Teacher: Mandile / Underwood (Circle)

Student Number:

or Student Name: ______________________

Danebank

An Anglican School for Girls

Mathematics Extension 1

Year 12 Examination Block 2023

Task 4 - Trial HSC Examination

General Instructions	
Write using black or blue pen Graphs can be drawn in pencil No liquid paper or correction tape are to be used.	
Reading Time	10 minutes
Working time	2 hours

Structure of the Paper		
Section 1	/10 Marks	Allow about 15 minutes for this section
Section 2	/ 60 Marks	Allow about 1 hour and 45 minutes for this section
Total	**/70 Marks**	**Weighting: 30 %**

Special Instructions
NESA approved Calculator may be used. A reference sheet is provided at the back of this paper. A separate multiple choice answer sheet is provided. Attempt all questions. This examination must not be removed from the Examination Room. In Questions 11-14, show relevant mathematical reasoning and/or calculations. Answer each question in a separate writing booklet.

Section I

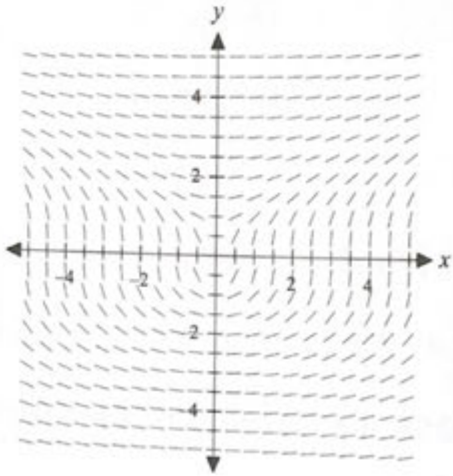
10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

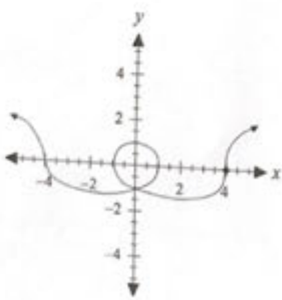
Use the multiple-choice answer sheet for Questions 1 – 10.

1 The graph of the direction field of a differential is shown below.



Which of the following best represents the particular solution which passes through the point $(4, 0)$?

A



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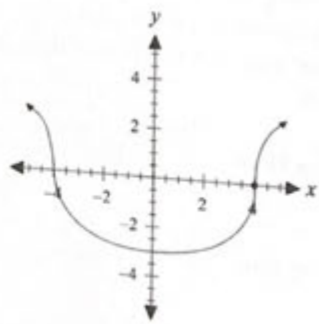
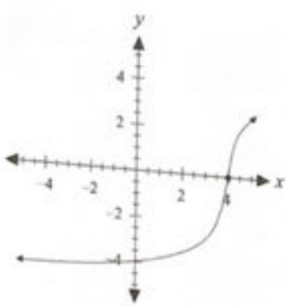
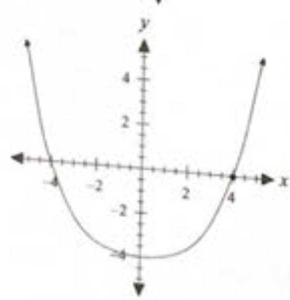


C



D



2 What is the domain and range of $y = 4\cos^{-1}\frac{3x}{2}$?

A Domain: $-\frac{2}{3} \leq x \leq \frac{2}{3}$
Range: $-4\pi \leq y \leq 4\pi$.

B Domain: $-\frac{3}{2} \leq x \leq \frac{3}{2}$
Range: $-4\pi \leq y \leq 4\pi$.

C Domain: $-\frac{2}{3} \leq x \leq \frac{2}{3}$
Range: $0 \leq y \leq 4\pi$.

D Domain: $-\frac{2}{3} \leq x \leq \frac{2}{3}$
Range: $0 \leq y \leq 4$.

3 Consider the equation $3x^3 + 2x^2 - x + 5 = 0$.
What is the sum of the reciprocals of the roots of this equation?

A $\frac{5}{2}$

B $\frac{1}{5}$

C $\frac{2}{5}$

D $\frac{5}{3}$

4 What does the integral simplify to?

$$\int_0^{\frac{\pi}{7}} \sin 3x \sin 4x \, dx$$

A $2\sin\frac{\pi}{7}$

B $2\sin\frac{\pi}{7} - \frac{1}{7}$

C $\frac{1}{2}\sin\frac{\pi}{7}$

D $\frac{1}{2}\sin\frac{\pi}{7} - \frac{1}{7}$

5 How many distinct ways are there to arrange the letters in the word "kangaroo" with the two "o" letters next to each other?

