KNOX GRAMMAR SCHOOL

2022 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 1

General Instructions:

- Reading Time – 10 mins
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided as on a separate document
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6–12)

- Attempt Questions 11–14
- Allow about 1 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the provided answer sheet for Questions 1-10.

1 Which is equal to $\int \frac{2}{2+x^2}\,dx$?

A. $\tan^{-1}\frac{x}{\sqrt{2}}+c$

B. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+c$

C. $\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}+c$

D. $2\tan^{-1}\frac{x}{\sqrt{2}}+c$

2 Which of the following are solutions to $\frac{2-x}{x}\le 0$?

A. $(-\infty,0]$ or $[2,\infty)$

B. $(-\infty,0)$ or $[2,\infty)$

C. $(0,2]$

D. $[2,\infty)$

3 Consider the function $f(x)=x^3+x+8$.
Which of the following is the point of intersection of the function $f(x)$ and its inverse $f^{-1}(x)$?

A. $(-8,-8)$

B. $(-2,-2)$

C. $(0,0)$

D. $(2,2)$

4 The Year 8 Mathematics class has 25 students. Their teacher marks each student's birthday on a class calendar. It is known that every month there is at least one birthday.

What is the greatest number of students who could be born in the same month?

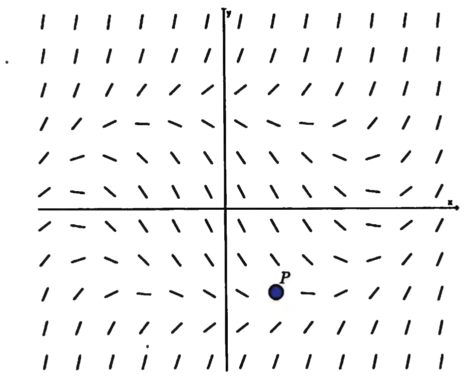
A. 12

B. 13

C. 14

D. 25

5 Consider the slope field shown which shows a missing line element at the point P.



Which of the following line elements would be appropriate at the point P?

A. 

B. 

C. 

D. 

6 The random variable X is distributed binomially such that $X \sim Bin(n, 0.2)$.

If the mean of X is double the size of the standard deviation of X, what is the size of n?

A. 2

B. 8

C. 12

D. 16

7 In how many ways can the letters of the word EXCELLENT be arranged?

A. 3024

B. 30240

C. 60480

D. 362880

8 Which of the following vectors is perpendicular to $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and has a magnitude of 5?

A. $5\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

B. $\frac{5}{\sqrt{17}}\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

C. $\frac{5}{\sqrt{15}}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

D. $\frac{5}{\sqrt{17}}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

9 What is the domain of $f(x)=2\cos^{-1}(\ln x)$?

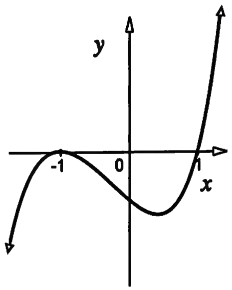
A. $-1\le x\le 1$

B. $0<x\le 1$

C. $e^{-1}\le x\le e$

D. $-e<x\le e$

10 The graph of the polynomial $y=f'(x)$ is shown.



Which of the following MUST be correct about the function $y=f(x)$?

A. $f(1)<f(-1)$

B. $y=f(x)$ is a polynomial of degree 4

C. $x=-1$ is a root of multiplicity 2

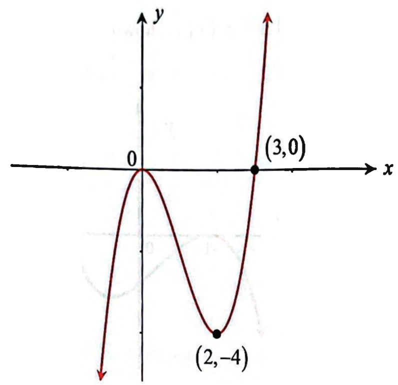
D. $x=-1$ is a root of odd multiplicity

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

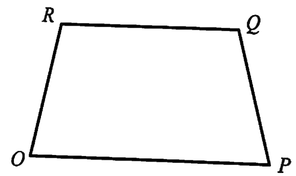
Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $y=f(x)$.



