NESA Number: ______________________ **Name:** ______________________________

Teacher: **MA** **MN** **RC** **LD**

Ascham School

VI·ET·ANIMO

Mathematics Extension 1

Trial HSC Examination

Monday 28th July 2025

2 hours

General Instructions

- Reading time – 10 minutes.
- Working time – 2 hours.
- Write using black non-erasable pen.
- NESA-approved calculators may be used.
- A Reference Sheet is provided.
- In Questions 11–14, show relevant mathematical reasoning and/or calculations.

Total marks – 70

Section I Pages 3 – 7

10 marks

- Use the Multiple-Choice Answer Sheet provided to answer Q1-10.
- Allow about 15 minutes for this section.

Section II Pages 8 – 13

60 marks

- Answer Questions 11-14.
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher's initials.
- Allow about 1 hour and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which is the cartesian equation representing the following parametric equations?

$$x = 3t - 1$$
$$y = \frac{1}{t} \qquad \text{where } t \neq 0$$

(A) $y = \dfrac{3}{x} + 1$

(B) $y = \dfrac{3}{x} - 1$

(C) $y = \dfrac{3}{x+1}$

(D) $y = \dfrac{x+1}{3}$

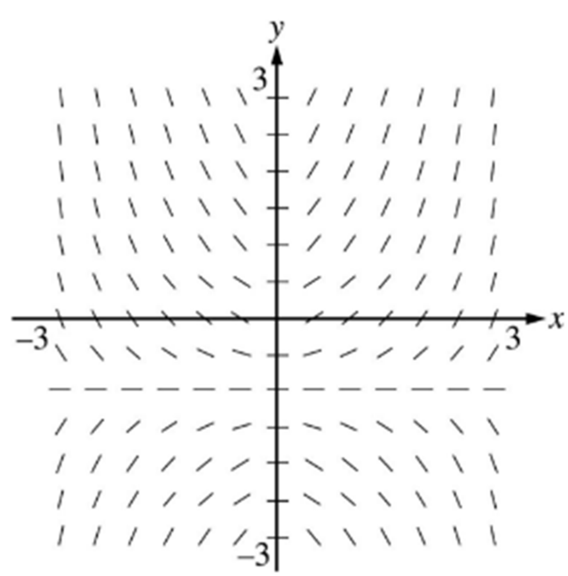
2 Which differential equation represents the slope field graphed?

(A) $\dfrac{dy}{dx} = (x+1)^2$

(B) $\dfrac{dy}{dx} = y + 1$

(C) $\dfrac{dy}{dx} = xy + y$

(D) $\dfrac{dy}{dx} = xy + x$



3 The figure shows a trapezium $ABCD$, where <u>AD is parallel to BC</u>.

$\overrightarrow{BD} = 5\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{DC} = \underset{\sim}{i} - 10\underset{\sim}{j}$ and $\overrightarrow{AD} = 4\underset{\sim}{i} + k\underset{\sim}{j}$, where k is an integer.

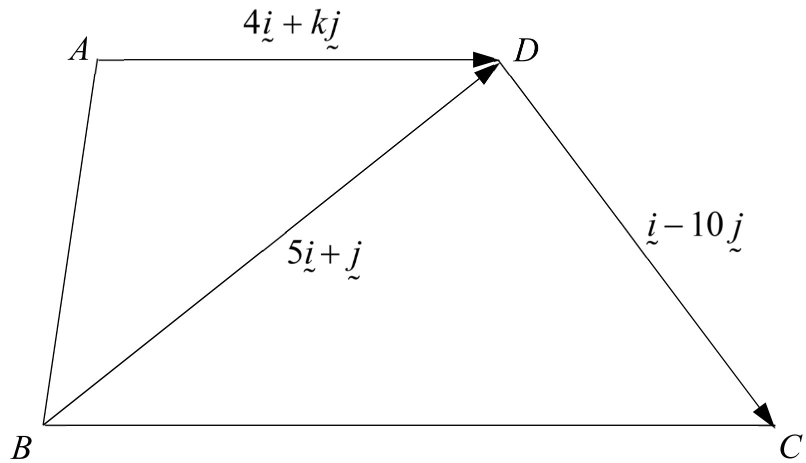
What is the correct value of k?

(A) $k = -9$

(B) $k = -6$

(C) $k = 6$

(D) $k = 9$



NOT TO SCALE

4 Which of the following expressions is equivalent to $2\sqrt{3}\cos x - 2\sin x$?

(A) $4\cos\left(x + \frac{\pi}{6}\right)$

(B) $4\sqrt{3}\cos\left(x + \frac{\pi}{6}\right)$

(C) $4\cos\left(x + \frac{\pi}{3}\right)$

(D) $4\sqrt{3}\cos\left(x + \frac{\pi}{3}\right)$

5 Two adults and six children are seated at a circular dining table.
In how many ways can they be arranged so that the adults are not seated next to each other?