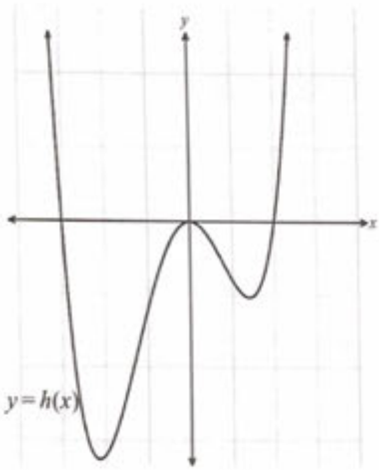
A $\frac{7!}{2!2!}$

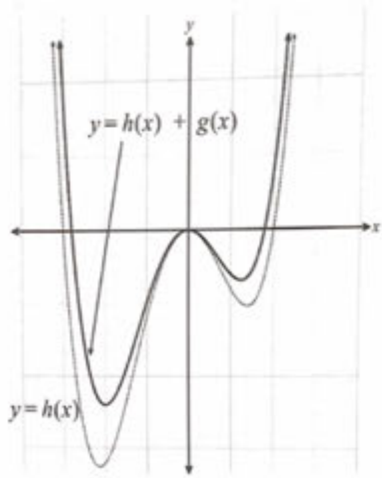
B $\frac{7!}{2!}$

C $\frac{8!}{2!2!}$

D $\frac{7!}{4!}$

6 Below, the graph on the left shows $y = h(x)$ and that on the right shows $y = h(x)$ and also $y = h(x) + g(x)$.





Which equation could describe $y = g(x)$?

A $y = x$

B $y = x^2$

C $y = x^2 + 6$

D $y = x^3$

7 A curve is defined parametrically as $f(t) = \begin{cases} x(t) = 2\cos t \\ y(t) = 2\sin t \end{cases}$, for $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

What is the gradient of the tangent to the curve at the point $(-2, 0)$?

A -1
B 0
C 1
D The gradient is undefined at this point.

A particle is moving in a straight line such that its displacement (x metres) from the origin after t seconds is given by $x = -3\cos^2 t$.

Which of the following best describes the motion of the particle when $t = \frac{5\pi}{6}$

A The particle is moving to the right with increasing speed.

B The particle is moving to the right with decreasing speed.

C The particle is moving to the left with increasing speed.

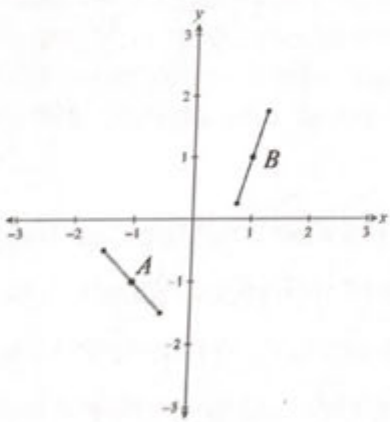
D The particle is moving to the left with decreasing speed.

A direction field is to be drawn for the differential equation:

$$\frac{dy}{dx} = 2x + \frac{y}{x}.$$

Which of the graphs below shows the correct slope lines at the points $A\,(-1,-1)$ and $(1, 1)$?

A



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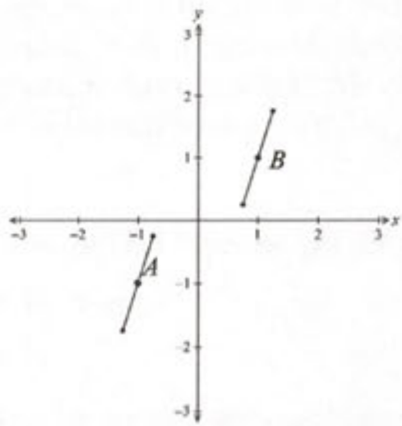