In your writing booklet, sketch the graphs of $y=f(x)$ and $y=\dfrac{1}{f(x)}$ on the same axes. Show any asymptotes and intercepts, together with the location of the points corresponding to the labelled points on $y=f(x)$. **2**

(b) The diagram shows a trapezium *OPQR*. **2**



If $\overrightarrow{OP}=\underset{\sim}{p}$, $\overrightarrow{OR}=\underset{\sim}{r}$ and using that fact that $\overrightarrow{OP}=3\overrightarrow{RQ}$, find an expression for $\overrightarrow{PQ}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{r}$.

(c) Find $\int_{\frac{\pi}{9}}^{\frac{\pi}{6}} 2\cos^2 3x\,dx$ **3**

Question 11 continues on page 7

Question 11 (continued)

(d) Find the coefficient of x^3 in the expansion of $\left(2x-\frac{3}{x^2}\right)^9$. **2**

(e) The ampere (or amp), is a unit used to measure electric current.

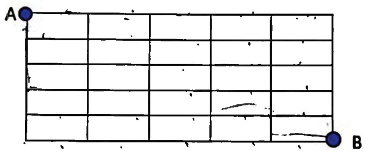
The current i in amperes, at time t, in a circuit is calculated using the equation

$$i = 12\sin t + 5\cos t$$

(i) Write an expression for i in the form $R\sin(t+\alpha)$ where $R>0$ and $0 \le \alpha \le \frac{\pi}{2}$. Give the value of α in radians correct to two decimal places. **2**

(ii) Using the result in part (i) or otherwise, find the maximum current in the circuit and the first time it occurs. Give your answer to two decimal places. **2**

(f) The diagram shows a grid consisting of unit squares. Chris needs to travel from point A to point B but can only do so by moving right or down along the grid lines. **2**



How many paths are there for Chris?

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underset{\sim}{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and θ is the acute angle between them. **4**

Show that the exact value of $\sin 2\theta$ is $\frac{4}{5}$. Give clear reasoning for your answer.

(b) A projectile is launched at an angle 30° to the horizontal and at an initial speed of 60 m/s By taking the point of projection as the origin, the projectile's displacement vector $\underset{\sim}{d}$ at any time t seconds is

$$\underset{\sim}{d} = \left(30\sqrt{3}t\right)\underset{\sim}{i} + \left(30t - 5t^2\right)\underset{\sim}{j} \qquad \textbf{Do not prove this.}$$

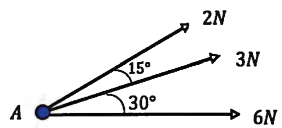
(i) Find the maximum height of the projectile. **2**

(ii) Find the exact speed of the projectile two seconds after it was launched. **2**

(c) Use the principle of mathematical induction to show that for all integers $n \ge 1$, **3**

$3 \times 5^{2n+1} + 2^{3n+1}$ is divisible by 17.

(d) Forces of $6N$, $3N$ and $2N$ act on an object, considered point A, as shown in the diagram.



The vector sum of these forces acting on the object at point A is called the resultant force. The force 6N is acting along a horizontal plane.

(i) Find the magnitude of the resultant force, correct to one decimal place. **3**

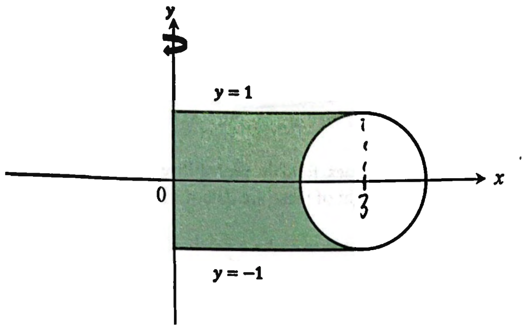
(ii) Find the direction of this resultant force, correct to one decimal place. **1**

Question 13 (14 marks) Use a SEPARATE writing booklet.

(a) A solid rim of two centimetre thickness is to be made out of steel. **4**

To make the rim, the region between the circle $(x-3)^2+y^2=1$, the lines $y=-1$ and $y=1$ and the y-axis is rotated around the y-axis.

The diagram below shows the region to be rotated about the y-axis.



Find the exact volume of the steel needed to make the rim.