(A) 2880

(B) 3600

(C) 4320

(D) 25 200

6 An anti-derivative of $\dfrac{1}{x^2-4x+13}$ is:

(A) $\tan^{-1}\left(\dfrac{x-2}{3}\right)$

(B) $\tan^{-1}\left(\dfrac{x-3}{2}\right)$

(C) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x-3}{2}\right)$

(D) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x-2}{3}\right)$

7 Suppose $x^3-4x^2+a \equiv (x+1)Q(x)-2$, where $Q(x)$ is a polynomial.
The correct value of a is:

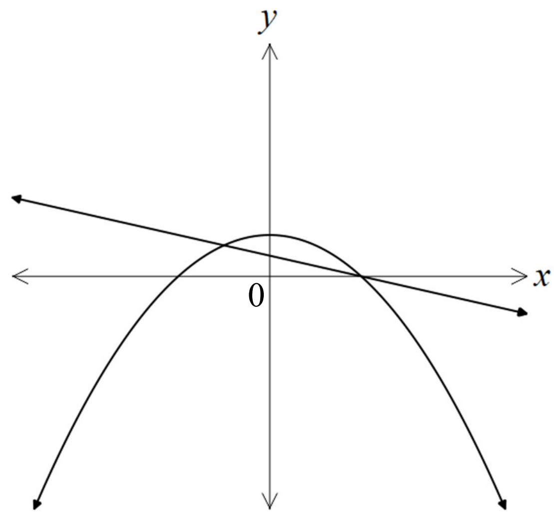
(A) -3

(B) 1

(C) 3

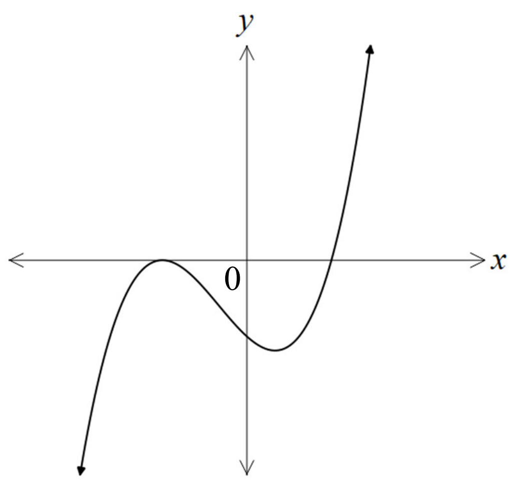
(D) 5

8 The graphs of $y = f(x)$ and $y = g(x)$ are shown below.

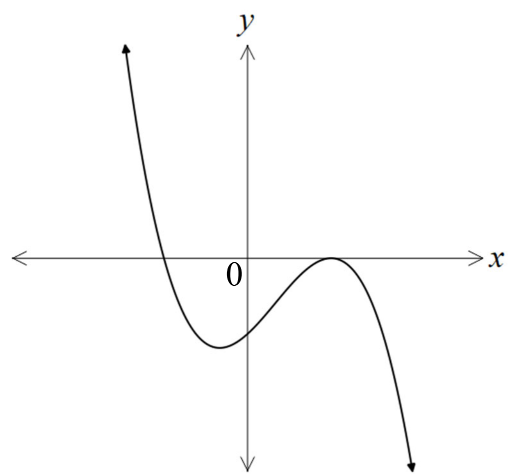


Which of the following graphs could present $y = f(x)g(x)$?

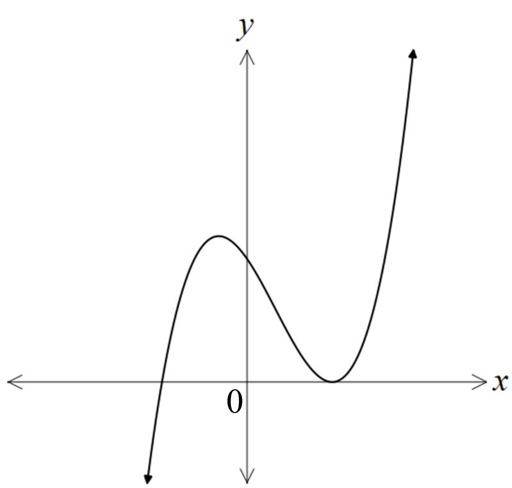
(A)



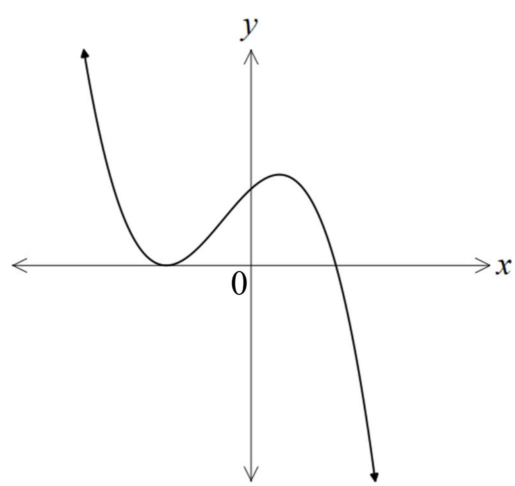
(B)



(C)



(D)



9 A bag contains 3 red marbles, 4 green marbles, 6 pink marbles and 7 black marbles. Marbles are drawn at random one at a time without replacement from the bag. What is the least number of marbles that need to be drawn to be certain that 6 marbles of the same colour are amongst those selected?

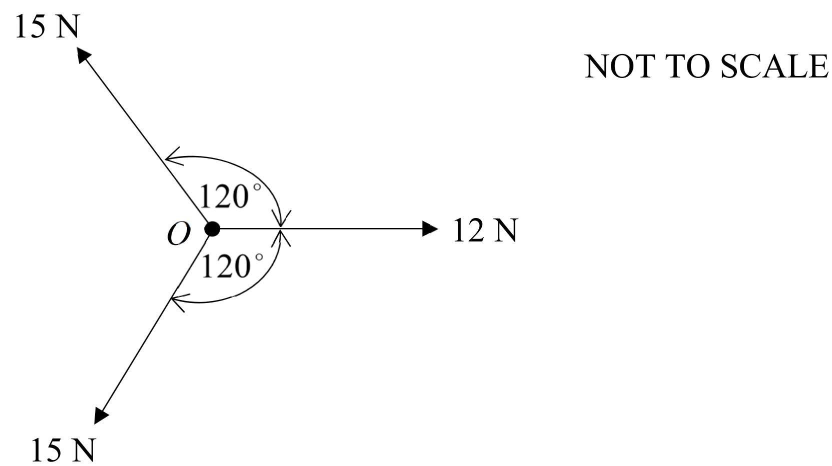
(A) 17

(B) 18

(C) 19

(D) 21

10 The figure below shows three forces which lie on the same plane, acting on a particle at O. The magnitude of these forces and their relative directions are shown.



What is the magnitude of the resultant of the three forces?

(A) 3 N

(B) 9 N

(C) 12 N

(D) 27 N

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In Questions 11–14, you should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find:

(i) $\int \frac{x}{\sqrt{1-x^4}}\,dx$. **2**

(ii) $\int 12 \sin 4x \sin 2x \; dx$. **2**

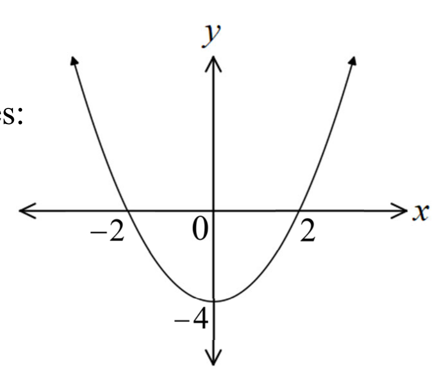
(b) Solve $\frac{1-2x}{x-5} \geq 1$. **3**

(c) (i) Write down the expansion of $(x-2)^5$. **2**

(ii) Hence or otherwise, find the coefficient of x in the expansion of $(2x+1)^2(x-2)^5$. **1**

(d) The polynomial $P(x) = 10x^3 + ax^2 - 20x + b$ has $x = 2$ as a root of multiplicity 2. Find the values of a and b. **2**

(e) Consider the graph of the function $f(x) = x^2 - 4$.
Sketch the following functions showing all key features:



(i) $y = |f(x)|$ **1**

(ii) $y = \frac{1}{|f(x)|}$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) 8 females and 3 males are members of a mixed netball training squad. However, only 7 players make up a starting team. How many ways can a starting team be formed if there must be at least 2 males playing? **2**

(b) Consider the function $y = 2\cos^{-1}(2x-1)$.