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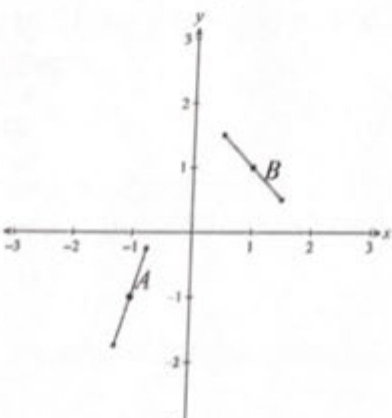
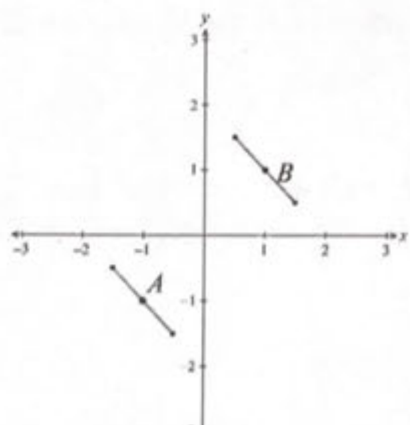


D



10 Let θ be the angle between unit vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

If $|\underset{\sim}{a} - \underset{\sim}{b}| < 1$, what values may θ take?

You may use the formula $|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b}$

A $\quad 0 \le \theta \le \frac{\pi}{2}$

B $\quad 0 \le \theta \le \frac{\pi}{3}$

C $\quad 0 \le \theta \le \frac{\pi}{4}$

D $\quad 0 \le \theta \le \frac{\pi}{6}$

Section II

60 marks
Attempt Questions 11 – 14.
Allow about 1 hour and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a **SEPARATE** writing booklet.

(a) Find the exact value of $\cos^{-1}\left(\sin\frac{4\pi}{3}\right)$. 2

(b) A Mathematics department consists of 12 teachers. Nine of the teachers wear glasses and three of the teachers do not wear glasses. Five of these teachers go out to dinner together. In how many ways will there be more teachers who wear glasses than teachers who do not wear glasses in the group who go out for dinner? 2

(c) Harry sets up a playlist which has ten tracks on the media player on his phone. 2

Three of the tracks are by G Flip and two are by D'Arcy.

He sets the ten tracks to play in a random order.

What is the probability that the three tracks by G Flip are played together and the two by D'Arcy are also played together? (The five tracks do not need to be together)

(d) If the polynomials $P(x) = 2x^3 + mx^2 + x - 3$ and $Q(x) = x^2 + 6x - 5$ have the same remainder when divided by $x + 1$, find the value of m. 2

(e) (i) Starting with the left-hand side, show that: 1

$$\cos x + \cos 2x + \cos 3x = \cos 2x + 2\cos 2x \cos x$$

(ii) Hence solve the equation $\cos x + \cos 2x + \cos 3x = 0, \quad 0 \le x \le 2\pi$. 3

(f) Find the equation of the tangent to $y = 4\tan^{-1}\frac{x}{2}$ at $(-2, -\pi)$ 3

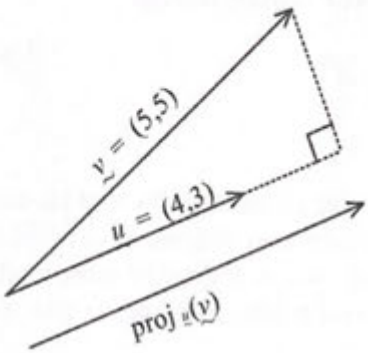
End of Question 11

Question 12 (15 marks) Use a **SEPARATE** writing booklet.

(a) Evaluate the definite integral using the substitution $u = log_e x$ 3

$$\int_e^{e^2} \frac{2}{x(log_e x)^2}\,dx$$

(b) The diagram shows the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ and the projection of $\underset{\sim}{v}$ onto $\underset{\sim}{u}$ denoted by $\text{proj}_{\underset{\sim}{u}}(\underset{\sim}{v})$. 3



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Determine the ordered pair for the projection vector, using the formula:

$$\text{proj}_{\underset{\sim}{u}}(\underset{\sim}{v}) = \frac{\underset{\sim}{u}.\underset{\sim}{v}}{|\underset{\sim}{u}|^2}\underset{\sim}{u}.$$

(c) Express $9\sin x + 40\cos x$ in the form $A\sin(x + \alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$, and hence or otherwise solve $9\sin x + 40\cos x = 6$ for $0 \le x \le \pi$. 3
(x and α in radians, correct to three significant figures).

