Student Number: | | | | | | | | | |

Class: _____

2026 **YEAR 12 TASK 4: TRIAL HSC**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your student number and Mathematics class at the top of this page

Total marks: 70

Section I – 10 marks (pages 3–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8–16)

- Attempt Questions 11–17
- Allow about 1 hour and 45 minutes for this section

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 A random variable X follows a binomial distribution $X \sim \text{Bin}(n, p)$. If $E(X) = 12$ and $\text{Var}(X) = 8$, which of the following is the value of n?

A. 18

B. 24

C. 30

D. 36

2 A box contains at least 20 blocks of each of the colours red, blue, green, and yellow. What is the minimum number of blocks that must be drawn from the box to guarantee that at least 4 of the blocks drawn are of the same colour?

A. 10

B. 12

C. 13

D. 16

3 Let $p(x) = 1 + x + x^2 + x^3 + \cdots + x^{12}$ where x is a real number.

Which of the following gives the coefficient of x^9 in the expansion of $p(x - 1)$?

A. -286

B. -174

C. 174

D. 286

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4 For $\frac{\pi}{2} \leq \theta \leq \pi$, which one of the following is equal to $\sin^{-1}(\sin\theta)$?

A. θ

B. $-\theta$

C. $\pi - \theta$

D. $\theta - \pi$

5 Define vectors

$$\underset{\sim}{u} = \begin{pmatrix} k+2 \\ -9 \end{pmatrix} \qquad \text{and} \qquad \underset{\sim}{v} = \begin{pmatrix} k-1 \\ 2 \end{pmatrix}$$

where k is a real constant. Given $\underset{\sim}{u} \perp \underset{\sim}{v}$, which of the following could be the value of k?

A. 5

B. -5

C. -4

D. None of the above.

6 The polynomial

$$P(x) = x^3 + x^2 - 5x + 3$$

has a factor $(x - \alpha)$ with multiplicity 2. Which one of the following is the value of α?

A. -1

B. $-\frac{5}{3}$

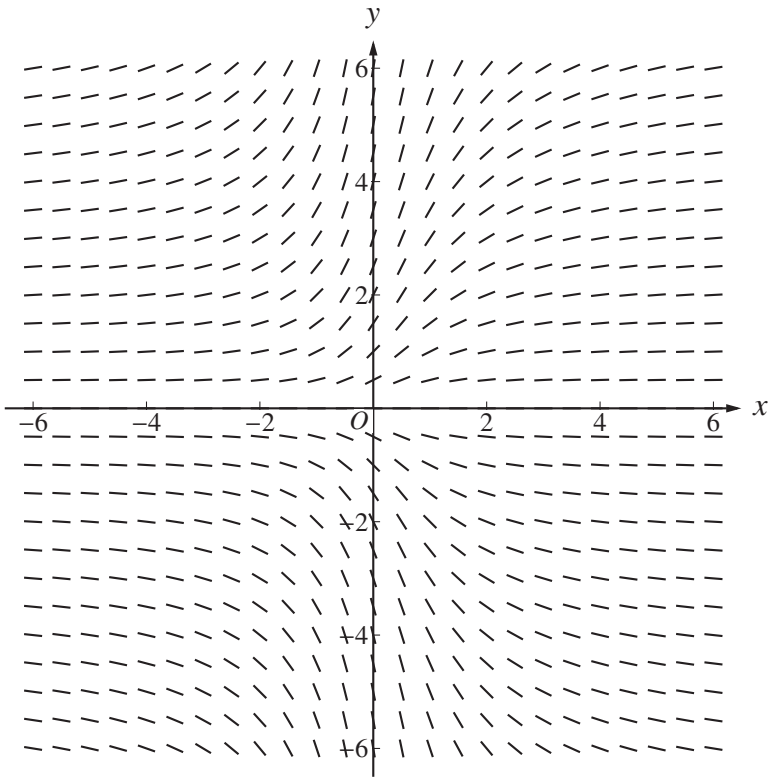
C. $\frac{5}{3}$

D. 1

Multiple choice continues on page 5

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7 Which one of the following differential equations could be represented by the slope field drawn below?



