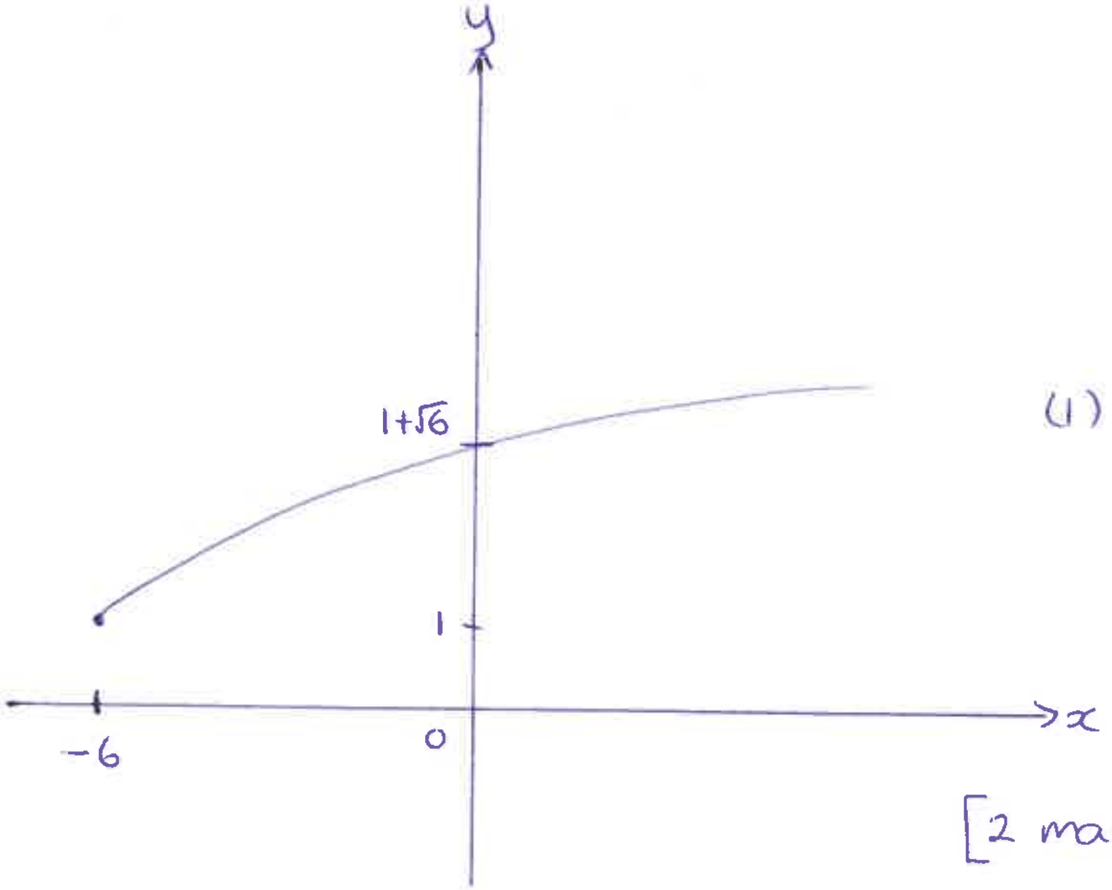
$y \geq 1$ for $f^{-1}(x)$

$\therefore \quad f^{-1}(x)=1+\sqrt{x+6}$ (1) [1 mark]

12.) c.) iii.)

Domain : $[-6, \infty)$ or $x \geq -6$ (1)

Range : $[1, \infty)$ or $y \geq 1$



(1)

[2 marks]

d.) $\underset{\sim}{u} = -12\underset{\sim}{i} - \underset{\sim}{j}$, $\underset{\sim}{v} = 15\underset{\sim}{i} - 4\underset{\sim}{j}$

$\underset{\sim}{u} + \underset{\sim}{v} = 3\underset{\sim}{i} - 5\underset{\sim}{j}$ (1)

$|\underset{\sim}{u} + \underset{\sim}{v}| = \sqrt{3^2 + (-5)^2}$

$= \sqrt{34}$ (1)

$\tan\theta = \frac{-5}{3}$ in fourth quadrant.

