Name:____________________ NESA NUMBER:____________

Teacher:_________

ASCHAM SCHOOL

# 2023 YEAR 12
# MATHEMATICS EXTENSION 1

# Trial Examination

GENERAL INSTRUCTIONS

Reading time – 10 minutes
Working time – 120 minutes (2 hours)
Use black pen, non-erasable
NESA-approved calculators may be used
Reference Sheet is provided

Total Marks – 70

Section A – Multiple Choice (1 mark each)

Attempt Questions 1 to 10.
Select answers on the separate multiple choice sheet provided.
Write your NESA number on the multiple choice sheet.

Section B – Questions 11 – 14 (15 marks each)

Start each question in a new booklet.
If you use a second booklet for a question, place it inside the first.
Label extra booklets for the same question as, for example, Q11-2 etc.
Write your NESA number and question number on each booklet.

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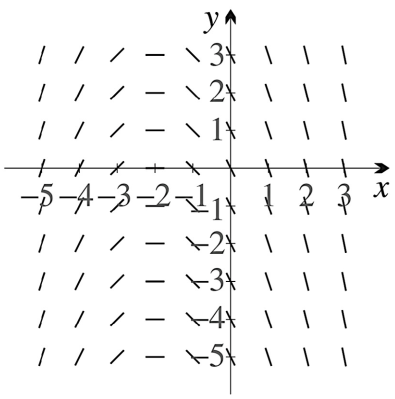
**Section A - Multiple Choice (10 marks)**
(Mark the correct answer on the sheet provided.)

1. Which differential equation best corresponds to the slope field shown?




A) $y' = x + 2$  B) $y' = -x - 2$

C) $y' = y + 2$  D) $y' = -y - 2$

2. Find $\displaystyle\int \frac{3}{\sqrt{9-4x^2}}\,dx$

A) $\sin^{-1}\left(\frac{2x}{3}\right) + C$  B) $3\sin^{-1}\left(\frac{2x}{3}\right) + C$

C) $\frac{3}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$  D) $6\sin^{-1}\left(\frac{2x}{3}\right) + C$

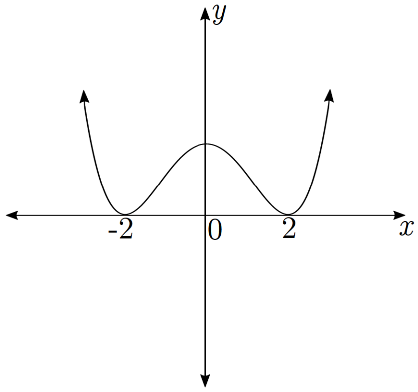
3. Which of the following polynomials $f(x)$ has a root of multiplicity 2 at $x = 1$ in the equation $f(x) = 0$?

A) $f(x) = x^2 - 5x + 4$  B) $f(x) = x^3 - 5x + 4$

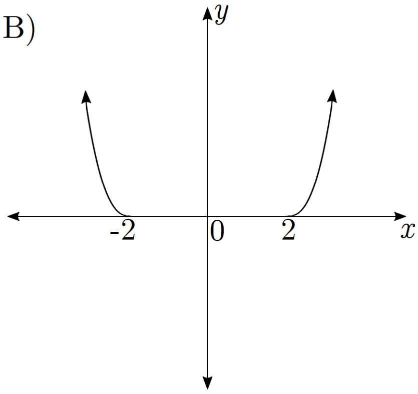
C) $f(x) = x^4 - 5x + 4$  D) $f(x) = x^5 - 5x + 4$

4. Which of the following is the correct sketch of $y=\sqrt{x^2-4}$?

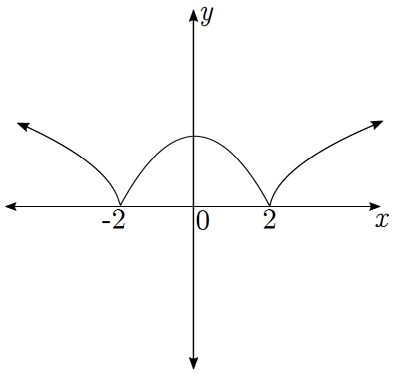
A)



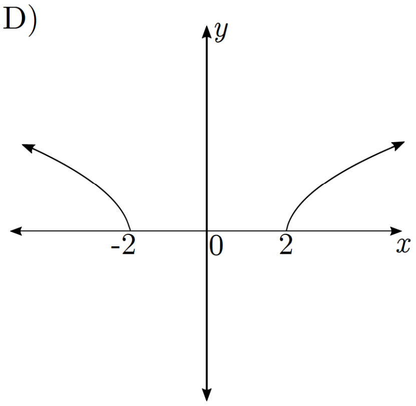

B)




C)




D)




5. The following parametric equations convert to a circle in Cartesian form.

$$x=4+3\cos\theta$$
$$y=2+3\sin\theta$$

What is the centre and radius of this circle?