(i) Find its domain and range. **2**

(ii) Sketch the graph of $y = f(x)$. **1**

(iii) How many solutions are there for the equation $2\cos^{-1}(2x-1) = -2\pi x + 2\pi$? **1**

(c) Using the substitution $t = \tan\dfrac{x}{2}$, or otherwise, solve the equation:

$$5\sin x - 12\cos x = 12 , \qquad \text{for } 0^\circ \le x \le 360^\circ .$$

Give your answers correct to the nearest minute. **3**

(d) Consider the position vector $\overrightarrow{OP} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ and the line ***l*** which has equation $x - 2y = 0$.

(i) Find the projection of the vector $\overrightarrow{OP}$ onto the line ***l*****.** **2**

(ii) Hence find the exact perpendicular distance from point P to the line ***l*****.** **1**

(e) The gradient at any point (x, y) on a curve $y = f(x)$ is given as:

$$\frac{dy}{dx} = k\sqrt{y}, \quad \text{where } k \text{ is a constant.}$$

Find the equation of $y = f(x)$ if it passes through the points $P(4, 4)$ and $Q(6, 16)$. **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

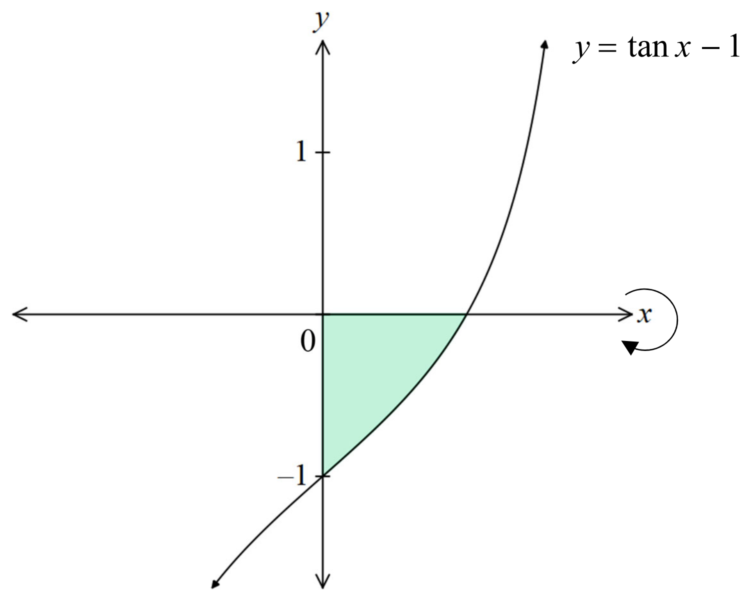
(a) A curve has equation $x = y\sqrt{2-y}$, where $y \le 2$. Find $\dfrac{dy}{dx}$ at the point $(-4, -2)$. **2**

(b) Prove by mathematical induction that for all integers $n \ge 1$:

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$ **3**

(c) The region bounded by the curve $y = \tan x - 1$, the x-axis and the y-axis is shaded below.

Find the exact volume of the solid formed if this region is rotated about the x-axis. **3**



(d) Liquid is pouring into a container at the constant rate of 12π cm^3/s . **3**
The container is initially empty and when the height of the liquid in the container is h cm, the volume of the liquid, V cm^3, is given by:

$$V = \pi h(h + 20).$$

Determine the rate at which the height of the liquid in the container is rising 8 seconds after the liquid started pouring in.

Question 13 continues on the following page.

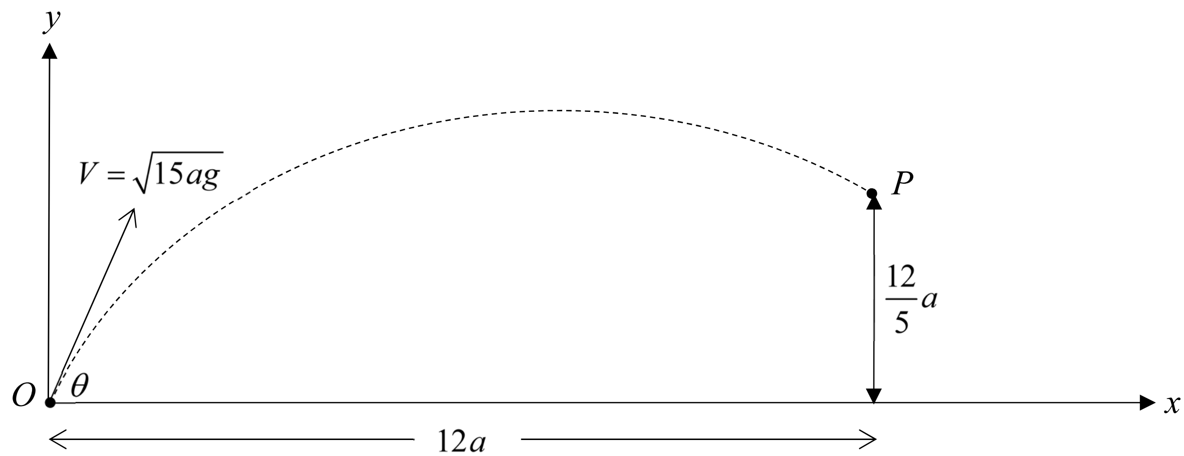
(e) A projectile is fired from a fixed point O at an angle of elevation θ.

It passes through the point P where $\overrightarrow{OP} = \begin{pmatrix} 12a \\ \frac{12}{5}a \end{pmatrix}$.

The speed of its projection is $V = \sqrt{15ag}$, where $a > 0$ and g m/s^2 is the acceleration due to gravity.

The position vector of the projectile is given below. It shows the horizontal and vertical displacements, x m and y m from O at time t seconds.

$$\underset{\sim}{s}(t) = \begin{pmatrix} Vt\cos\theta \\ Vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$



Find the possible values of θ correct to the nearest degree. **4**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^1 \sqrt{\frac{x}{1-x}}\, dx$ using the substitution $x = \sin^2\theta$. **4**

(Assume that $0 \leq \sin\theta \leq 1$).

(b) Solve the following equation for x:

$$\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{x+1}\right) = \frac{\pi}{4}$$ **3**

(c) The population P of a colony of birds, in thousands, varies according to the differential equation:

$$\frac{dP}{dt} = Pe^{-0.5t}$$

where $P > 0$, $t \geq 0$ and t is the time in years.