$\theta = 300° 58'$ (nearest minute) (1)

[3 marks]

Question 13

a) i) $RHS = \frac{2}{2x+1} - \frac{1}{x+3}$

$= \frac{2(x+3) - (2x+1)}{(2x+1)(x+3)}$

$= \frac{2x+6-2x-1}{2x^2+6x+x+3}$ (1)

$= \frac{5}{2x^2+7x+3}$

$= LHS.$ [1mark]

ii) $y' = \frac{5}{2x^2+7x+3}$

$\therefore \frac{dy}{dx} = \frac{2}{2x+1} - \frac{1}{x+3}$

$y = \int \frac{2}{2x+1}\,dx - \int \frac{1}{x+3}\,dx$

$y = \ln|2x+1| - \ln|x+3| + C$ (1)

when $x = \frac{1}{2}$, $y = 1$

$1 = \ln(2) - \ln\left(\frac{7}{2}\right) + C$

$1 = \ln\left(\frac{2}{\frac{7}{2}}\right) + C$

$C = 1 - \ln\frac{4}{7}$

$\therefore y = \ln\left|\frac{2x+1}{x+3}\right| - \ln\frac{4}{7} + 1$ (1)

[2 marks]

13 b.) i.)

$$\sin 4x \sin x = \frac{1}{2}\left[\cos(4x-x) - \cos(4x+x)\right]$$

$$2\sin 4x \sin x = \cos 3x - \cos 5x \qquad (1)$$

[1 mark]

ii.) $$\int_0^{\frac{\pi}{4}} \sin 4x \sin x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 3x - \cos 5x \, dx$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin 3x - \frac{1}{5}\sin 5x\right]_0^{\frac{\pi}{4}} \qquad (1)$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin\frac{3\pi}{4} - \frac{1}{5}\sin\frac{5\pi}{4} - 0\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{5}\left(-\frac{1}{\sqrt{2}}\right)\right] \qquad (1)$$

$$= \frac{1}{6\sqrt{2}} + \frac{1}{10\sqrt{2}}$$

$$= \frac{\sqrt{2}}{12} + \frac{\sqrt{2}}{20} \qquad (1)$$

[3 marks]

c) $y = 2\cos 3\theta + 1$ (2) [2 marks]

13.) d.)

$2\sin\theta - \cos\theta - 2 = 0$

Let $t = \tan\frac{\theta}{2}$

then

$\frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2} = 2$ (1)

$4t - 1 + t^2 = 2(1+t^2)$

$4t - 1 + t^2 = 2 + 2t^2$

$0 = t^2 - 4t + 3$

$(t-3)(t-1) = 0$

$t = 3$ or 1

$\therefore \tan\frac{\theta}{2} = 3$ or $\tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = 1.24904\ldots$ $\quad\quad \frac{\theta}{2} = \frac{\pi}{4}$

$\theta = 2.498091545$ (radians) $\quad\quad \theta = \frac{\pi}{2}$ (1)

[2 marks]

13.) e) $\frac{dP}{dt} = -0.05(P-33)$, $t=0$, $P=245$.

$$\frac{dt}{dP} = -\frac{1}{0.05(P-33)} \qquad (1)$$

$$t = -\frac{1}{0.05}\int \frac{1}{P-33}\,dP$$

$$t = -\frac{1}{0.05}\ln|P-33| + c \qquad (1)$$

$$0 = -\frac{1}{0.05}\ln(212) + c$$

$$c = \frac{1}{0.05}\ln 212$$

$$t = \frac{1}{0.05}\left[\ln 212 - \ln|P-33|\right]$$

$$t = \frac{1}{0.05}\ln\left(\frac{212}{P-33}\right) \qquad (1)$$

$$0.05t = \ln\left(\frac{212}{P-33}\right)$$

$$e^{0.05t} = \frac{212}{P-33}$$

$$(P-33)e^{0.05t} = 212 \qquad (1)$$

$$Pe^{0.05t} - 33e^{0.05t} = 212$$

$$P = 33 + 212e^{-0.05t}$$

[4 marks]