A) $(4,2),\quad r=3$

B) $(-4,-2),\quad r=3$

C) $(4,2),\quad r=\dfrac{1}{3}$

D) $(-4,-2),\quad r=\dfrac{1}{3}$

6. Which of the following is true in the domain $0\le x\le\pi$?

A) $\sin^{-1}\left(\sin x\right)=x$

B) $\sin\left(\sin^{-1}x\right)=x$

C) $\cos^{-1}\left(\cos x\right)=x$

D) $\cos\left(\cos^{-1}x\right)=x$

7. Which expression is identical to $2\sin 3x \sin 5x$ ?

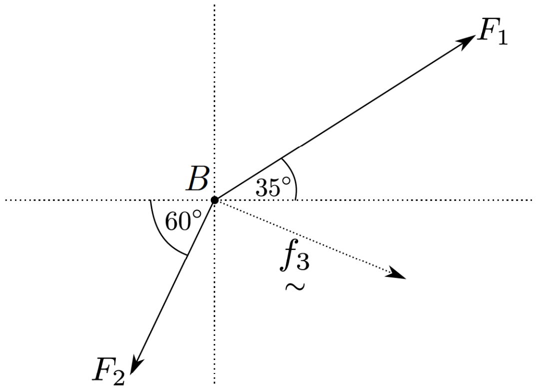
A) $-\cos 2x - \cos 8x$

B) $\cos 8x - \cos 2x$

C) $\cos 2x - \cos 8x$

D) $\cos 2x + \cos 8x$

8. Two forces are pulling on a boat at $B$ and their sum (resultant) is shown as vector $\underset{\sim}{f_3}$. The two forces have magnitudes $F_1$ and $F_2$ respectively, and their angles with the horizontal are shown below. Which is the correct expression for $\underset{\sim}{f_3}$?




A) $\underset{\sim}{f_3} = \left(F_1\cos 35° - \dfrac{F_2}{2}\right)\underset{\sim}{i} + \left(F_1\sin 35° - \dfrac{\sqrt{3}F_2}{2}\right)\underset{\sim}{j}$

B) $\underset{\sim}{f_3} = \left(\dfrac{F_2}{2} - F_1\cos 35°\right)\underset{\sim}{i} + \left(\dfrac{\sqrt{3}F_2}{2} - F_1\sin 35°\right)\underset{\sim}{j}$

C) $\underset{\sim}{f_3} = \left(F_1\sin 35° - \dfrac{\sqrt{3}F_2}{2}\right)\underset{\sim}{i} + \left(F_1\cos 35° - \dfrac{F_2}{2}\right)\underset{\sim}{j}$

D) $\underset{\sim}{f_3} = \left(\dfrac{\sqrt{3}F_2}{2} - F_1\sin 35°\right)\underset{\sim}{i} + \left(\dfrac{F_2}{2} - F_1\cos 35°\right)\underset{\sim}{j}$

9. At the Year 12 valedictory dinner, Abby and her mother, Betty and father, and Cara and her aunty sit down at a circular table. The three groups are otherwise not related. In how many ways can they sit so that each student sits adjacent to their relative?

A) $2!\times 2^3$

B) $3!\times 2^3$

C) $\dfrac{5!}{2^3}$

D) $5!$

10. For the function $y=f(x)$, it is known that:

$f(2)=5$, $f'(2)=-2$, and
$f(5)=3$, $f'(5)=\dfrac{1}{4}$

What is the gradient of the tangent on the inverse function $y=f^{-1}(x)$ at the point where $x=5$?

A) $-2$

B) $\dfrac{-1}{2}$

C) $\dfrac{1}{4}$

D) $4$

*(End of Multiple Choice. Question 11 begins on the next page.)*

**Section B (60 marks)**

**Question 11** (Begin and label a new booklet.) **(15 marks)**

a) Solve: $\dfrac{3x}{x+2} \geq 2$. [3]

b) The polynomial $P(x)$ is given by $P(x) = x^3 + ax + b$ for real numbers $a$ and $b$. $P(x) = 0$ has a root at $x = 2$. When $P(x)$ is divided by $(x+1)$ the remainder is $-15$. Find the values of $a$ and $b$. [3]

c) i) Sketch the graph of $y = 2\sin^{-1} x$ showing clearly the coordinates of the endpoints. [2]

ii) The area in the first quadrant bounded by the curve $y = 2\sin^{-1} x$, the y-axis and the line $y = \dfrac{\pi}{2}$ is rotated about the $y$-axis. Find the volume of the resulting solid. [3]

d) i) Show that $\dfrac{u^2}{1+u^2} = 1 - \dfrac{1}{1+u^2}$. [1]