A. $\dfrac{dy}{dx} = \dfrac{y}{1+x^2}$

B. $\dfrac{dy}{dx} = \dfrac{y}{1+y^2}$

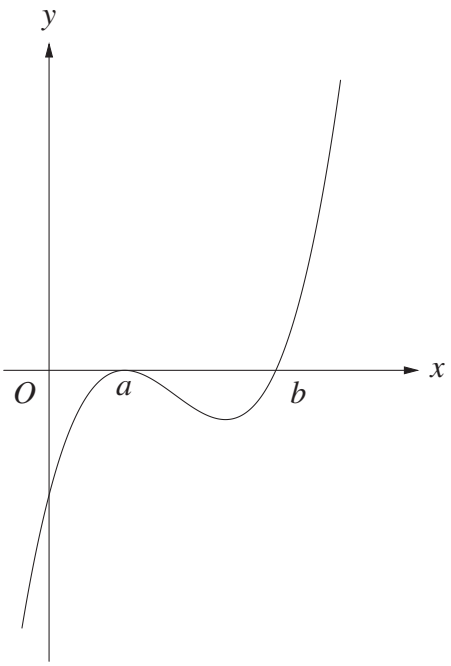
C. $\dfrac{dy}{dx} = \dfrac{x}{1+x^2}$

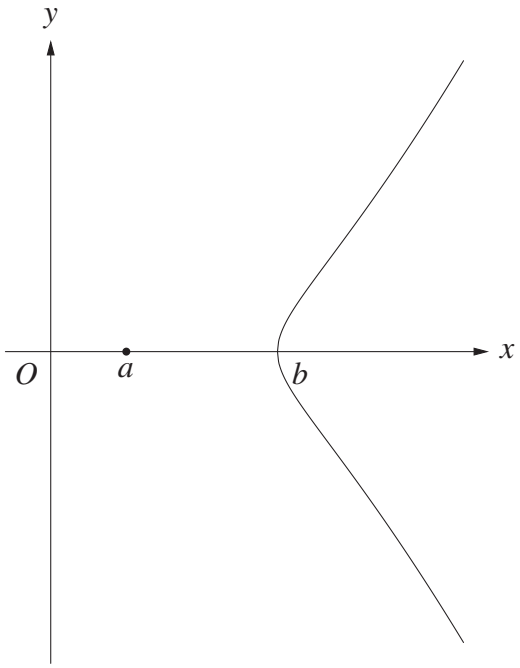
D. $\dfrac{dy}{dx} = \dfrac{x}{1+y^2}$

Multiple choice continues on page 6

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8 The graph $y = f(x)$ is shown on the left. A second graph, obtained from $y = f(x)$, is shown on the right.





Which of the following equations best represents the graph on the right?

A. $y^2 = |f(x)|$

B. $y = \sqrt{f(x)}$

C. $y = f(\sqrt{x})$

D. $y^2 = f(x)$

Multiple choice continues on page 7

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9 Given $0 < c \leq 1$, which of the following integrals is equal to 0?

A. $\displaystyle\int_{-c}^{c} x^2 \left(\cos^{-1}(x) - \frac{\pi}{2}\right) dx$

B. $\displaystyle\int_{-c}^{c} x \cos^{-1}(x)\, dx$

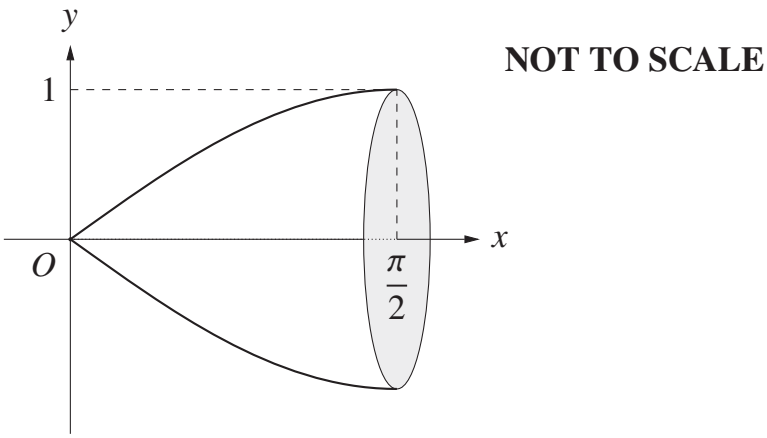
C. $\displaystyle\int_{-c}^{c} x^2 \cos^{-1}(x)\, dx$