Question 14

a) Prove true for $n=1$

$LHS = 1(3-1)$
$= 2$
$RHS = 1^2(1+1)$
$= 2$ $\therefore$ LHS=RHS True for $n=1$

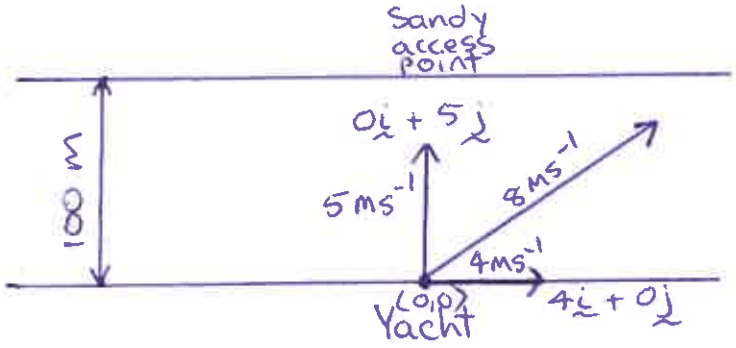
Assume true for $n=k$
ie $2+10+24+\cdots+k(3k-1) = k^2(k+1)$

Prove true for $n=k+1$ if true for $n=k$
ie $2+10+24+\cdots+k(3k-1)+(k+1)(3[k+1]-1)$
$= (k+1)^2(k+1+1)$

$RHS = (k+1)^2(k+2)$
$LHS = [2+10+24+\cdots+k(3k-1)]+(k+1)(3k+2)$
$= k^2(k+1)+(k+1)(3k+2)$
$= (k+1)(k^2+3k+2)$
$= (k+1)(k+1)(k+2)$
$= (k+1)^2(k+2)$
$= RHS$ $\therefore$ true for $n=k+1$ if true for $n=k$

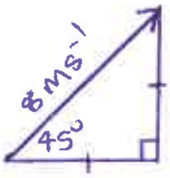
Conclusion:
True for $n=k+1$ if true for $n=k$, but true for $n=1$ $\therefore$ true for $n=2,3,4,\ldots$
Hence by Mathematical induction, true for all $n\geq 1$.

14.) b.) i.)



$8^2 = 2x^2$
$x^2 = 32$
$x = \sqrt{32}$
$= 4\sqrt{2}$

Wind



wind vector is $4\sqrt{2}\underset{\sim}{i} + 4\sqrt{2}\underset{\sim}{j}$

Rowing vector is $0\underset{\sim}{i} + 5\underset{\sim}{j}$ (1)

water flow vector is $4\underset{\sim}{i} + 0\underset{\sim}{j}$

Resultant vector $= 4\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{i} + 5\underset{\sim}{j} + 4\sqrt{2}\underset{\sim}{i} + 4\sqrt{2}\underset{\sim}{j}$ (1)

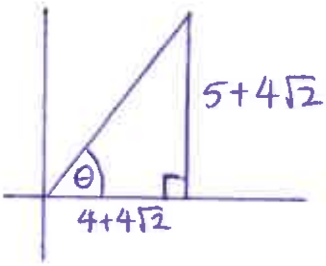
$= (4+4\sqrt{2})\underset{\sim}{i} + (5+4\sqrt{2})\underset{\sim}{j}$

[2 marks]

ii.) $V = \sqrt{(4+4\sqrt{2})^2 + (5+4\sqrt{2})^2}$

$\doteqdot 14.38 \text{ ms}^{-1}$ (1) [1 mark]

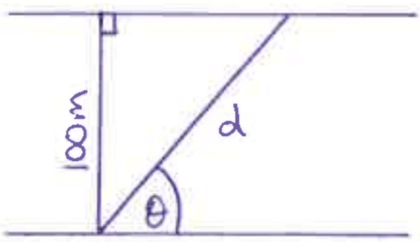
iii.)