ii) Using the substitution $x = u^2$, $u \geq 0$, and the result of part i), find: [3]

$$\int \frac{\sqrt{x}}{1+x}\, dx$$

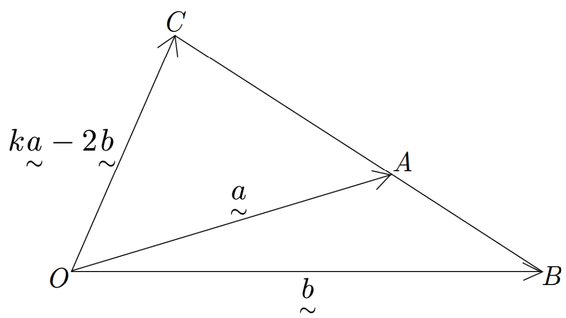
*(End of Question 11.)*

**Question 12** (Begin and label a new booklet.) **(15 marks)**

a) Given that $y = e^{-5x}$ satisfies the 2nd order differential equation $y + ky'' = 0$, find the value of $k$. [2]

b) Find the coefficient of $x^2$ in the expansion of $(2+x^2)\left(3-2x\right)^5$. [2]

c) In the diagram, $A$, $B$ and $C$ are collinear. Find the value of $k$.




[3]

d) Consider the differential equation $\dfrac{dy}{dx} = x \sec y$.

Find the solution that has a $y$-intercept of $\dfrac{\pi}{6}$, expressing your answer in the form $y = f(x)$. [3]

e) Find the number of ways in which the letters of the word EPSILON can be arranged in a straight line so that:
i) the arrangement begins with P and ends with S. [1]
ii) the three vowels are adjacent to each other. [1]
iii) N is somewhere to the left of L. [1]

*(Question 12 continues on the next page.)*

*(Question 12 continued...)*

f) The Dalton Company has introduced a collection of 10 new unique collectible cards. They are sold in foil-wrapped random packs of 3, with each card being equally likely and a pack cannot contain the same card more than once.

Jessie is an avid Dalton Company fan and would like to get the same pack three times so she can show off to her friends how lucky she is to get that pack three times. (Without telling her friends how many packs she actually bought.)

Jessie has calculated that the minimum number of packs she must buy to guarantee that she gets the same pack three times is 241.

Explain using the pigeonhole principle why Jessie is correct. [2]

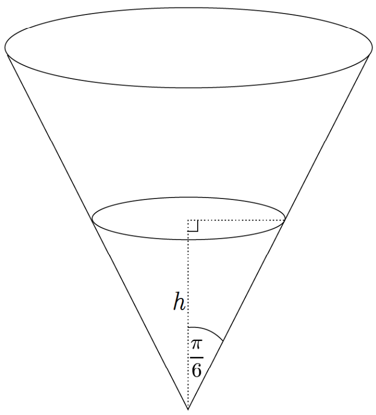
*(End of Question 12.)*

**Question 13** (Begin and label a new booklet.) **(15 marks)**

a) i) Show that $\tan 2x + \tan x = \dfrac{\sin 3x}{\cos 2x \cos x}$. [2]

ii) Hence solve the equation $\tan 2x + \tan x = 0$ for $0 \le x \le \dfrac{\pi}{2}$. [2]

b) A leaking funnel in the shape of an inverted cone has a semi-vertical angle of $\dfrac{\pi}{6}$. It contains sand to a depth of $h$ cm. Sand flows out of the apex of the cone at a constant rate of $0.5\ \text{cm}^3/\text{s}$.




i) Show that the volume $V$ cm$^3$ of sand in the cone is given by $V = \dfrac{1}{9}\pi h^3$. [1]

ii) Find the value of $h$ when the depth of sand is decreasing at a rate of $0.05$ cm/s, giving your answer correct to two decimal places. [3]

c) An employer wishes to hire three people to form a new team. There are ten applicants, four of whom are women and six of whom are men.

How many different teams can be created if at least two members are women? [2]

*(Question 13 continues on the next page.)*

*(Question 13 continued...)*

d) A dam can sustainably carry 10 million fish.

The fish population $N$ (in millions) in the dam follows the logistic differential equation $\dfrac{dN}{dt} = \dfrac{N(10-N)}{200}$ where $t$ is the time in months since the initial fish count.

A fish farming company was unaware of this carrying capacity and introduces 15 million fish. Prior to this, the dam did not have fish.

Find the number of fish remaining after 6 months. Answer to the nearest million fish. [5]

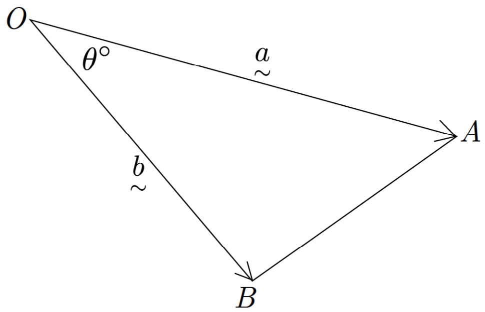
(You may use the identity $\dfrac{10}{N(10-N)} = \dfrac{1}{N} + \dfrac{1}{10-N}$.)

