CRANBROOK SCHOOL

Student Number: _____

2022 **TRIAL HSC EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using blue or black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- Use the Multiple-Choice Answer Sheet provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 69

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 59 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section 1

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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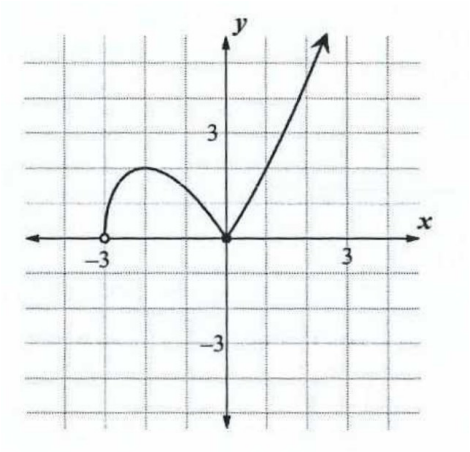
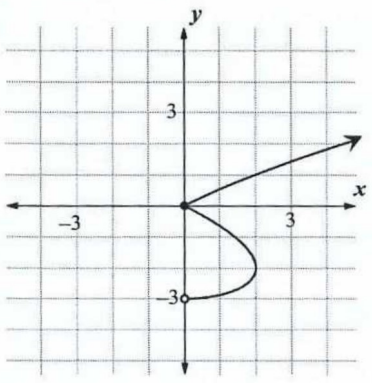
1 Below is the graph of a function.

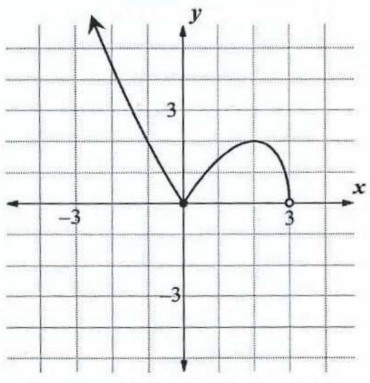


Which of the following is the correct graph of its inverse?

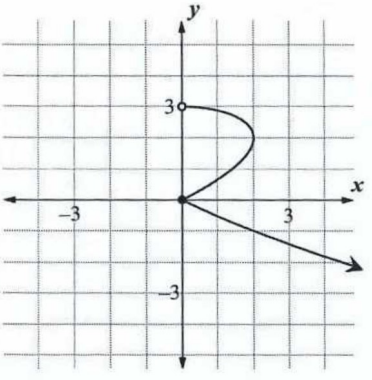
A.



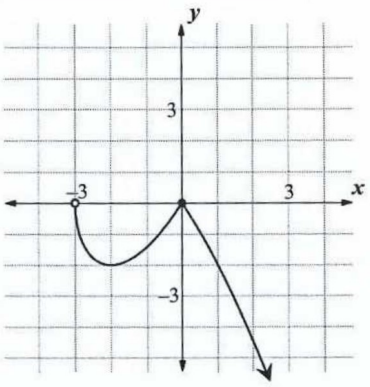
B.



C.



D.



2 Which of the following expressions in equivalent to $\int \frac{-2}{\sqrt{1-4x^2}}\,dx$?

A. $\frac{1}{2}\sin^{-1}(2x)+c$

B. $\sin^{-1}(2x)+c$

C. $\frac{1}{2}\cos^{-1}(2x)+c$

D. $\cos^{-1}(2x)+c$

3 Marie has attempted to differentiate the inverse cosine function, but has made a mistake in her working below.

$$y=\cos^{-1}x$$

$$x=\cos y \qquad \textbf{line A}$$

$$\frac{dx}{dy}=\sin y \qquad \textbf{line B}$$

$$\frac{dy}{dx}=\frac{1}{\sin y} \qquad \textbf{line C}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\cos^2 y}} \qquad \textbf{line D}$$

$$=\frac{1}{\sqrt{1-x^2}}$$

In which line has she made her mistake?