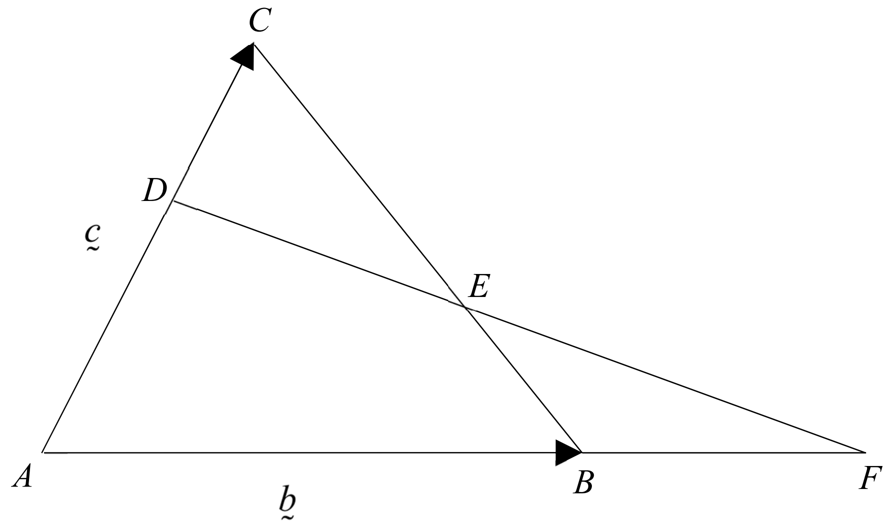
(i) Using separation of variables, solve the given differential equation to show that:

$$P = Ae^{-2e^{-0.5t}}$$ where A is a constant. **2**

(ii) P approaches its limiting value as $t \to \infty$.
Find the time when P will reach half its limiting value.
Give your answer in exact form. **2**

Question 14 continues on the following page.

(d)



In $\triangle ABC$ above, D is the point on AC such that $AD : DC = 2 : 1$ and E is the point on BC such that $BE : EC = 1 : 2$.

When DE is extended, it meets the extension of AB at F.

Let $\overrightarrow{AB} = \underset{\sim}{b}$ and $\overrightarrow{AC} = \underset{\sim}{c}$.

(i) Show that $\overrightarrow{DE} = \frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{c}$. **1**

(ii) By considering the collinearity of DEF and ABF, show that $AB : BF = 3 : 1$. **3**

End of Exam

Year 12 Ext. 1 Trial Exam 2025 SOLUTIONS

Section I

1 If $y = \frac{1}{t}$, then $t = \frac{1}{y}$

and if $x = 3t - 1$

then $x = 3\left(\frac{1}{y}\right) - 1$

$x + 1 = \frac{3}{y}$

$y = \frac{3}{x+1}$ $\therefore$ C

2 D

3 $\vec{BC} = \binom{5}{1} + \binom{1}{-10}$

$= \binom{6}{-9}$

If $\vec{BC} \parallel \vec{AD}$, then

$\binom{4}{k} = \lambda \binom{6}{-9}$

or gradients being parallel means

$\frac{k}{4} = \frac{-9}{6}$

$k = -\frac{9}{6} \times 4$

$\therefore k = -6$ $\therefore$ B

4
Let $2\sqrt{3}\cos x - 2\sin x = R\cos(x + \alpha)$

$2\sqrt{3}\cos x - 2\sin x = R[\cos x\cos\alpha - \sin x\sin\alpha]$

$\therefore$ Equating Co-efficients:

$R\cos\alpha = 2\sqrt{3}$ ... (1)

$R\sin\alpha = 2$ ... (2)

$(1)^2 + (2)^2 \Rightarrow R^2\cos^2\alpha + R^2\sin^2\alpha = (2\sqrt{3})^2 + 2^2$

$R^2(1) = 4 \times 3 + 4$

$R^2 = 16$

$\therefore R = 4$

$(2) \div (1)$: $\frac{R\sin\alpha}{R\cos\alpha} = \frac{2}{2\sqrt{3}}$

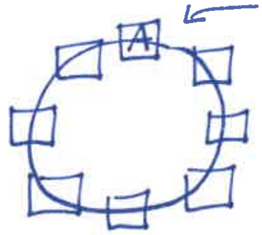
$\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \pi/6$

i.e. $2\sqrt{3}\cos x - 2\sin x = 4\cos(x + \pi/6)$

$\therefore$ A

5



Fix 1 adult

2nd adult can be seated in 5 spots

Then remaining 6 children can be arranged in 6! ways

$5 \times 6!$

$= 3600$ ways

$\therefore$ B

$\boxed{6}$

$$\frac{1}{x^2-4x+13}$$

$$= \frac{1}{x^2-4x+4+13-4}$$

$$= \frac{1}{(x-2)^2+9}$$

$$\therefore \int \frac{1}{x^2-4x+13}\,dx$$

$$= \int \frac{1}{(x-2)^2+3^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)$$

$\therefore$ D

$\boxed{7}$ Substitute $x=-1$ into the identity:

$$x^3-4x^2+a \equiv (x+1)Q(x)-2$$

i.e. $(-1)^3-4(-1)^2+a \equiv (-1+1)Q(-1)-2$

$$-1-4+a \equiv 0-2$$

$$-5+a \equiv -2$$

$$\therefore a = 3$$

$\therefore$ C

$\boxed{8}$ C

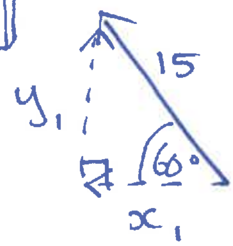
$\boxed{9}$ Least number of marbles to ensure 6 of same colour

$= 3$ red $+ 4$ green $+ 5$ pink $+ 5$ black $+ 1$ (pink or black)

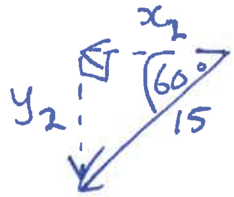
$= 18$

$\therefore$ B

$\boxed{10}$



$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -15\cos 60^\circ \\ 15\sin 60^\circ \end{pmatrix}$$



$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -15\cos 60^\circ \\ -15\sin 60^\circ \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1+x_2+x_3 \\ y_1+y_2+y_3 \end{pmatrix} = \begin{pmatrix} -15\cos 60^\circ - 15\cos 60^\circ + 12 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -30\cos 60^\circ + 12 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -30\left(\frac{1}{2}\right) + 12 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$\therefore$ Magnitude $= 3$

$\therefore$ A

NESA Number: ________________

Ascham School

VI·ET·ANIMO

Name: SOLUTIONS

Teacher: MA MN RC LD

SECTION I

Year 12 Extension 1 Mathematics
Multiple Choice Answer Sheet

2025 TRIAL EXAMINATION

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D
6	A	B	C	D
7	A	B	C	D
8	A	B	C	D
9	A	B	C	D
10	A	B	C	D