*(End of Question 13.)*

**Question 14** (Begin and label a new booklet.) **(15 marks)**

a) Using the substitution $t = \tan x$, solve $3\sin 2x - 5\cos 2x = 5$ for $0 \le x \le 2\pi$. [3]

b) The diagram below shows $\triangle OAB$ where $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$, and $\angle AOB = \theta$.




By considering the dot product identity $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, show that the area of the triangle can be expressed as $A = \dfrac{1}{2}\sqrt{(\underset{\sim}{a} \cdot \underset{\sim}{a}) \times (\underset{\sim}{b} \cdot \underset{\sim}{b}) - (\underset{\sim}{a} \cdot \underset{\sim}{b})^2}$. [3]

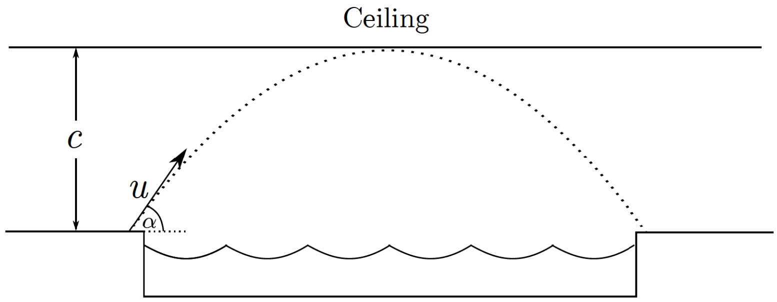
c) Use mathematical induction to show that for all positive integers $n \ge 1$,

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}$$ [4]

*(Question 14 continues on the next page.)*

*(Question 14 continued...)*

d) Queenie brings a football to a swimming pool and wishes to kick it to the other side without it landing in the water or bouncing off the ceiling. The ceiling is $c$ metres high and the football will be kicked with an initial speed of $u$ m/s and angle of elevation of $\alpha$.




The equations of motion of the football are given by, using $g = 10$ m/s$^2$:

$$\dot{x} = u\cos\alpha \qquad \dot{y} = -10t + u\sin\alpha$$
$$x = ut\cos\alpha \qquad y = -5t^2 + ut\sin\alpha$$
(Do not prove these)

i) In order to improve the ball's chances of landing on the other side, Queenie kicks the ball so that it just touches the ceiling at its highest point in the trajectory.

Show that the speed required is: $u = \dfrac{2\sqrt{5c}}{\sin\alpha}$ m/s. [2]

ii) The length of the pool is $4\sqrt{3}$ times the height of the ceiling. Find the steepest angle at which Queenie can successfully kick the ball to the other side. [3]

*(End of Examination.)*

# 2023 Extension 1 Trial Solutions.

## Section A - Multiple Choice

1. For $x=-2$, $y'=0$, either A or B.
at $(0,0)$ $y'<0$, not A.
$\therefore$ B: $y'=-x-2$ (B)

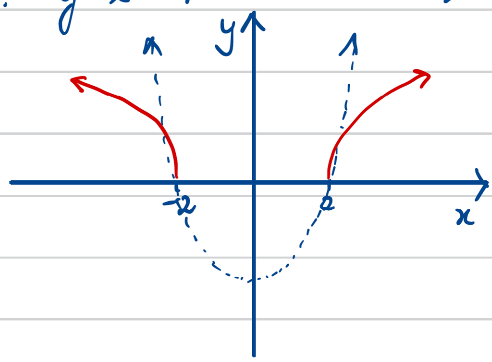
2. $\int \frac{3}{\sqrt{9-4x^2}}dx$

$= \frac{3}{2}\int \frac{2\times 1}{\sqrt{9-(2x)^2}}dx$ (C)

$= \frac{3}{2}\sin^{-1}\left(\frac{2x}{3}\right)+C$

3. $f(1)=0$ : all of them
$f'(1)=0$ : $f'(x)=2x-5$, $3x^2-5$
$4x^3-5$, $5x^4-5$ (D)
only $5(1)^4-5=0$. $\therefore$ D

4. $y=x^2-4 = (x-2)(x+2)$




gradient of $\sqrt{x^2-4}$ at roots: (D)

$\frac{d}{dx}\sqrt{x^2-4} = \frac{2x}{2\sqrt{x^2-4}}$

undefined (vertical) at $x=2$.