(d) By considering the coefficient of the x^6 term in the expansion of $(1 + x)^3(1 + x)^6$ show that: 3

$$\binom{3}{0}\binom{6}{6} + \binom{3}{1}\binom{6}{5} + \binom{3}{2}\binom{6}{4} + \binom{3}{3}\binom{6}{3} = \binom{9}{6}.$$

(e) (i) Write $\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$ in terms of $\tan\theta$. 2

(ii) Hence, sketch the graph $y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$ on the domain $[-\pi, \pi]$, showing any x-intercepts and asymptotes. 1

End of Question 12

Question 13 (15 marks) Use a **SEPARATE** writing booklet.

(a) Use the principle of mathematical induction to show that for all integers $n \geq 1$, 3

$$1 + 5 + 25 + \ldots\ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)$$

(b) Use the substitution $u^2 = x + 1$ to find the volume of the solid formed by rotating the area bounded by the curve $y = \frac{x-1}{\sqrt{x+1}}$, the x axis and the lines $x = 3$ and $x = 8$ about the x axis. Express your answer in exact form. 4

(c) Find in the form $y = f(x)$ the solution of the differential equation $\frac{dy}{dx} = \frac{2e^{-\frac{1}{2}y}}{\cos^2 x}$ given that $y = \ln 3$ when $x = \frac{\pi}{3}$. 3

(d) The polynomial $P(x) = 2x^3 + mx^2 + nx - 20$ has a double root at $x = 2$. Find the values of m and n and find the other root of $P(x)$. 3

(e) At a film festival there are nine feature films. On the opening night there are three sessions, at 6 pm, 8 pm and 10 pm. Every feature film is shown at each of the three sessions. 2

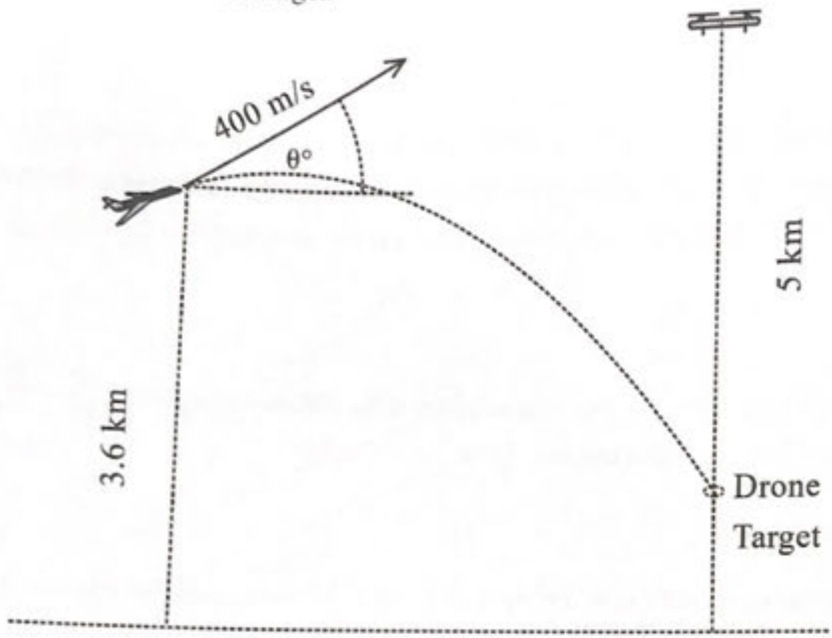
There are 200 film critics at the festival and each one attends three different films on the opening night and writes a review of each.

Use the pigeonhole principle to show that at least one combination of three films will be reviewed by at least three different critics.

End of Question 13

Question 14 (15 marks) Use a **SEPARATE** writing booklet.

(a) A drone, which is hovering at a height of 5 km, releases a target object which falls under gravity. At the same time, a jet, which is at a height of 3.6 km and is 5km west of the drone, fires a projectile at a speed of 400 m/s at an angle of θ^{o} to the horizontal toward the target.