Section II

Question 11

a) i) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

$= \displaystyle\int \frac{x}{\sqrt{1-(x^2)^2}}\,dx$

$= \displaystyle\frac{1}{2}\int \frac{2x}{\sqrt{1-(x^2)^2}}\,dx$

$= \frac{1}{2}\sin^{-1}(x^2)+C$

(2 marks)

ii) $\displaystyle\int 12\sin 4x \sin 2x\,dx$

$= 12\displaystyle\int \frac{1}{2}\left[\cos(4x-2x)-\cos(4x+2x)\right]$

$= 6\displaystyle\int (\cos 2x - \cos 6x)\,dx$

$= 6\left[\frac{\sin 2x}{2} - \frac{\sin 6x}{6}\right]+C$

$= 3\sin 2x - \sin 6x + C$

(2 marks)

b) $\displaystyle\frac{1-2x}{x-5} \geq 1 \qquad x \neq 5$

$\displaystyle\frac{(1-2x)(x-5)^2}{x-5} \geq (x-5)^2$

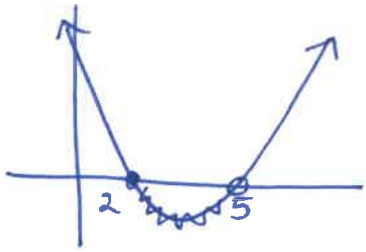
$(1-2x)(x-5)^2 \geq (x-5)^2$

$(x-5)^2-(1-2x)(x-5) \leq 0$

$(x-5)[x-5-(1-2x)] \leq 0$

$(x-5)(3x-6) \leq 0$

$3(x-5)(x-2) \leq 0$



$\therefore\ 2 \leq x < 5$

(3 marks)

c) i) $(x-2)^5 = \binom{5}{0}x^5(-2)^0+\binom{5}{1}x^4(-2)^1+\binom{5}{2}x^3(-2)^2+\binom{5}{3}x^2(-2)^3$
$+\binom{5}{4}x(-2)^4+\binom{5}{5}(-2)^5$

$= x^5-10x^4+40x^3-80x^2+80x-32$

(2 marks)

ii) $(2x+1)^2(x-2)^5$

$= (4x^2+4x+1)(x^5-10x^4+40x^3-80x^2+80x-32)$

Term in x is: $4x(-32)+1(80x)$

$= -48x \qquad \therefore$ coefficient is -48

(1 mark)

d) If $P(x)$ has $x=2$ as a root of multiplicity 2, then

$P(2)=0$ and $P'(2)=0$

$P(2) = 10(2)^3 + a(2)^2 - 20(2) + b$ $\left(\frac{1}{2}\right)$

$= 80 + 4a - 40 + b$

$\therefore P(2) = 4a + b + 40$

$P'(x) = 30x^2 + 2ax - 20$

$\therefore P'(2) = 30(2)^2 + 2a(2) - 20$ $\left(\frac{1}{2}\right)$

$= 120 + 4a - 20$

$\therefore P'(2) = 100 + 4a$

Now if $P(2)=0$, then

$4a + b + 40 = 0$

$4a + b = -40 \quad \ldots (1)$

And if $P'(2)=0$, then

$100 + 4a = 0 \quad \ldots (2)$

$\therefore \underline{a = -25}$ $\left(\frac{1}{2}\right)$

Sub. into (1):

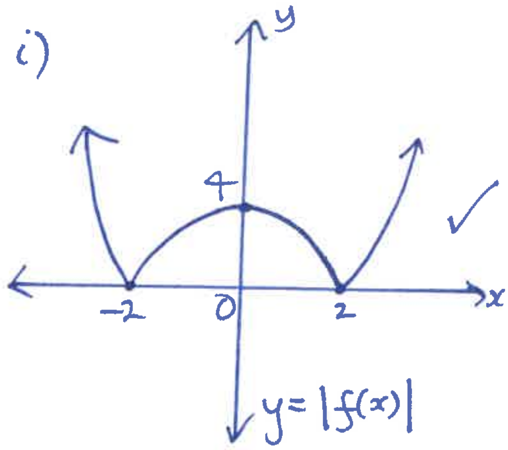
$4(-25) + b = -40$

$b = -40 + 100$

$\therefore \underline{b = 60}$ $\left(\frac{1}{2}\right)$

(2marks)

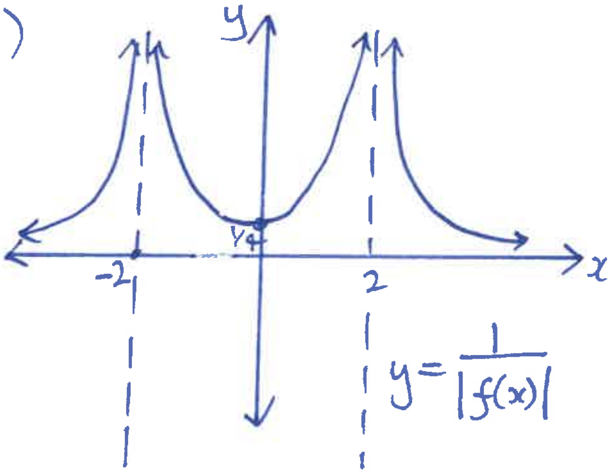
e) i)



✓

(1mark)

ii)



✓ shape (2marks)

✓ asymptotes/intercepts

Question 12

a) Either 2 males / 5 females

or 3 males / 4 females

No: ways $= {}^3C_2 \times {}^8C_5 + {}^3C_3 \times {}^8C_4$

$= 3 \times 56 + 1 \times 70$

$= 238$ ways

✓ (at least 1 pair)

✓ (correct answer)

(2 marks)

b) i) Domain: $-1 \leq 2x - 1 \leq 1$

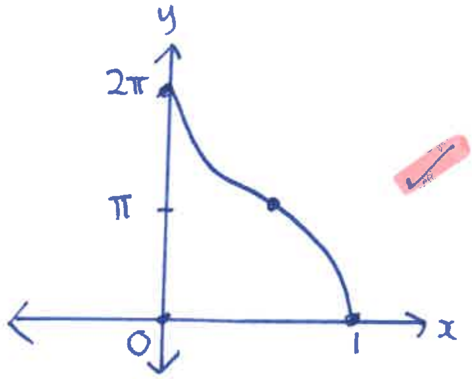
$0 \leq 2x \leq 2$

$\therefore \ 0 \leq x \leq 1$ ✓

Range: $0 \leq y \leq 2\pi$ ✓

(2 marks)