5. $x=4+3\cos\theta \rightarrow \frac{x-4}{3}=\cos\theta$
$y=2+3\sin\theta \rightarrow \frac{y-2}{3}=\sin\theta$ (A)

$\therefore \left(\frac{x-4}{3}\right)^2+\left(\frac{y-2}{3}\right)^2=1$ centre $(4,2)$
$(x-4)^2+(y-2)^2=9$ radius 3

6. B, D are undefined for $x>1$

Test 2nd quadrant angles in A or C: ⓒ

$x=\frac{2\pi}{3}$, $\sin^{-1}(\sin\frac{2\pi}{3}) = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$ ✗

$\cos^{-1}(\cos\frac{2\pi}{3}) = \cos^{-1}(\frac{-1}{2}) = \frac{2\pi}{3}$ ✓

Range of $\cos^{-1}$ makes it work.

7. Using ref sheet: $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

Put $A=3x$ $B=5x$ ⓒ

$\therefore 2\sin 3x \sin 5x = \cos(3x-5x) - \cos(3x+5x)$

$= \cos(-2x) - \cos 8x$

$= \cos 2x - \cos 8x$

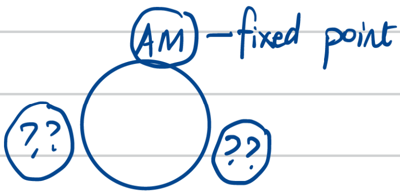
8. $\underset{\sim}{f_1} = F_1 \cos 35° \underset{\sim}{i} + F_1 \sin 35° \underset{\sim}{j}$

$\underset{\sim}{f_2} = -F_2 \cos 60° \underset{\sim}{i} - F_2 \sin 60° \underset{\sim}{j}$ Ⓐ

$= -\frac{F_2}{2}\underset{\sim}{i} - \frac{\sqrt{3}F_2}{2}\underset{\sim}{j}$

$\underset{\sim}{f_3} = \underset{\sim}{f_1} + \underset{\sim}{f_2} = \left(F_1 \cos 35° - \frac{F_2}{2}\right)\underset{\sim}{i} + \left(F_1 \sin 35° - \frac{\sqrt{3}F_2}{2}\right)\underset{\sim}{j}$

9. 3 groups: AM BF CN Ⓐ




$2! \times 2^3$

$2!$ ↑ for positions of groups

$2^3$ ↑ for arrangement within each group.

10. On $y = f^{-1}(x)$ where $x = 5$:

corresponds to point on $y = f(x)$ where $y = 5$, ie $(2,5)$

$\therefore \quad y = f^{-1}(5) = 2$ because $5 = f(2)$

Must use $(5,2)$ on the inverse

Now: $y = f(x)$ becomes $x = f(y)$

$$\frac{dx}{dy} = f'(y)$$

B

For the inverse: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)}$

sub $y=2$: $m = \frac{1}{f'(2)}$

$= \frac{1}{-2}$

Optionally: you can directly take reciprocal of tangent gradient on $y = f(x)$ at $(2,5)$: ie $\frac{1}{f'(2)}$.

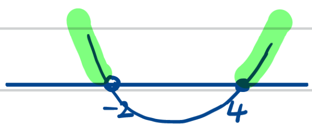
# Section B - Written Solutions

Q11. a) $\dfrac{3x}{x+2} > 2 \quad , x \neq -2$

M1: $3x(x+2) \geqslant 2(x+2)^2$

$(x+2)[3x - 2(x+2)] \geqslant 0$

$(x+2)(x-4) \geqslant 0$




$\therefore \ x < -2$ or $x \geqslant 4$

M2: Critical Values:

$x \neq -2 \ ,\qquad 3x = 2(x+2)$

$3x - 2x = 4$

$x = 4$

$\therefore \ x < -2$ or $x \geqslant 4$


b) $P(x) = x^3 + ax + b$

$P(2) = 0 \Rightarrow 8 + 2a + b = 0 \quad \ldots ①$

$P(-1) = -15 \Rightarrow -1 - a + b = -15 \quad \ldots ②$

$① - ②: \quad 9 + 3a = 15$

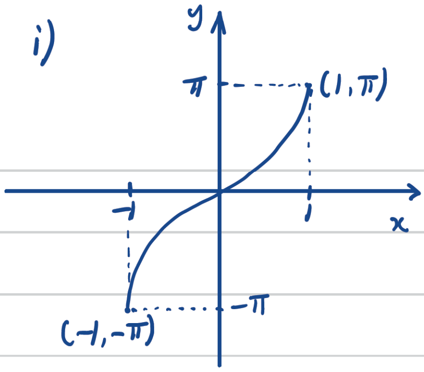
$3a = 6$

$a = 2$

into ①: $8 + 4 + b = 0$

$b = -12$

Q11 c) i)




ii) $V = \pi \int_a^b x^2 \, dy$

$y = 2\sin^{-1} x$

$\frac{y}{2} = \sin^{-1} x$

$x = \sin\left(\frac{y}{2}\right)$

$= \pi \int_0^{\frac{\pi}{2}} \sin^2\left(\frac{y}{2}\right) dy$

$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos y) \, dy$

$= \frac{\pi}{2}\left[y - \sin y\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - \frac{\pi}{2}(0 - 0)$