(b) (i) Show that $\dfrac{d}{dx}\left[\tan^{-1}\left(\dfrac{e^x-e^{-x}}{2}\right)\right]=2\dfrac{d}{dx}\left[\tan^{-1}e^x\right]$ **3**

(ii) Hence, or otherwise, state the relationship between $\tan^{-1}\left(\dfrac{e^x-e^{-x}}{2}\right)$ and $\tan^{-1}e^x$. **1**

Question 13 continues on page 10

(c) Tomatoes are considered to be either determinate or indeterminate.

Mario buys tomatoes from his local store where the tomatoes are twice as likely to be sourced from Farm A than Farm B.

It is known that 60% of Farm A's tomatoes are determinate while 70% of Farm B's tomatoes are determinate. Mario cannot tell the difference between these tomatoes from their appearance.

(i) Show that the probability that a randomly selected tomato is determinate is $\dfrac{19}{30}$. **1**

(ii) Mario buys ten tomatoes. Find the probability, correct to two decimal places, that no more than eight of these are determinate. **2**

(iii) Mario buys 33 tomatoes to make a batch of tomato paste and knows from his grandmother that at least 55% of the tomatoes should be determinate to achieve a fine paste. **3**

By using a normal approximation to the sample proportion, determine the approximate probability that the tomato paste produced will be considered fine.

End of Question 13

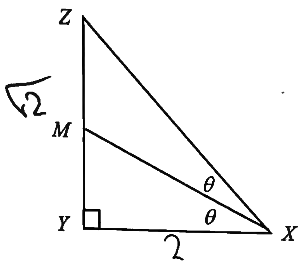
Question 14 (16 marks) Use a SEPARATE writing booklet.

(a) Consider the polynomial $P(x) = Ax^n + 3x^{n-2} - 12$, where $A \neq 0$. **3**

When $P(x)$ is divided by $(x-1)$, the remainder is -2 and when $P(x)$ is divided by $(x+1)$, the remainder is -22.

Show that $P(x)$ is of odd degree, greater than or equal to 3.

(b) ΔXYZ is a right-angled triangle, with M located along the side YZ as shown. The lengths of YZ and YX are 2 cm and $\sqrt{2}$ cm respectively. It is also known that $\angle MXY = \angle ZXM = \theta$. **3**



Find the exact length of MY.

(c) A 500 L tank contains 200 L of brine (salt in water) with 50 kg of salt dissolved.

Pure water is pumped into the tank at 20 L/min. At the same time, the perfectly mixed brine in the tank is pumped out of the tank at 15 L/min.

(i) Explain why the amount of salt m kg in the tank after t minutes can be modelled by the differential equation **1**

$$\frac{dm}{dt} = -\frac{3m}{40+t}$$

(ii) Hence, find m as a function of t. **3**

(iii) How many kilograms of salt is in the tank when it begins to overfill? Give your answer correct to one decimal place. **2**

Question 14 continues on page 12

(d) (i) Show that $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$ **1**

(ii) Use the substitution $u = \sqrt{x-1}$ to find the exact value of **3**

$$\int_2^5 \frac{x-1}{x-1+\sqrt{x-1}}\,dx$$

End of paper

2022 Mathematics Extension 1 Year 12 Trial HSC Marking Guidelines

Section 1

Multiple-choice Answer Key

Question	Answer
1	C
2	B
3	B
4	C
5	A
6	D
7	B
8	D
9	C
10	A

Multiple choice possible solutions:

1. Rearrange integral to give;

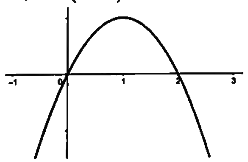
$$I = 2\int \frac{dx}{\left(\sqrt{2}\right)^2 + x^2} = \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

2. After multiplying both sides of the inequality by x^2, solve:

$x(2-x) \le 0$, where $x \ne 0$.

When sketching the parabola $y = x(2-x)$, consider the section below the x-axis.



$\therefore x < 0$ or $x \ge 2$

In interval notation, correct answer is B

3. For the point of intersection of $f(x)$ and $f^{-1}(x)$, solve:

$f(x) = x$

$x^3 + x + 8 = x$

$x^3 = -8$

$x = -2, y = -2$

4. Since every month has at least one birthday, using the pigeonhole principle, we have $25 - 12 = 13$ birthdays to place on the calendar. If all remaining birthdays are inserted into one month, the greatest number of students born in one month is:

$1 + 13 = 14$

5. Consider the line segments on either side of point P and check the change in slope. To the RHS of point P, slope increases. Correct answer is A.