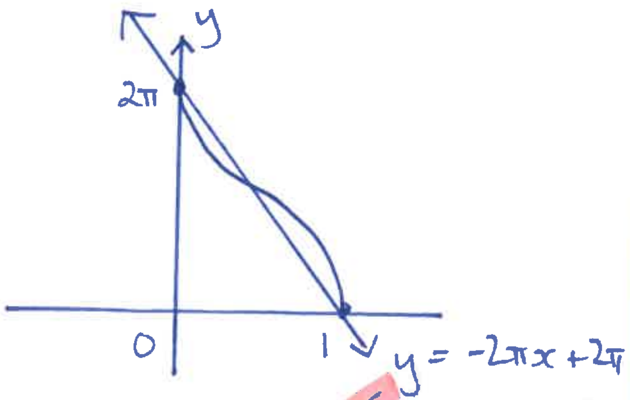
(ii)



✓

(1 mark)

(iii)



$\therefore$ 3 solutions ✓ (1 mark)

due to 3 points of intersection

c) $5\sin x - 12\cos x = 12$

$\Rightarrow 5\left(\frac{2t}{1+t^2}\right) - 12\left(\frac{1-t^2}{1+t^2}\right) = 12$

$\left[\text{when } t = \tan\frac{x}{2}\right]$

$10t - 12(1-t^2) = 12(1+t^2)$

$10t - 12 + 12t^2 = 12 + 12t^2$

$10t - 12 - 12 = 0$

$10t = 24$ ← linear

$\therefore$ test $x = 180°$

$\therefore \ t = 12/5$ ✓

$\therefore \ \tan\left(\frac{x}{2}\right) = 12/5$

where if $0° \leq x \leq 360°$

then $0° \leq x/2 \leq 180°$

$\therefore \ \frac{x}{2} = 67.3\ldots°$ (1st quad only)

$x = 67.3\ldots \times 2$

$x = 134°45'36.97''$

$\therefore \ x \doteqdot 134°46'$ ✓

(nearest minute)

Also need to test $x = 180°$.

$LHS = 5\sin(180°) - 12\cos(180°)$

$= 5(0) - 12(-1)$

$= 12$

$= RHS$

$\therefore \ x = 180°$ is also soln ✓

i.e. 2 solutions are

$x = 180°$ or $134°46'$

(3 marks)

d) The line $x - 2y = 0$

$$\Rightarrow \quad 2y = x$$
$$\Rightarrow \quad y = \tfrac{1}{2}x$$

has gradient $= \tfrac{1}{2}$ ($\tfrac{1}{2}$)

$\therefore$ vector parallel to this line is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

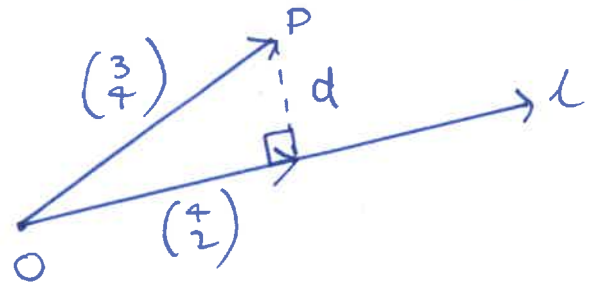
i) projection vector $= \dfrac{\begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$= \frac{6+4}{4+1}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$= 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$= \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ or $4\underset{\sim}{i} + 2\underset{\sim}{j}$ ($\tfrac{1}{2}$)

(2 marks)

ii)



$$\left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right| = \sqrt{3^2+4^2} = \sqrt{25} = 5$$

$$\left|\begin{pmatrix} 4 \\ 2 \end{pmatrix}\right| = \sqrt{4^2+2^2} = \sqrt{20}$$

$$d^2 + (\sqrt{20})^2 = 5^2$$
$$d^2 + 20 = 25$$
$$d^2 = 5$$

$\therefore d = \sqrt{5}$ units

(1 mark)

e) $\dfrac{dy}{dx} = k\sqrt{y}$

$$\frac{1}{\sqrt{y}}\,dy = k\,dx$$

$$\int y^{-1/2}\,dy = \int k\,dx$$

$$\frac{y^{1/2}}{1/2} = kx + c$$

$$\boxed{2\sqrt{y} = kx + c}$$

• when $x = 4$, $y = 4$

$\therefore 2\sqrt{4} = 4k + c$

$4k + c = 4 \quad \ldots (1)$

• when $x = 6$, $y = 16$

$\therefore 2\sqrt{16} = 6k + c$

$6k + c = 8 \quad \ldots (2)$

$(2) - (1)$: $2k = 4$

$\therefore k = 2$

$\Rightarrow (1)$: $4(2) + c = 4$

$\therefore c = -4$

$\therefore 2\sqrt{y} = 2x - 4$

$$\sqrt{y} = \frac{2(x-2)}{2}$$

$\sqrt{y} = x - 2$ $\quad x \geq 2$ $\tfrac{1}{2}$

$\therefore y = (x-2)^2$ $\quad \tfrac{1}{2}$ $\quad x \geq 2$

(3 marks)

Question 13

a) If $x = y(2-y)^{\frac{1}{2}}$

$$\frac{dx}{dy} = u'v + v'u$$

where $u = y$ $\quad v = (2-y)^{1/2}$

$\therefore u' = 1 \quad v' = \frac{1}{2}(2-y)^{-1/2} \times -1$

$$\therefore \frac{dx}{dy} = \sqrt{2-y} - \frac{y}{2\sqrt{2-y}}$$ ✓

when $y = -2$:

$$\frac{dx}{dy} = \sqrt{2-(-2)} - \frac{-2}{2\sqrt{2-(-2)}}$$

$$= \sqrt{4} + \frac{2}{2\sqrt{4}}$$

$$= 2 + \frac{1}{2}$$

$$= 5/2$$

$\therefore \frac{dy}{dx}$ at $y = -2$ is $\frac{2}{5}$ ✓

(2 marks)

b) R.T.P. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$ for $n \geq 1$ (where $n \in \mathbb{Z}$)

Prove true for $n=1$:

$$LHS = \frac{1}{2^1} = \frac{1}{2}$$

$$RHS = 2 - \frac{1+2}{2^1}$$

$$= 2 - \frac{3}{2}$$

$$= \frac{1}{2} = LHS \quad \therefore \text{ true for } n=1$$

($\frac{1}{2}$)

Assume true for $n=k$:

i.e. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$ ✱ ($\frac{1}{2}$)

Prove true for $n=k+1$:

R.T.P $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$

L.H.S. $= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$ [using ✱] ✓

$$= 2 - \left(\frac{k+2}{2^k} - \frac{k+1}{2^{k+1}}\right)$$

$$= 2 - \left[\frac{2(k+2) - (k+1)}{2^{k+1}}\right]$$

$$= 2 - \left(\frac{2k+4-k-1}{2^{k+1}}\right)$$

$$= 2 - \frac{k+3}{2^{k+1}}$$ ✓

$$= RHS$$

(3 marks)