A. Line A

B. Line B

C. Line C

D. Line D

4 Which of the following expressions in NOT equivalent to $2\sin\alpha\sin 2\alpha$?

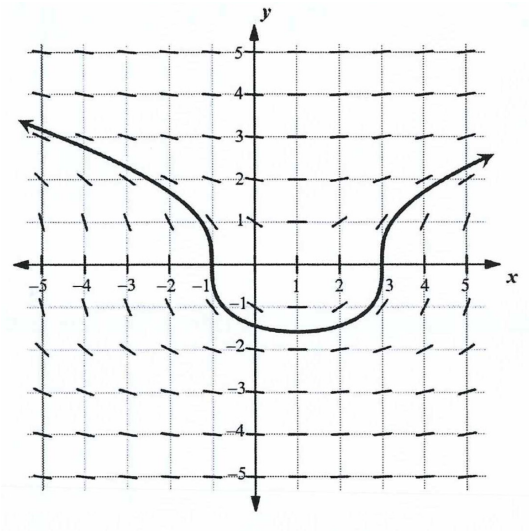
A. $4(\cos\alpha - \cos^3\alpha)$

B. $4\cos\alpha\sin^2\alpha$

C. $\cos\alpha - \cos 3\alpha$

D. $2(\cos\alpha - \cos 3\alpha)$

5 The direction field diagram below has a particular solution shown by the curve sketched on it.



Which of the following differential equations and Cartesian equations satisfies the solution curve sketched on this direction field?

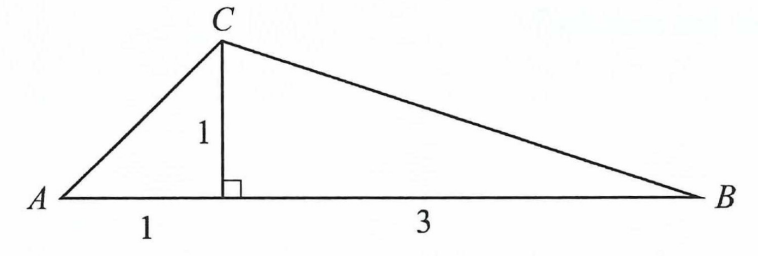
A. $\dfrac{dy}{dx} = \dfrac{2x-2}{3y^2}$ and $y^3 = x^2 - 2x + 3$

B. $\dfrac{dy}{dx} = \dfrac{2x-2}{3y^2}$ and $y^3 = x^2 - 2x - 3$

C. $\dfrac{dy}{dx} = \dfrac{2x-2}{y^2}$ and $y^3 = 3(x^2 - 2x + 3)$

D. $\dfrac{dy}{dx} = \dfrac{2x-2}{y^2}$ and $y^3 = 3(x^2 - 2x - 3)$

6 In triangle ABC, what is the exact value of $\sin(A+B)$?



A. $\dfrac{2}{\sqrt{5}}$

B. $\dfrac{\sqrt{5}}{2}$

C. $\dfrac{5}{\sqrt{5}}$

D. $\dfrac{\sqrt{5}}{5}$

7 Which of the following is an odd function?

A. $a(x) = \frac{\pi}{2} + \cos^{-1} x$

B. $b(x) = \frac{\pi}{2} - \cos^{-1} x$

C. $c(x) = \frac{\pi}{2} + \cos^{-1}(1-x)$

D. $d(x) = \frac{\pi}{2} - \cos^{-1}(1-x)$

8 Which of the following is the length of the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$?

A. $\dfrac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{\underset{\sim}{a} \cdot \underset{\sim}{a}}$

B. $\dfrac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{\underset{\sim}{b} \cdot \underset{\sim}{b}}$

C. $\dfrac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{|\underset{\sim}{a}|}$

D. $\dfrac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{|\underset{\sim}{b}|}$

9 Consider the vectors

$\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

If $\underset{\sim}{c} = \dfrac{2\underset{\sim}{b} + \underset{\sim}{a}}{3}$, which of the following is true?

A. $AC = 2BC$

B. $2AC = BC$

C. $2AC = 3BC$

D. $3AC = 2BC$

10 What is the value of ${}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^nC_2$ for $n \geq 2$?

A. $3n - 5$

B. $(n-1)^2$

C. ${}^{n+1}C_3$

D. 4^{n-2}

Section II

59 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Solve $\dfrac{3}{x-2} \leq 1$. **3**

(b) If $\theta = \sin^{-1}\dfrac{1}{3}$, show that $\cos 2\theta = \dfrac{7}{9}$. **2**

(c) Using the substitution $u = \sqrt{x}$ evaluate $\displaystyle\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$. **3**

(d) Given $\underset{\sim}{a}$ is a unit vector and $(\underset{\sim}{x}-\underset{\sim}{a})\cdot(\underset{\sim}{x}+\underset{\sim}{a}) = 24$, evaluate $|\underset{\sim}{x}|$. **2**

(e) (i) Write the expression $2\sqrt{3}\sin x + 2\cos x$ in the form $R\sin(x+\alpha)$ where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$. **2**

(ii) Hence find the value of k, where $0 \leq k \leq 2\pi$, for which **3**

$$\int_0^k (2\sqrt{3}\cos x - 2\sin x)\, dx = 2$$

.