6. $E(X) = 2 \times \sigma$

$n \times 0.2 = 2 \times \sqrt{n \times 0.2 \times 0.8}$

$0.1n = \sqrt{0.16n}$

$0.01n^2 - 0.16n = 0$

$\therefore n = 16$

7. There are 9 letters in the word EXCELLENT. However, there are 3 Es and 2 Ls.

$\therefore \frac{9!}{3! \times 2!} = 30240$

8. Magnitude of given vector = $\sqrt{4^2 + (-1)^2} = \sqrt{17}$

For a vector perpendicular to $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$, we need a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ such that the dot product,

$\begin{pmatrix} 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$

$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\therefore$ unit vector perpendicular to given vector $= \frac{1}{\sqrt{17}}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

To have a magnitude 5, required solution is $5 \times \frac{1}{\sqrt{17}}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

9. The domain for $y = 2\cos^{-1} X$ is $-1 \le X \le 1$.

$\therefore$ for $f(x) = 2\cos^{-1}(\ln x)$, we need:

$-1 \le \ln x \le 1 \Rightarrow e^{-1} \le x \le e$

10. A root for $f'(x) = 0$ does *not* guarantee a root for $f(x) = 0$.

Also, $f'(x)$ looks like a cubic function but this is not necessarily true. The function could have a degree 5.

Using the fundamental theorem of calculus,

$$f(1)-f(-1)=\int_{-1}^{1}f'(x)\,dx \Rightarrow f(1)=f(-1)+\int_{-1}^{1}f'(x)\,dx$$

Note – Using the graph, $\int_{-1}^{1}f'(x)\,dx<0$

$\therefore f(1)<f(-1)$

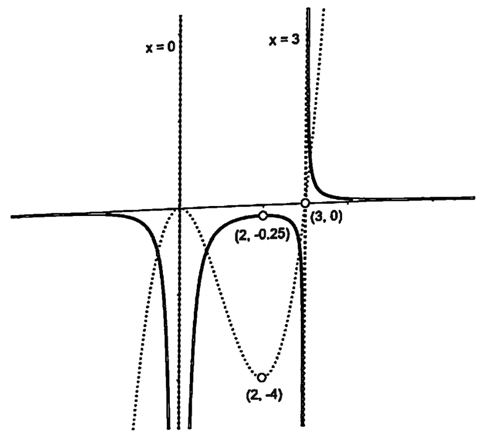
Section II

Note: An incorrect answer in a previous part will not necessarily preclude students from achieving full marks in a later part. Answers here are based on correct prior part answers. Marking will need to adapt to pursue correct method with the use of incorrect prior parts.

Question 11 (a)

Criteria	Marks
• Provides correct sketch with the point $(2,-0.25)$ clearly labelled	2
• Provides correct shape and asymptotes or correct asymptotes and *y*-intercept, or equivalent merit	1

Sample answer:



Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Provides correct expression for the vector sum of $\overrightarrow{PQ}$	1

Sample answer:

$$\overline{PQ}=\overline{PO}+\overline{OR}+\overline{RQ}$$
$$=-\underset{\sim}{p}+\underset{\sim}{r}+\frac{1}{3}\underset{\sim}{p}$$
$$=\underset{\sim}{r}-\frac{2}{3}\underset{\sim}{p}$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Gives correct expression for $\int_{\frac{\pi}{9}}^{\frac{\pi}{6}}(1+\cos 6x)\,dx$	2
• Correctly rewrites integral in terms of $\cos 6x$	1

Sample answer:

$$\int_{\frac{\pi}{9}}^{\frac{\pi}{6}} 2\cos^2 3x\,dx$$

$$=\int_{\frac{\pi}{9}}^{\frac{\pi}{6}} 2\times\frac{1}{2}(1+\cos 6x)\,dx$$

$$=\left[x+\frac{\sin 6x}{6}\right]_{\frac{\pi}{9}}^{\frac{\pi}{6}}$$

$$=\frac{\pi}{6}+\frac{\sin\pi}{6}-\left[\frac{\pi}{9}+\frac{\sin\left(\frac{2\pi}{3}\right)}{6}\right]$$

$$=\frac{\pi}{18}-\frac{\sqrt{3}}{12}$$

Question 11 (d)