D. $\displaystyle\int_{-c}^{c} x \left(\cos^{-1}(x) - \frac{\pi}{2}\right) dx$

10 The diagram shows the solid obtained by rotating, about the x-axis, the region between the curve

$$y = \sin x, \qquad 0 \leq x \leq \frac{\pi}{2},$$

and the x-axis. The volume of this solid is V units^3.



Let $a > 0$, $\omega > 0$ and $\delta \in \mathbb{R}$. A second solid is obtained by rotating, about the x-axis, the region between the curve

$$y = a \sin(\omega x - \delta), \qquad \frac{\delta}{\omega} \leq x \leq \frac{\delta + \pi/2}{\omega},$$

and the x-axis.

By considering the substitution $u = \omega x - \delta$, which of the following gives the volume of the second solid?

A. $\dfrac{a}{\omega} V$

B. $\dfrac{a^2}{\omega} V$

C. $a^2 \omega V$

D. $\dfrac{a^2}{\omega^2} V$

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Section II

60 marks
Attempt Questions 11–17
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (8 marks) Use the Question 11 Writing Booklet

(a) By first completing the square on the quadratic, evaluate **3**

$$\int_{-1}^{3} \frac{1}{\sqrt{7 + 2x - x^2}}\, dx.$$

(b) Differentiate **2**

$$y = \cos^{-1}\left(\frac{1}{\sqrt{1 + 4x^2}}\right), \qquad x > 0.$$

(c) In triangle ABC, $AB = 10\,\text{cm}$, $AC = 12\,\text{cm}$, and $\angle BAC$ is increasing at a rate of $0.1\ \text{rad s}^{-1}$. At what rate is the area of $\triangle ABC$ increasing when $\angle BAC = \dfrac{\pi}{3}$? **2**

(d) How many distinct arrangements of the letters of the word **K U G E L B L I T Z** have K and Z adjacent to each other? **1**

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Question 12 (8 marks) Use the Question 12 Writing Booklet

(a) Using the substitution $t = \tan\theta$, prove that **2**

$$\frac{2\sin 2\theta - \sin 4\theta}{2\sin 2\theta + \sin 4\theta} = \tan^2\theta$$

for all real θ satisfying $\sin 2\theta \cos\theta \neq 0$.

(b) Using the substitution $u = 2 - \sqrt{x}$, find **2**

$$\int \sqrt{2 - \sqrt{x}}\, dx.$$

(c) A rugby ball is modelled as the solid obtained when the region enclosed between $y = 0$ and the upper half of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{20} = 1$$

is rotated about the x-axis.

(i) By dilating the circle $x^2 + y^2 = 1$, sketch the graph of $\dfrac{x^2}{25} + \dfrac{y^2}{20} = 1$. **1**

(ii) Find the volume of the rugby ball, leaving your answer in simplified exact form. **2**

(iii) Find the maximum area of a cross-section of the rugby ball that is perpendicular to the x-axis, leaving your answer in simplified exact form. **1**

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Question 13 (8 marks) Use the Question 13 Writing Booklet

(a) Let m and r be non-negative integers such that $r < m$.