$= \frac{\pi^2}{4} - \frac{\pi}{2}$ units$^3$.

d) i) $RHS = 1 - \frac{1}{1+u^2}$

$= \frac{1 + u^2 - 1}{1+u^2}$

$= \frac{u^2}{1+u^2}$

$= LHS$

ii) $\int \frac{\sqrt{x}}{1+x} dx$

$x = u^2$

$dx = 2u \, du$

$= \int \frac{\sqrt{u^2}}{1+u^2} \cdot 2u \, du$

$\sqrt{u^2} = u$, since $u \geq 0$

$= 2\int \frac{u^2}{1+u^2} du$

$= 2\int 1 - \frac{1}{1+u^2} du$

$= 2(u - \tan^{-1} u) + C$

$u = \sqrt{x}$, since $u \geq 0$

$= 2(\sqrt{x} - \tan^{-1}\sqrt{x}) + C$

Q12. a) $y = e^{-5x}$

$y' = -5e^{-5x}$

$y'' = 25e^{-5x}$

$\therefore \quad e^{-5x} + k \times 25e^{-5x} = 0$

$e^{-5x}(1 + 25k) = 0$

since $e^{-5x} \neq 0$, $\quad 1 + 25k = 0$

$k = \frac{-1}{25}$.

b) $(2+x^2)(3-2x)^5$

$= (2+x^2)(3^5 + {}^5C_1 3^4(-2x) + {}^5C_2 3^3(-2x)^2 + \dots)$

not displayed since power $>2$.

coeff of $x^2$: $\quad 2 \times {}^5C_2 \times 3^3 \times (-2)^2 + 3^5$

$= 2160 + 243$

$= 2403$

c) Collinear means $\vec{BC} = \lambda \vec{BA}$ for some real $\lambda$.

or $\lambda \vec{AC}$

$\vec{BA} = \vec{OA} - \vec{OB}$ $\qquad \vec{BC} = \vec{OC} - \vec{OB}$

$= \underset{\sim}{a} - \underset{\sim}{b}$ $\qquad = k\underset{\sim}{a} - 2\underset{\sim}{b} - \underset{\sim}{b}$

$= k\underset{\sim}{a} - 3\underset{\sim}{b}$

$\therefore \quad k\underset{\sim}{a} - 3\underset{\sim}{b} = \lambda(\underset{\sim}{a} - \underset{\sim}{b})$

Coeff of $\underset{\sim}{b}$: $\quad -3 = -\lambda$

$\lambda = 3$

$\therefore \quad k\underset{\sim}{a} = 3\underset{\sim}{a}$

$k = 3$

$\vec{AC} = k\underset{\sim}{a} - 2\underset{\sim}{b} - \underset{\sim}{a}$

$= (k-1)\underset{\sim}{a} - 2b = \lambda(\underset{\sim}{a} - b)$

$\therefore \quad \lambda = 2$

$\therefore \quad k - 1 = 2$

$k = 3$

Q12 d) $\frac{dy}{dx} = x \sec y$

$= \frac{x}{\cos y}$

$\int \cos y \, dy = \int x \, dx$

$\sin y = \frac{x^2}{2} + c$

$x=0, \; y = \frac{\pi}{6}$

$\therefore \; \frac{1}{2} = c$

$\sin y = \frac{x^2}{2} + \frac{1}{2}$

$y = \sin^{-1}\left(\frac{x^2}{2} + \frac{1}{2}\right)$

e) i) E P S I L O N

$P \underbrace{\_ \; \_ \; \_ \; \_ \; \_}_{\text{5 letters to arrange.}} S \qquad 5! = 120$

ii) $\boxed{E I O}$ P S L N $\qquad 5! \times 3! = 720$

iii) Equally as common as L to the left of N:

$\frac{7!}{2} = 2520$

f) There are ${}^{10}C_3 = 120$ possible packs, ie categories/holes.

Worst case is she gets each pack twice : $120 \times 2$

So $120 \times 2 + 1$ is minimum packs to guarantee the same pack appears 3 times.

$120 \times 2 + 1 = 241$.