Criteria	Marks
• Provides correct solution	2
• Correctly uses the binomial theorem to write a general term in the given expansion, or equivalent merit	1

Sample answer:

For the term in x^3, we need to consider,

$$\binom{9}{2}\times(2x)^7\times\left(-\frac{3}{x^2}\right)^2=41472x^3$$

$\therefore$ coefficient of $x^3=41472$

Question 11 (e) (i)

Criteria	Marks
• Correctly provides both the value of R and the value of α	2
• Correctly provides either the value of R or the value of α	1

Sample answer:

$$R=\sqrt{12^2+5^2}=13$$

$\alpha=\tan^{-1}\left(\frac{5}{12}\right)=0.39$ (2 decimal places)

$$\therefore i=13\sin(t+0.39)$$

Question 11 (e) (ii)

Criteria	Marks
• Correctly provides value of the maximum current and first time it occurs	2
• Provides the value of the maximum current	1

Sample answer:

Since the amplitude $=13$, the maximum value of $i=13$.
This occurs when:

$$\sin(t+0.39)=1$$

$$t+0.39=\frac{\pi}{2},\frac{5\pi}{2},\ldots$$

$\therefore$ first time maximum current occurs is at $t=1.18$ seconds (2 decimal places).

Question 11 (f)

Criteria	Marks
• Provides correct solution	2
• Makes significant progress towards solution	1

Sample answer:

Each path which Chris can take from point A to B consists of a sequence of 5 units downwards and 5 rightwards.

$\therefore$ number of paths Chris could take $=\frac{10!}{5!\times 5!}=252$

Question 12 (a)

Criteria	Marks
• Provides the correct solution	4
• Correctly finds the exact value of $\cos\theta$ and $\sin\theta$	3
• Correctly finds the exact value of $\cos\theta$	2
• Correctly finds the value of the dot product of vectors $\underline{u}$ and $\underline{v}$, or equivalent merit	1

Sample answer:

Consider:

$$\cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

$$= \frac{2\times 0 + 1\times 3}{\sqrt{2^2+1^2}\times\sqrt{0^2+3^2}}$$

$$= \frac{3}{\sqrt{5}\times\sqrt{9}}$$

$$= \frac{1}{\sqrt{5}}$$

Using the identity $sin^2\theta + cos^2\theta = 1$, we have:

$$sin^2\theta + \left(\frac{1}{\sqrt{5}}\right)^2 = 1$$

$$sin^2\theta = \frac{4}{5}$$

$\sin\theta = \frac{2}{\sqrt{5}}$, where $\sin\theta > 0$ since θ is acute.

Using $\sin 2\theta = 2\sin\theta\cos\theta$,

$$= 2\times\frac{2}{\sqrt{5}}\times\frac{1}{\sqrt{5}}$$

$$= \frac{4}{5}$$

Question 12 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly finds time to reach maximum height	1

Sample answer:

For the maximum height, we need to find the vertical component for velocity $(\dot{y})$.
From the given displacement vector, the vertical component (height at any time t) is $30t - 5t^2$.

Differentiate to give: $\dot{y} = 30 - 10t$

Time to reach maximum height occurs when $\dot{y} = 0$.

$\therefore t = 3$ seconds.

$\therefore$ height at $t = 3$ is $30(3) - 5(3)^2 = 45$ metres.

Question 12 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly finds either the horizontal or vertical components for velocity at $t=2$	1

Sample answer:

When $t=2$, the horizontal component for velocity $\dot{x}=30\sqrt{3}$ and the vertical component for velocity $\dot{y}=10$.

$\therefore$ speed of projectile at $t=2$ is

$$=\sqrt{\left(30\sqrt{3}\right)^2+\left(10\right)^2}$$

$=\sqrt{2800}=20\sqrt{7}$ m/s

Question 12 (c)

Criteria	Marks
• Provides correct solution	3
• Proves true for $n=1$ and incorporates the assumption $P(k)$ into $P(k+1)$	2
• Proves true for $n=1$	1

Sample answer:

Required to prove: $3\times5^{2n+1}+2^{3n+1}$ is divisible by 17.

Consider $P(1)$.

$$3\times5^{2(1)+1}+2^{3(1)+1}=391=17\times23$$

$\therefore$ true for $n=1$

Assume $P(k)$ true, for $k\in\mathbb{Z}^+$

i.e. $3\times5^{2k+1}+2^{3k+1}=17M$, where M is some integer.