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Emma is the driver of a car and is planning to take six of her friends to a local soccer match. She can take at most four passengers in her car. She will need to drive two trips. **2**

Determine the number of different ways in which her friends can be selected for the first trip.

(b) Find the value of the constant a such that $8750x^4$ is a term in the Binomial expansion of $(a+2x)^7$. **3**

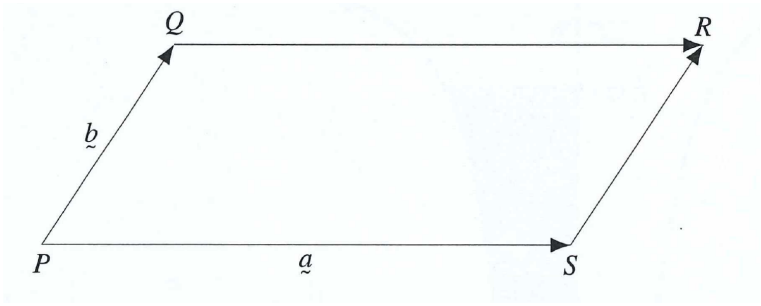
(c) Given the pair of parametric equations $x = 3\cos^2\theta$ and $y = \sin\theta\cos\theta$ for $\theta \in [0, \frac{\pi}{2}]$, find a Cartesian equation in the form $y = f(x)$. **3**

(d) Prove by mathematical induction that **3**

$$1\times 2+2\times 2^2+3\times 2^3+\ldots+n\times 2^n = 2+(n-1)2^{n+1}$$

for all positive integers n.

(e) For the quadrilateral $PQRS$, $\overrightarrow{PS} = \overrightarrow{QR} = \underset{\sim}{a}$ and $\overrightarrow{PQ} = \overrightarrow{SR} = \underset{\sim}{b}$, as shown in the diagram below. **4**



If $\left|\overrightarrow{PR}\right| = \left|\overrightarrow{SQ}\right|$, use vector methods to prove that $PQRS$ is a rectangle.

Question 13 (14 marks) Use the Question 13 Writing Booklet.

(a) The first camel arrived in Australia in 1840. Camels were released into the wild in the 1920s and have become a pest that is a threat to fragile native ecosystems. The growth rate of the population P millions of wild camels in Central Australia at time t months after the end of 2021 is given by

$$\frac{dP}{dt} = \frac{1}{80}P\left(1 - \frac{P}{3}\right)$$

The wild camel population was reported to be 1 million at the end of 2021. That is, when $t = 0$, $P = 1$.

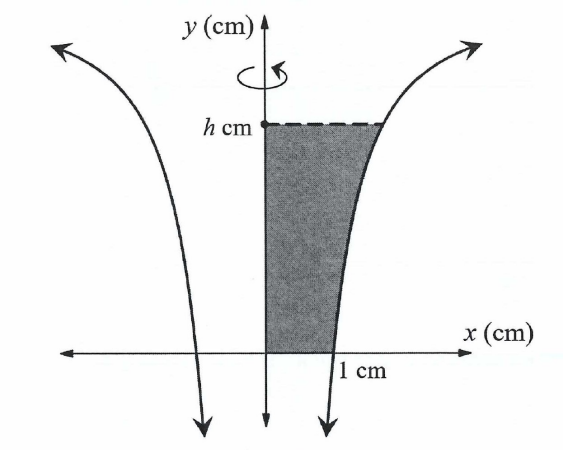
(i) Show that $\dfrac{3}{P(3-P)} = \dfrac{1}{P} + \dfrac{1}{3-P}$. **1**

(ii) Solve the differential equation to show that $P = \dfrac{3}{2e^{-\frac{t}{80}} + 1}$. **3**

(iii) Write down the limiting population of the wild camels. **1**

(iv) How many years will it take for the population to become double what it was at the end of 2021? **2**

(b) Consider the shaded region bounded by the curve $y = 9 - \dfrac{9}{x^2}$, the coordinate axes, and the line $y = h$ as shown in the diagram below. **4**