$\tan\theta = \dfrac{5+4\sqrt{2}}{4+4\sqrt{2}}$

$\theta = \tan^{-1}\left(\dfrac{5+4\sqrt{2}}{4+4\sqrt{2}}\right)$

$\doteqdot 48°$ (1)



$\sin\theta = \dfrac{100}{d}$

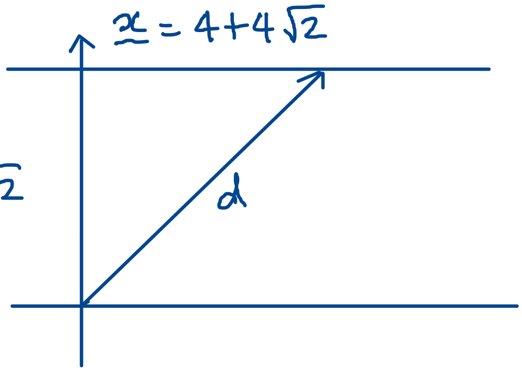
$d = \dfrac{100}{\sin 48°}$

$\doteqdot 135\text{m}$ (1)

Time taken $= 134.9493465\text{m} \div 14.38 \text{ms}^{-1}$

$\doteqdot 9.4$ seconds (1)

$y = 100$
$\underset{\sim}{y} = 5 \vert 2 \sqrt{2}$



$$s = \frac{d}{t}$$

$$t = \frac{d}{s}$$

$$= \frac{100}{5+4\sqrt{2}}$$

$$= 9.383632\cdots$$

$$\doteqdot 9.4 \text{ s}$$

$$d = s \times t$$

$$= 9.38\cdots \times 14.38\cdots$$

$$= 135 \text{ m}$$

14.) b.) iii) cont.

Time taken $= 134.9493465\,m \div 14.38\,ms^{-1}$

$\doteqdot 9.4$ seconds (1)

[3 marks]

c) i) $\underset{\sim}{s} = 90t\cos\theta\,\underset{\sim}{i} + (90t\sin\theta - 3t^2)\,\underset{\sim}{j}$ $\quad g = 9.8\,ms^{-2}$

Max range when $y = 0$

$90t\sin\theta - 3t^2 = 0$

$t(90\sin\theta - 3t) = 0$

$t = 0$ or $90\sin\theta - 3t = 0$

$t = \dfrac{90\sin\theta}{3} = 30\sin\theta$ (1)

Max range when $t = 30\sin\theta$.

$x = 90(30\sin\theta)\cos\theta$

$= 1350\sin 2\theta$ (1) [2 marks]

ii) $400 < 1350\sin 2\theta < 450$

$\dfrac{400}{1350} < \sin 2\theta < \dfrac{450}{1350}$

$17°14'7.03'' < 2\theta < 19°28'16.39''$

$8°37'3.52'' < \theta < 9°44'8.2''$ (1) [1 mark]

or

$180° - 19°28'16.39'' < 2\theta < 180° - 17°14'7.03''$

$160°31'43.61'' < 2\theta < 162°45'52.97''$

$80°15'51.81'' < \theta < 81°22'56.49''$

14.) c.) iii.)

$t = 3.4$ s $\quad y = 9$m (Drone height)

$9 = 90(3.4)\sin\theta - 3(3.4)^2$

$9 = 306\sin\theta - 34.68$

$\sin\theta = \dfrac{9 + 34.68}{306}$

$\theta = \sin^{-1}(0.142745098)$

$\theta = 8°12'24.21''$ (angle ball was hit at) (1)

Using time of flight from part (i) $t = 30\sin\theta$

Distance travelled with $\theta = 8°12'24.21''$:

$t = 30\sin(8°12'24.21'')$ (1)

$= 4.282353183$

$x = 90(4.282353183)\cos(8°12'24.21'')$

$= 381.4649707$

$\therefore$ The ball would not have landed on the green. (1)

[3 marks]