Q13 a) i) $\tan 2x + \tan x = \dfrac{\sin 2x}{\cos 2x} + \dfrac{\sin x}{\cos x}$

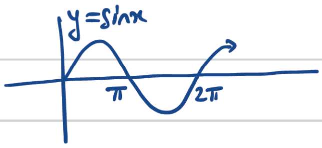
$$= \frac{\sin 2x \cos x + \sin x \cos 2x}{\cos 2x \cos x}$$

$$= \frac{\sin(2x + x)}{\cos 2x \cos x}$$

$$= \frac{\sin 3x}{\cos 2x \cos x}$$

ii) $\therefore \dfrac{\sin 3x}{\cos 2x \cos x} = 0$

$\therefore \sin 3x = 0, \quad x \neq \frac{\pi}{2}, \frac{3\pi}{2} \ldots$ and $2x \neq \frac{\pi}{2}, \frac{3\pi}{2} \ldots$




$3x = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$

$x = 0, \frac{\pi}{3}$ for $0 \leq x \leq \frac{\pi}{2}$

b) i) $V = \frac{1}{3}\pi r^2 h$ , $\frac{r}{h} = \tan\frac{\pi}{6}$

$$= \frac{1}{\sqrt{3}}$$

$$r = \frac{h}{\sqrt{3}}$$

$\therefore V = \frac{1}{3}\pi \cdot \frac{h^2}{3} h$

$$= \frac{\pi h^3}{9}$$

ii) $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$ $\quad\quad \dfrac{dV}{dh} = \dfrac{3\pi h^2}{9} = \dfrac{\pi h^2}{3}$

$\dfrac{dV}{dt} = \dfrac{\pi h^2}{3} \times \dfrac{dh}{dt}$

$\dfrac{dV}{dt} = -0.5$ , $\dfrac{dh}{dt} = -0.05$

$-0.5 = \dfrac{\pi h^2}{3} \times -0.05$

$10 = \dfrac{\pi h^2}{3}$

$h^2 = \dfrac{30}{\pi}$

$h = \sqrt{\dfrac{30}{\pi}} \approx 3.09$ cm

Q13 c) At least 2 women:

Case 1: 2 women 1 man: $^4C_2 \times {}^6C_1 = 36$

Case 2: 3 women: $^4C_3 = 4$

Total = 40 ways.

OR: $^{10}C_3 - \left( {}^6C_3 + {}^6C_2 \, {}^4C_1 \right) = 40$

d) $$\frac{dN}{dt} = \frac{N(10-N)}{200}$$

$$\frac{dt}{dN} = \frac{200}{N(10-N)}$$

$$t = \frac{200}{10} \int \frac{1 \times 10}{N(10-N)} \, dN$$

$$= 20 \int \frac{1}{N} + \frac{-1}{10-N} \, dN$$

$$= 20 \left( \ln|N| - \ln|10-N| \right) + c$$

$$\frac{t-c}{20} = \ln\left|\frac{N}{10-N}\right|$$

$$\left|\frac{N}{10-N}\right| = e^{\frac{t-c}{20}}$$

$$= Ae^{\frac{t}{20}}, \quad A = e^{\frac{-c}{20}}$$

$$\therefore \frac{N}{10-N} = Be^{\frac{t}{20}}, \quad B = \pm e^{\frac{-c}{20}}$$

$t = 0$, $N = 15$:

$$\frac{15}{10-15} = Be^0$$

$$\frac{15}{-5} = B$$

$$B = -3$$

$$\therefore \frac{N}{10-N} = -3e^{\frac{t}{20}}$$

$t = 6$: $$\frac{N}{10-N} = -3e^{\frac{6}{20}}$$

$$N = -30e^{0.3} + 3Ne^{0.3}$$

$$N(1-3e^{0.3}) = -30e^{0.3}$$

$$N = \frac{-30e^{0.3}}{1-3e^{0.3}}$$

$$= 13.279\ldots$$

$$\approx 13 \text{ million}$$

Q14 a) $3\sin 2x - 5\cos 2x = 5$,

Let $t = \tan x$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$$3\left(\frac{2t}{1+t^2}\right) - 5\left(\frac{1-t^2}{1+t^2}\right) = 5$$

$$\frac{6t - 5 + 5t^2}{1+t^2} = 5$$

$$6t - 5 + 5t^2 = 5 + 5t^2$$

$$6t = 10$$

$$t = \frac{5}{3}$$

$$\tan x = \frac{5}{3}$$

$$x = \tan^{-1}\frac{5}{3} \text{ or } \pi - \tan^{-1}\frac{5}{3}$$

Also testing: $x = \frac{\pi}{2}$: LHS $= 3\sin\pi - 5\cos\pi$

$= 5$ ✓

$x = \frac{3\pi}{2}$: LHS $= 3\sin 3\pi - 5\cos 3\pi$

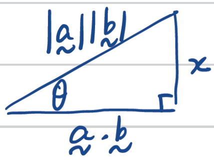
$= 5$ ✓

$\therefore$ 4 solutions: $x = \tan^{-1}\frac{5}{3}, \pi - \tan^{-1}\frac{5}{3}, \frac{\pi}{2}, \frac{3\pi}{2}$.