Prove $P(k+1)$ true.

i.e. $3\times5^{2(k+1)+1}+2^{3(k+1)+1}$

$=3\times5^{2k+3}+2^{3k+4}$

$=3\times5^2\times5^{2k+1}+2^3\times2^{3k+1}$

$=3\times25\times5^{2k+1}+25\times2^{3k+1}-17\times2^{3k+1}$

$=25\left(3\times5^{2k+1}+2^{3k+1}\right)-17\times2^{3k+1}$

Now using assumption,

$=25(17M)-17\times2^{3k+1}$

$=17\left(25M-2^{3k+1}\right)$, where $25M-2^{3k+1}$ is an integer.

$\therefore$ $P(k+1)$ is true if $P(k)$ is true.

Hence, by mathematical induction, $P(n)$ is true for all positive integers $n\geq1$

Question 12 (d) (i)

Criteria	Marks
• Provides correct solution	3
• Correctly finds the sum of the horizontal and vertical components of the resultant force	2
• Correctly finds either the sum of the horizontal or vertical components of the resultant force	1

Sample answer:

Resolve forces into horizontal and vertical components.
For horizontal components, the sum gives:

$$6+3\cos 30°+2\cos 45°=10.0122...N$$

For vertical compinents, perpendicular to force $6N$, the sum gives:

$$3\sin 30°+2\sin 45°=2.9142...N$$

$\therefore$ magnitude of the resultant force $=\sqrt{(10.01)^2+(2.91)^2}=10.4N$

Question 12 (d) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

To find the direction of the resultant force which is a vector, we need the angle, θ, it makes with the horizontal.

$$\therefore \tan\theta=\frac{2.9142...}{10.0122...}$$
$$\theta=16.2°$$

Question 13 (a)

Criteria	Marks
• Provides the correct solution	4
• Integrates correctly and makes one error in determining the exact volume of metal needed	3
• Provides correct integral for volume	2
• Gives correct equation of left semi-circle, or equivalent merit	1

Sample answer:

$$(x-3)^2+y^2=1$$
$$(x-3)^2=1-y^2$$
$$x=3\pm\sqrt{1-y^2}$$

However, the left semi-circle has equation:

$x=3-\sqrt{1-y^2}$

Volume $=\pi\int_{-1}^{1}x^2\,dy$

Due to symmetry, we have:

Volume $=2\pi\int_{0}^{1}x^2\,dy$

$$=2\pi\int_0^1\left(3-\sqrt{1-y^2}\right)^2dy$$

$$=2\pi\int_0^1\left(9-6\sqrt{1-y^2}+1-y^2\right)dy$$

$$=2\pi\int_0^1\left(10-y^2\right)dy-2\pi\times 6\int_0^1\sqrt{1-y^2}\,dy$$

$$=2\pi\left[10y-\frac{y^3}{3}\right]_0^1-12\pi\times\left(\frac{\pi\times 1^2}{4}\right),$$

since $\int_0^1\sqrt{1-y^2}\,dy$ is the area of one quarter of a circle with radius 1.

$\therefore$ Volume $=2\pi\left(10-\frac{1}{3}\right)-3\pi^2$

$$=\frac{58}{3}\pi-3\pi^2$$

Question 13 (b) (i)

Criteria	Marks
• Provides the correct solution	3
• Makes significant progress towards the required solution	2
• Attempts to differentiate $\tan^{-1}\left(\frac{e^x-e^{-x}}{2}\right)$ using $\frac{d}{dx}\left[\tan^{-1}(f(x))\right]=\frac{f'(x)}{1+(f(x))^2}$ or using the chain rule	1

Sample answer:

Using $\frac{d}{dx}\left[\tan^{-1}(f(x))\right]=\frac{f'(x)}{1+(f(x))^2}$,

$$\frac{d}{dx}\left[\tan^{-1}\left(\frac{e^x-e^{-x}}{2}\right)\right]=\frac{1}{1+\left(\frac{e^x-e^{-x}}{2}\right)^2}\times\frac{e^x+e^{-x}}{2}$$

$$=\frac{e^x+e^{-x}}{2\left[1+\left(\frac{e^x-e^{-x}}{2}\right)^2\right]}$$

$$=\frac{2(e^x+e^{-x})}{4+(e^x-e^{-x})^2}$$

$$=\frac{2(e^x+e^{-x})}{e^{2x}+2+e^{-2x}}$$

$$=\frac{2(e^x+e^{-x})}{(e^x+e^{-x})^2}$$

$$=\frac{2}{(e^x+e^{-x})}=\frac{2e^x}{(e^x)^2+1}=2\frac{d}{dx}\left[\tan^{-1}(e^x)\right]$$