$\therefore$ If true for $n=k$, then it is true for $n=k+1$. Since it is true for $n=1$, then it is true for $n=2$ and so on for all integers $n \geq 1$ by Mathematical Induction.

c) when $y=0$: $\tan x - 1 = 0$

$\tan x = 1$

$\therefore$ x-int. is $\pi/4$ ✓ ½

$$V = \pi \int_0^{\pi/4} (\tan x - 1)^2 \, dx$$ ✓ ½

$$V = \pi \int_0^{\pi/4} (\tan^2 x - 2\tan x + 1) \, dx$$

$$V = \pi \int_0^{\pi/4} \left(\sec^2 x - \frac{2 \sin x}{\cos x}\right) dx$$

$$V = \pi \Big[\tan x + 2 \ln|\cos x| \Big]_0^{\pi/4}$$ ✓

$$= \pi \left[\tan \pi/4 + 2\ln|\cos \pi/4| - (\tan 0 + 2\ln|\cos 0|) \right]$$

$$= \pi \left[1 + 2 \ln\left(\tfrac{1}{\sqrt{2}}\right) \right] \text{ units}^3$$ ✓

(3 marks)

or

$$= \pi[1 + 2 \ln 2^{-1/2}]$$

$$= \pi[1 + 2(-\tfrac{1}{2}) \ln 2]$$

$$= \pi[1 - \ln 2] \text{ units}^3$$

d) $\frac{dV}{dt} = 12\pi$

If $V = \pi h(h+20)$

$V = \pi h^2 + 20\pi h$

then $\frac{dV}{dh} = 2\pi h + 20\pi$

Now $\boxed{\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}}$

$\frac{dh}{dt} = \frac{1}{2\pi h + 20\pi} \times 12\pi$ ✓

i.e. $\frac{dh}{dt} = \frac{12\pi}{2\pi(h+10)} = \frac{6}{h+10}$

If the rate is 12π cm^3/s, then after 8 seconds, the volume increases by $8 \times 12\pi$ cm^3

i.e. $V = 96\pi$ cm^3

solve for h: $96\pi = \pi h(h+20)$ ✓ [using $V = \pi h(h+20)$]

$96 = h^2 + 20h$

$h^2 + 20h - 96 = 0$

$(h+24)(h-4) = 0$

$h = -24$ or $h = 4$

$\therefore$ $h = 4$ only $(h>0)$

Sub. into $\frac{dh}{dt}$: $\frac{dh}{dt} = \frac{6}{4+10}$

$= \frac{6}{14}$

$= \frac{3}{7}$ cm/s ✓

(3 marks)

e) * Equating horizontal components:

$$Vt\cos\theta = 12a$$

$$\therefore t = \frac{12a}{V\cos\theta} \quad \dots (1)$$

* Equating vertical components:

$$Vt\sin\theta - \frac{1}{2}gt^2 = \frac{12a}{5} \quad \dots (2)$$

sub. (1) → (2):

$$V\left(\frac{12a}{V\cos\theta}\right)\sin\theta - \frac{1}{2}g\left(\frac{12a}{V\cos\theta}\right)^2 = \frac{12a}{5}$$

✓

$$12a\tan\theta - \frac{1}{2}g\left(\frac{144a^2}{V^2\cos^2\theta}\right) = \frac{12a}{5}$$

$$12a\tan\theta - \frac{72g\,a^2\sec^2\theta}{V^2} = \frac{12a}{5}$$

* Sub. $V = \sqrt{15ag}$:

$$12a\tan\theta - \frac{72a^2g(1+\tan^2\theta)}{15ag} = \frac{12a}{5}$$

✓ $\div a$

$$12\tan\theta - \frac{24(1+\tan^2\theta)}{5} = \frac{12}{5}$$

$$60\tan\theta - 24 - 24\tan^2\theta = 12$$

$$24\tan^2\theta - 60\tan\theta + 36 = 0$$

$\div 12$

$$2\tan^2\theta - 5\tan\theta + 3 = 0$$

✓

* Let $u = \tan\theta$: $2u^2 - 5u + 3 = 0$

$$(2u-3)(u-1) = 0$$

$$\therefore u = 3/2 \quad \text{or} \quad u = 1$$

$$\tan\theta = 3/2 \quad \text{or} \quad \tan\theta = 1$$

$$\therefore \theta \doteqdot 56° \quad \text{or} \quad 45°$$

✓

(4 marks)

Question 14

a) If $x = (\sin\theta)^2$

$\begin{cases} \text{when } x = 0 \\ \theta = 0 \\ \text{when } x = 1 \\ \theta = \frac{\pi}{2} \end{cases}$

$$\frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\therefore dx = 2\sin\theta\cos\theta \, d\theta \qquad \checkmark \tfrac{1}{2}$$

$$\int_0^1 \sqrt{\frac{x}{1-x}} \, dx$$

$$= \int_0^{\pi/2} \left(\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \times 2\sin\theta\cos\theta \right) d\theta \qquad \checkmark \tfrac{1}{2}$$

$$= \int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta \right) d\theta$$

$$= \int_0^{\pi/2} 2\sin^2\theta \, d\theta \qquad \checkmark \tfrac{1}{2}$$

$$= \int_0^{\pi/2} 2 \times \tfrac{1}{2}(1-\cos 2\theta) \, d\theta \qquad \checkmark \tfrac{1}{2}$$

$$= \int_0^{\pi/2} (1-\cos 2\theta) \, d\theta$$

$$= \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \qquad \checkmark$$

$$= \frac{\pi}{2} - \frac{\sin(2 \times \pi/2)}{2} - \left(0 - \frac{\sin 0}{2}\right)$$

$$= \frac{\pi}{2} - \frac{0}{2} - 0 + 0$$

$$= \frac{\pi}{2} \qquad \checkmark$$

(4 marks)

b) $\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{x+1}\right) = \pi/4$

Let $A = \tan^{-1}\left(\frac{1}{x}\right)$ and $B = \tan^{-1}\left(\frac{1}{x+1}\right)$

then $\tan A = \frac{1}{x}$ and $\tan B = \frac{1}{x+1}$

$\therefore$ solving $A + B = \pi/4$

$\Rightarrow \tan(A+B) = \tan \pi/4$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\frac{\frac{1}{x} + \frac{1}{x+1}}{1 - \frac{1}{x} \times \frac{1}{x+1}} = 1$$

$$\frac{x+1+x}{x(x+1)} = 1 - \frac{1}{x(x+1)}$$

$$\frac{2x+1}{x(x+1)} = \frac{x(x+1)-1}{x(x+1)}$$

$$2x+1 = x^2+x-1$$

$$x^2 + x - 2x - 1 - 1 = 0$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$\therefore x = 2$ or $x = -1$ (½)

but $\underline{x=2}$ is $\underline{\text{only valid}}$ solution (½)