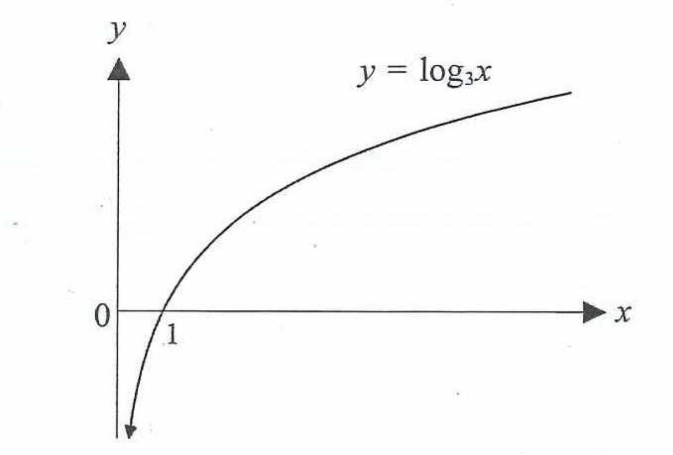


A hollow vase is generated by rotating the section of this curve in the first quadrant about the y-axis. The vase is being filled with water at a constant rate of 50 cm^3/min. Find the depth h cm of the water at the instant when it is rising at a rate of $\dfrac{50}{3\pi}$ cm/min.

Question 13 continues on page 10

Question 13 (continued)

(c) The diagram below shows the graph of the function $f(x) = \log_3 x$.



On separate diagrams, sketch the graphs of

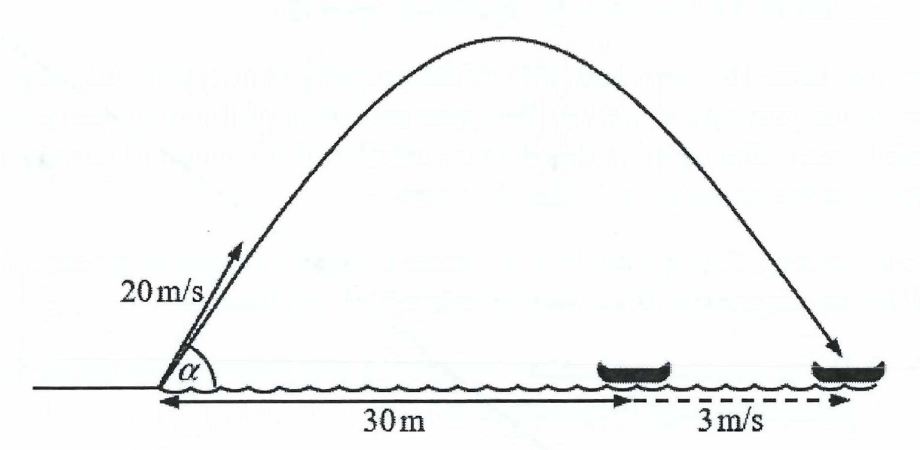
(i) $y = |f(x)|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

in each case showing clearly any intercepts on the axes and the equations of any asymptotes.

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) A projectile is thrown at 20 m/s at an angle of inclination α from a lake's edge towards a canoe that is 30 metres away and moving away from shore at 3 m/s as shown in the diagram below.



The position vector $\underset{\sim}{r}(t)$ of the projectile t seconds after it is thrown is given by

$$\underset{\sim}{r}(t) = \begin{bmatrix} 20t\cos\alpha \\ -5t^2 + 20t\sin\alpha \end{bmatrix}$$

Do NOT prove this.

(i) Show that the projectile lands in the canoe if $20\sin 2\alpha - 6\sin\alpha - 15 = 0$. **2**

(ii) Find integers a, b, and c, such that **1**

$15x^4 + 92x^3 + 30x^2 - 68x + 15 = (3x^2 + 16x - 5)(ax^2 + bx + c)$.

(iii) Using factorisation in (ii) and the substitution $x = \tan\left(\frac{\alpha}{2}\right)$, find solutions for α so that the projectile lands in the canoe. Give your answers correct to the nearest degree. **3**

Question 14 continues on page 12

Question 14 (continued)

(b) Consider the polynomial $P(x) = x^3 + kx^2 + mx + n$ with distinct zeros α, β and γ.

Let $Q(x)$ be a monic polynomial with zeros α^2, β^2 and γ^2.

(i) Show that $Q(x^2) = -P(x)P(-x)$. **2**

(ii) Hence show that $Q(x) = x^3 - (k^2 - 2m)x^2 + (m^2 - 2kn)x - n^2$. **2**

(c) Consider the polynomial $P(x) = x^3 - 3x^2 - x + 1$ with distinct zeros α, β and γ.

(i) By using two applications of the result in part (b) (ii), or otherwise, show that the polynomial $R(x) = x^3 - 107x^2 + 27x - 1$ has zeros α^4, β^4 and γ^4. **2**

(ii) By assuming that $\alpha > 0$, and that the magnitude of α is significantly greater than the magnitudes of β and γ, and by considering the sum of zeros of $R(x)$ above, explain why $\alpha^4 \approx 107$. **1**

(iii) Hence find an approximation for α to 2 decimal places and verify it is a good approximation for a zero of $P(x)$. **2**

End of paper

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