Using a point on the ground directly below the jet as the origin, the positions of the projectile and target at time t seconds after the projectile is launched are as follows:

Projectile

$$\underset{\sim}{p}(t) = \begin{bmatrix} 400t\cos\theta \\ 3600 + 400t\sin\theta - 5t^2 \end{bmatrix}$$

Drone Target

$$\underset{\sim}{d}(t) = \begin{bmatrix} 5000 \\ 5000 - 5t^2 \end{bmatrix}$$

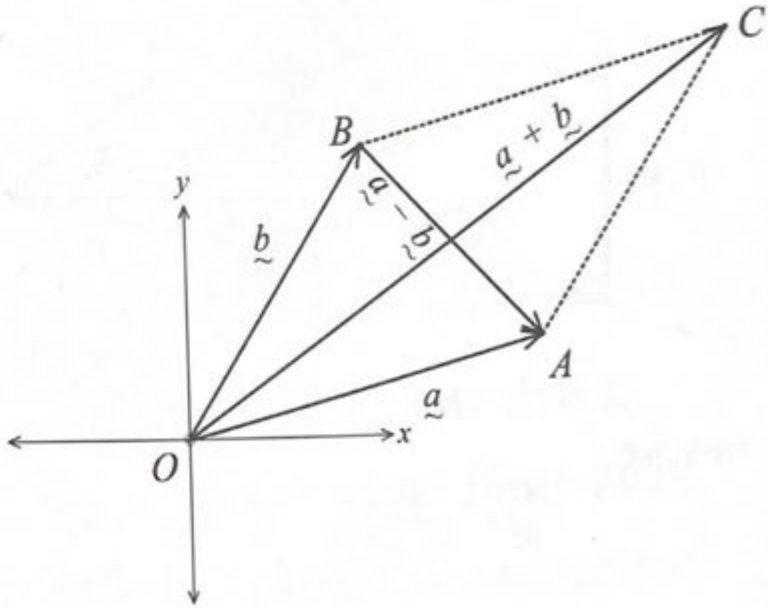
(i) Calculate the size of angle θ, if the projectile is to hit the target. 2

(ii) Determine how many seconds after the projectile is fired that it hits the target, the height of the target at that time, and the speed at which the target it is falling when it is struck by the projectile. 3
Answer to the nearest whole number.

b) (i) For a vector $\underset{\sim}{a} = (x_1, y_1)$ show that $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$. **1**

(ii) The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ form two adjacent sides of a parallelogram. **1**

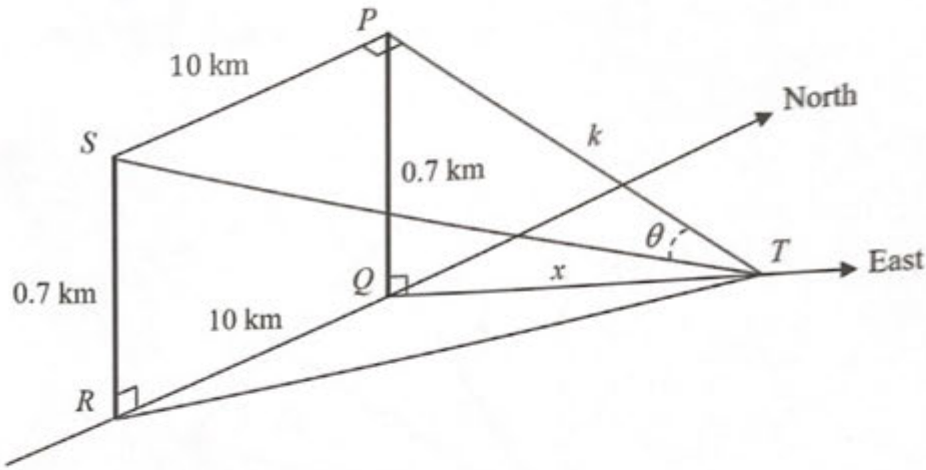
The diagonals of the parallelogram are the vectors $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$, as shown.



Show that for the parallelogram, $(\underset{\sim}{a} + \underset{\sim}{b}).(\underset{\sim}{a} - \underset{\sim}{b}) = |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$

(iii) Hence show that if $OBCA$ is a Rhombus then the diagonals are perpendicular. **2**

PQ and SR are two towers of height 0.7 km. Q is 10 km due North of R. A vehicle at T is travelling due East away from Q at a constant speed of 10 km/h. Let the distance QT be x and the distance PT be k.



(i) Show that: 2

$$\frac{dk}{dt} = \frac{10x}{\sqrt{x^2 + 0.49}}$$

(ii) By first finding an expression for k in terms of θ, show that: 1

$$\frac{dk}{d\theta} = -10 \operatorname{cosec}^2\theta$$

(iii) Find the exact rate at which θ is changing when the vehicle is 2.4 km from Q. 3

End of Examination