[since $\tan^{-1}\left(\frac{1}{-1+1}\right)$ is undefined] (3 marks)

c) i) $\frac{dP}{dt} = Pe^{-0.5t}$

$$\frac{1}{P}\,dP = e^{-0.5t}\,dt$$

$$\int \frac{1}{P}\,dP = \int e^{-0.5t}\,dt$$

$$\ln|P| = \frac{e^{-0.5t}}{-0.5} + c \quad \checkmark$$

$$\ln|P| = -2e^{-0.5t} + c$$

$$|P| = e^{-2e^{-0.5t} + c}$$

$$P = \pm e^{c} \times e^{-2e^{-0.5t}}$$

$$\therefore P = Ae^{-2e^{-0.5t}} \quad \checkmark$$

[where $A = \pm e^{c}$] (2 marks)

(ii) As $t \to \infty$

$$P \to Ae^{-2\times 0}$$

$$\therefore P \to A$$

Now find t when $P = \frac{1}{2}A$:

$$\frac{1}{2}A = Ae^{-2e^{-0.5t}} \quad \checkmark$$

$$0.5 = e^{-2e^{-0.5t}}$$

$$\ln(0.5) = -2e^{-0.5t}$$

$$\ln(2)^{-1} = -2e^{-0.5t}$$

$$\ln 2 = 2e^{-0.5t}$$

$$\ln\left[\frac{\ln 2}{2}\right] = -0.5t$$

$$t = \ln\left[\frac{\ln 2}{2}\right] \div -0.5$$

$$\therefore t = -2\ln\left[\frac{\ln 2}{2}\right] \quad \text{(ok. here)} \quad \checkmark$$

$\boxed{\text{or}}$ $t = -2\ln(\ln\sqrt{2})$ ✓

or $t = 2\ln\left[\frac{2}{\ln 2}\right]$ (2 marks)

d) i) Since $AD:DC = 2:1$

then $\overrightarrow{AD} = \frac{2}{3}\underset{\sim}{c}$

and $\overrightarrow{DC} = \frac{1}{3}\underset{\sim}{c}$

Since $BE:EC = 1:2$

then $\overrightarrow{BE} = \frac{1}{3}\overrightarrow{BC}$

and $\overrightarrow{EC} = \frac{2}{3}\overrightarrow{BC}$

Now $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= -\underset{\sim}{b} + \underset{\sim}{c}$

$\therefore \overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b}$ *

And $\overrightarrow{DE} = \overrightarrow{DC} + \overrightarrow{CE}$

$\overrightarrow{DE} = \overrightarrow{DC} - \overrightarrow{EC}$

$\overrightarrow{DE} = \overrightarrow{DC} - \frac{2}{3}\overrightarrow{BC}$ $\frac{1}{2}$

$\overrightarrow{DE} = \frac{1}{3}\underset{\sim}{c} - \frac{2}{3}(\underset{\sim}{c} - \underset{\sim}{b})$ using *

$\overrightarrow{DE} = \frac{1}{3}\underset{\sim}{c} - \frac{2}{3}\underset{\sim}{c} + \frac{2}{3}\underset{\sim}{b}$ $\frac{1}{2}$

$\therefore \overrightarrow{DE} = \frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{c}$ (as required)

(1 mark)

ii) Method (1)

$\overrightarrow{DF} = k\overrightarrow{DE}$

$= k\left(\frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{c}\right)$

$\overrightarrow{AF} = \lambda \overrightarrow{AB}$

$= \lambda \underset{\sim}{b}$

Now $\overrightarrow{DF} = \overrightarrow{DA} + \overrightarrow{AF}$

$k\left(\frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{c}\right) = -\frac{2}{3}\underset{\sim}{c} + \lambda\underset{\sim}{b}$

Equating coefficients:

$-\frac{k}{3} = -\frac{2}{3}$ (for vector $\underset{\sim}{c}$)

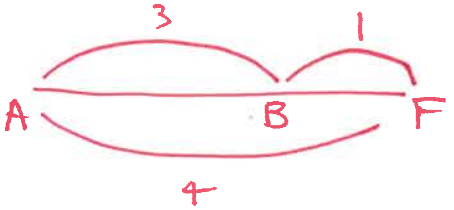
$\therefore k = 2$

and

$\frac{2}{3}k = \lambda$ (for vector $\underset{\sim}{b}$)

$\therefore \lambda = \frac{2}{3} \times 2$

$\lambda = \frac{4}{3}$



$\therefore AB : BF$

$= 3 : 1$

(as required) (3 marks)

OR

Method (2)

Since DEF is collinear and

$\overrightarrow{DE} = \frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{c}$

then $\overrightarrow{EF} = \lambda\left(\frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{c}\right)$

since ABF is collinear and $\overrightarrow{AB} = \underset{\sim}{b}$

then $\overrightarrow{BF} = k\underset{\sim}{b}$

Also, since $BE : EC = 1:2$,

then $\overrightarrow{BE} = \frac{1}{3}\overrightarrow{BC}$

i.e. $\overrightarrow{BE} = \frac{1}{3}(\underset{\sim}{c} - \underset{\sim}{b})$

Now $\overrightarrow{BF} = \overrightarrow{BE} + \overrightarrow{EF}$

$\overrightarrow{BF} = \frac{1}{3}(\underset{\sim}{c} - \underset{\sim}{b}) + \lambda\left(\frac{2}{3}\underset{\sim}{b} - \frac{1}{3}\underset{\sim}{c}\right)$

$= \frac{1}{3}\underset{\sim}{c} - \frac{1}{3}\underset{\sim}{b} + \frac{2\lambda}{3}\underset{\sim}{b} - \frac{\lambda}{3}\underset{\sim}{c}$

$= \left(\frac{1-\lambda}{3}\right)\underset{\sim}{c} + \left(\frac{2\lambda - 1}{3}\right)\underset{\sim}{b}$

but $\overrightarrow{BF} = k\underset{\sim}{b} + 0\underset{\sim}{c}$

$\therefore$ equating scalar coefficients:

$\frac{1-\lambda}{3} = 0$ (of $\underset{\sim}{c}$)

$\therefore \lambda = 1$

and $\frac{2\lambda - 1}{3} = k$ (of $\underset{\sim}{b}$)

$\frac{2(1) - 1}{3} = k$

$\therefore k = 1/3$

$\therefore \overrightarrow{BF} = \frac{1}{3}\underset{\sim}{b}$

$\therefore \overrightarrow{AB} : \overrightarrow{BF} = \underset{\sim}{b} : \frac{1}{3}\underset{\sim}{b}$

$= 3 : 1$ (3 marks)

(as required)