b) $A = \frac{1}{2}ab\sin C$

$= \frac{1}{2}|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$

But $\cos\theta = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$




$x = \sqrt{|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2 - (\underset{\sim}{a}\cdot\underset{\sim}{b})^2}$

$= \sqrt{\underset{\sim}{a}\cdot\underset{\sim}{a} \times \underset{\sim}{b}\cdot\underset{\sim}{b} - (\underset{\sim}{a}\cdot\underset{\sim}{b})^2}$

$\therefore \frac{1}{2}|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta = \frac{1}{2}\cancel{|\underset{\sim}{a}||\underset{\sim}{b}|}\cdot\dfrac{\sqrt{\cdots}}{\cancel{|\underset{\sim}{a}||\underset{\sim}{b}|}}$

$= \frac{1}{2}\sqrt{(\underset{\sim}{a}\cdot\underset{\sim}{a})\times(\underset{\sim}{b}\cdot\underset{\sim}{b}) - (\underset{\sim}{a}\cdot\underset{\sim}{b})^2}$ as required

c) STEP 1: Test for $n=1$,

$LHS = \frac{1}{1}$ $\quad RHS = \frac{2\times 1}{1+1} = 1$

$\therefore LHS = RHS$. True for $n=1$.

STEP 2: Assume that for $n=k$:

$$\frac{1}{1}+\frac{1}{1+2}+\dots+\frac{1}{1+2+\dots+k} = \frac{2k}{k+1}.$$

STEP 3: RTP for $n=k+1$:

$$\frac{1}{1}+\frac{1}{1+2}+\dots+\frac{1}{1+2+\dots+k}+\frac{1}{1+2+\dots+k+1} = \frac{2(k+1)}{k+1+1}$$

$LHS = \frac{2k}{k+1} + \frac{1}{1+2+\dots+k+1}$ by assumption

$= \frac{2k}{k+1} + \frac{1}{\frac{k+1}{2}(2+(k+1-1))}$ $\quad \left[\begin{array}{l} a=1 \quad d=1 \quad AP \\ n=k+1 \end{array}\right]$

$= \frac{2k}{k+1} + \frac{2}{(k+1)(2+k)}$ $\rightarrow$ $= \frac{2}{k+1}\left(k+\frac{1}{2+k}\right)$

$= \frac{2k(2+k)+2}{(k+1)(2+k)}$ $\quad = \frac{2}{k+1}\left(\frac{2k+k^2+1}{2+k}\right)$

$\quad = \frac{2}{k+1}\cdot\frac{(k+1)^2}{2+k}$

$= \frac{4k+2k^2+2}{(k+1)(2+k)}$

$\quad = \frac{2(k+1)}{2+k}$

$= \frac{2(k+1)^2}{(k+1)(2+k)}$

$\quad = RHS$

$= \frac{2(k+1)}{(2+k)}$

$= RHS$

$\therefore$ State true by mathematical induction.

Q14 d) i) Highest point: $\dot{y}=0$,

$$-10t + u\sin\alpha = 0$$

$$t = \frac{u\sin\alpha}{10}$$

at this time, $y=c$:

$$c = -5\left(\frac{u\sin\alpha}{10}\right)^2 + u\sin\alpha\left(\frac{u\sin\alpha}{10}\right)$$

$$= -\frac{u^2\sin^2\alpha}{20} + \frac{u^2\sin^2\alpha}{10}$$

$$= \frac{u^2\sin^2\alpha}{20}$$

$$\frac{20c}{\sin^2\alpha} = u^2$$

$$u = \sqrt{\frac{20c}{\sin^2\alpha}} = \frac{2\sqrt{5c}}{\sin\alpha}$$

ii) To land: $y=0$:

$$-5t^2 + ut\sin\alpha = 0$$

$$ut\sin\alpha = 5t^2$$

$$u\sin\alpha = 5t$$

$$t = \frac{u\sin\alpha}{5}$$

horizontal range $= u\cos\alpha \cdot \frac{u\sin\alpha}{5}$

$$= \frac{u^2\sin\alpha\cos\alpha}{5}$$

Using $u = \frac{2\sqrt{5c}}{\sin\alpha}$ to maximise range for any given $\alpha$:

$$\text{range} = \frac{\sin\alpha\cos\alpha}{5} \times \frac{20c}{\sin^2\alpha}$$

$$= \frac{4c\cos\alpha}{\sin\alpha}$$

$$= \frac{4c}{\tan\alpha}$$

we want range $> 4\sqrt{3}c$

$$\therefore \frac{4c}{\tan\alpha} > 4\sqrt{3}c$$

$$\frac{1}{\sqrt{3}} > \tan\alpha, \quad 0<\alpha<\frac{\pi}{2}$$

$$\therefore \alpha < 30^\circ.$$

The steepest angle is $30^\circ$.