Question 13 (b) (ii)

Criteria	Marks
• Provides the correct solution	1

Sample answer:

Since $\frac{d}{dx}\left[\tan^{-1}\left(\frac{e^x-e^{-x}}{2}\right)\right]=2\frac{d}{dx}\left[\tan^{-1}e^x\right]$,

Integrating both sides gives,

$$\tan^{-1}\left(\frac{e^x-e^{-x}}{2}\right)-2\tan^{-1}e^x=C$$

To find the value of the constant C, let $x=0$ which gives,

$$\tan^{-1}0-2\tan^{-1}1=C$$

$$\therefore C=-\frac{\pi}{2}$$

$$\therefore \tan^{-1}\left(\frac{e^x-e^{-x}}{2}\right)=2\tan^{-1}e^x-\frac{\pi}{2}$$

Question 13 (c) (i)

Criteria	Marks
• Provides the correct solution.	1

Sample answer:

$$P(determinate)=P(Farm\ A\ and\ determinate)+P(Farm\ B\ and\ determinate)$$

$$=\frac{2}{3}\times 60\%+\frac{1}{3}\times 70\%$$

$$=\frac{19}{30}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides the correct solution.	2
• Provides correct expression for $P(X\le 8)$	1

Sample answer:

Let X be the random variable representing the number of determinate tomatoes.

$P(X\le 8)=1-P(X=9)-P(X=10)$

$$= 1 - {}^{10}C_9\left(\frac{19}{30}\right)^9\left(\frac{11}{30}\right)^1 - {}^{10}C_{10}\left(\frac{19}{30}\right)^{10}\left(\frac{11}{30}\right)^0$$
$$= 0.93$$

Question 13 (c) (iii)

Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards finding the required probability	2
• States the value of the standard deviation of $\hat{p}$	1

Sample answer:

Let $\hat{p}$ be the random variable representing the sample proportion of determinate tomatoes.

$$E(\hat{p}) = p = \frac{19}{30}$$

Standard deviation $(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{\frac{19}{30}\times\frac{11}{30}}{33}} = \sqrt{\frac{19}{2700}}$

We need to find

$$P(\hat{p} \geq 0.55) = P\left(Z \geq \frac{0.55 - {}^{19}\!/_{30}}{\sqrt{{}^{19}\!/_{2700}}}\right)$$
$$= P(Z \geq -1)$$

$= 0.34 + 0.5$, using z-score empirical rules

$$= 0.84$$

Question 14 (a)

Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards required proof	2
• Uses remainder theorem to find value for A or equivalent merit	1

Sample answer:

Using the remainder theorem,

$$P(1) = A + 3 - 12 = -2$$
$$\therefore A = 7$$

Using the remainder theorem once more,

$$P(-1) = A(-1)^n + 3(-1)^{n-2} - 12 = -22$$

However, since $A = 7$, we have:

$$7(-1)^n + 3(-1)^{n-2} - 12 = -22$$

Rearrange to give:

$$7(-1)^n + 3(-1)^n(-1)^{-2} = -10$$
$$7(-1)^n + 3(-1)^n = -10$$
$$10(-1)^n = -10$$

$$(-1)^n = -1$$

$\therefore n$ must be an odd integer.

However, we also need $n-2 \geq 1$

$$\therefore n \geq 3$$

Question 14 (b)

Criteria	Marks
• Provides correct solution	3
• Finds the exact value for $\tan\theta$	2
• Correctly uses double angle identity to obtain quadratic in $\tan\theta$	1

Sample answer:

From the diagram, $\tan 2\theta = \dfrac{2}{\sqrt{2}}$

Using double angle properties,

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2}{\sqrt{2}}$$

$$2\sqrt{2}\tan\theta = 2 - 2\tan^2\theta$$

$$2\tan^2\theta + 2\sqrt{2}\tan\theta - 2 = 0$$

$(\div 2)$ to give

$\tan^2\theta + \sqrt{2}\tan\theta - 1 = 0$

$$\therefore \tan\theta = \frac{-\sqrt{2} \pm \sqrt{6}}{2}$$

However, since ΔXYZ is a right-angled triangle, θ must be acute, hence $\tan\theta$ must be equal to $\dfrac{-\sqrt{2}+\sqrt{6}}{2}$.

Using the diagram, $\tan\theta = \dfrac{MY}{YX}$

Since $YX = \sqrt{2}$, we have:

$\dfrac{-\sqrt{2}+\sqrt{6}}{2} = \dfrac{MY}{\sqrt{2}}$

$\therefore MY = \left(-1+\sqrt{3}\right)$ cm

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$\frac{dm}{dt}$ = mass rate of salt pumped into the tank - mass rate of salt pumped out of the tank

= concentration pumped in (in kg/L) × volume rate pumped in (in L/min) - concentration pumped out (in kg/L) × volume rate pumped out (in L/min)

$$=0\times 20-\frac{m}{V_{tank}}\times 15$$

$$=\frac{-15m}{V_{tank}}$$

$$=\frac{-15m}{200+5t}$$

$$=\frac{-3m}{40+t}$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards finding the correct expression for m	2
• Correctly separates variables and attempts to integrate both sides or equivalent merit	1

Sample answer:

Separating variables and integrating both sides gives;

$$\int \frac{1}{m}dm=-3\int \frac{1}{40+t}dt \quad (m\neq 0)$$

$$\ln|m|=-3\ln|40+t|+C$$

$$\ln|m|+3\ln|40+t|=C$$

$$\ln|m(40+t)^3|=C$$

$$m(40+t)^3=e^C$$

$m=\frac{A}{(40+t)^3}$, where $A=e^C$

When $t=0, m=50$,

$$\therefore 50=\frac{A}{40^3}\Rightarrow A=50\times 40^3$$

$$m=\frac{50\times 40^3}{(40+t)^3}=50\left(\frac{40}{40+t}\right)^3$$

Question 14 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Finds time when tank overfills, i.e. when $t=60$ minutes	1

Sample answer:

Tank overfills when:

$$200+5t=500$$

$$5t=300$$

$$t=60\,mins$$

$$\therefore m=50\left(\frac{40}{40+60}\right)^3$$

$$=3.2\,kg$$

Question 14 (d) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

To show $\dfrac{x^2}{x+1}=x-1+\dfrac{1}{x+1}$, take R.H.S which gives:

$$x-1+\frac{1}{x+1}=\frac{(x-1)(x+1)+1}{x+1}$$

$$=\frac{x^2-1+1}{x+1}$$

$$=\frac{x^2}{x+1}$$ = L.H.S, as required.

Question 14 (d) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly integrates $2\int_1^2 \frac{u^2}{u+1}du$ using identity in part(i), or equivalent merit	2
• Correctly rewrites integral in terms of u, including limits, and obtains $2\int_1^2 \frac{u^2}{u+1}du$	1

Sample answer:

$$I=\int_2^5 \frac{x-1}{x-1+\sqrt{x-1}}dx$$

Let $u=\sqrt{x-1} \Rightarrow u^2=x-1$

$\dfrac{du}{dx}=\dfrac{1}{2\sqrt{x-1}}=\dfrac{1}{2u} \Rightarrow dx=2u\,du$

Also when $x=5, u=2$

$$x=2, u=1$$

$$\therefore I=\int_1^2 \frac{u^2}{u^2+u}\times 2u\,du$$

$$=2\int_1^2 \frac{u^2}{u+1}du$$

Rewrite $\dfrac{u^2}{u+1}$ using part (i), i.e.

$$\frac{u^2}{u+1}=u-1+\frac{1}{u+1}$$

$$\therefore I=2\int_1^2\left(u-1+\frac{1}{u+1}\right)du$$

$$=2\times\left[\frac{1}{2}u^2-u+\ln|u+1|\right]_1^2$$

$$=2\times\left[2-2+\ln 3-\left(\frac{1}{2}-1+\ln 2\right)\right]$$

$$=2\times\left[\ln 3-\ln 2+\frac{1}{2}\right]$$

$$=2\ln\frac{3}{2}+1$$