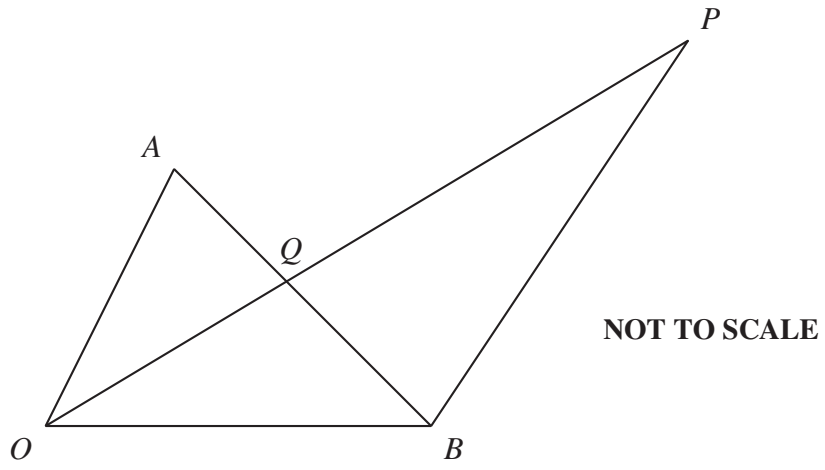
(i) Simplify the binomial expression **1**

$$\binom{m+1}{r+1} - \binom{m}{r+1}.$$

(ii) Hence simplify the following, expressing your answer as a difference of two binomial coefficients, where $n \geq 7$ is an integer: **1**

$$\binom{n-3}{3} + \binom{n-2}{3} + \binom{n-1}{3} + \binom{n}{3}.$$

(b) In the diagram, the position vectors of points A and B relative to O are $\underline{a}$ and $\underline{b}$, respectively. The points O, A, B are non-collinear. The line segments AB and OP intersect at Q.



NOT TO SCALE

We are given that

$$\overrightarrow{BP} = 2\left(\underline{a} + \underline{b}\right), \qquad \overrightarrow{AQ} = \lambda\overrightarrow{AB}, \qquad \overrightarrow{OQ} = \mu\overrightarrow{OP},$$

where λ and μ are constants.

(i) Express $\overrightarrow{OQ}$ in terms of λ, $\underline{a}$ and $\underline{b}$. **1**

(ii) Hence, or otherwise, find the values of λ and μ. **2**

Question 13 continues on page 11

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(c) Prove, by mathematical induction, that **3**

$$(1^2 \times 2) + (2^2 \times 3) + (3^2 \times 4) + \cdots + n^2(n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

for all integers $n \geq 1$.

End of Question 13

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Question 14 (8 marks) Use the Question 14 Writing Booklet

(a) For $-1 \leq x \leq 1$, show that **2**

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}(x).$$

(b) A hospital emergency department records patient arrivals over a day divided into $n = 86\,400$ one-second intervals. The number of patient arrivals in each interval is modelled as a Bernoulli random variable. These random variables are independent, and the probability of one arrival in any interval is $p = \dfrac{1}{710}$. **3**

Let $\hat{p}$ denote the proportion of intervals containing an arrival.

Using a normal approximation to the sampling distribution of $\hat{p}$, find the smallest integer m such that the probability that more than m patients arrive during the day is less than 5%.

Do not use a continuity correction. Interpolation is not required.

(c) A virus population $V(t)$ in a laboratory culture is modelled, while $V(t) > 0$, by

$$\frac{dV}{dt} = rV\left(1 - \frac{V}{k}\right) - c, \qquad V(0) > 0,$$

where r, k, c are all positive, $0 < c < \dfrac{rk}{4}$, and c is the constant removal rate due to antiviral treatment. If $V(t)$ reaches zero, the virus is considered eliminated and the model terminates.

Do not solve the differential equation.

(i) Find the equilibrium virus populations. **1**

(ii) Hence determine the range of initial virus populations for which the virus is eventually eliminated. Justify your answer. **2**

End of Question 14

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Question 15 (10 marks) Use the Question 15 Writing Booklet

(a) Solve the equation **3**

$$\sin 3x + \sin x = \sqrt{3}\,(\cos 3x + \cos x)$$

for $0 \leq x < \pi$. Leave your answers in exact form.

(b) Solve the differential equation **3**

$$x + 2y\sqrt{x^2 + 1}\,\frac{dy}{dx} = 0,$$

expressing y in terms of x, given that $y = -2$ when $x = 0$.

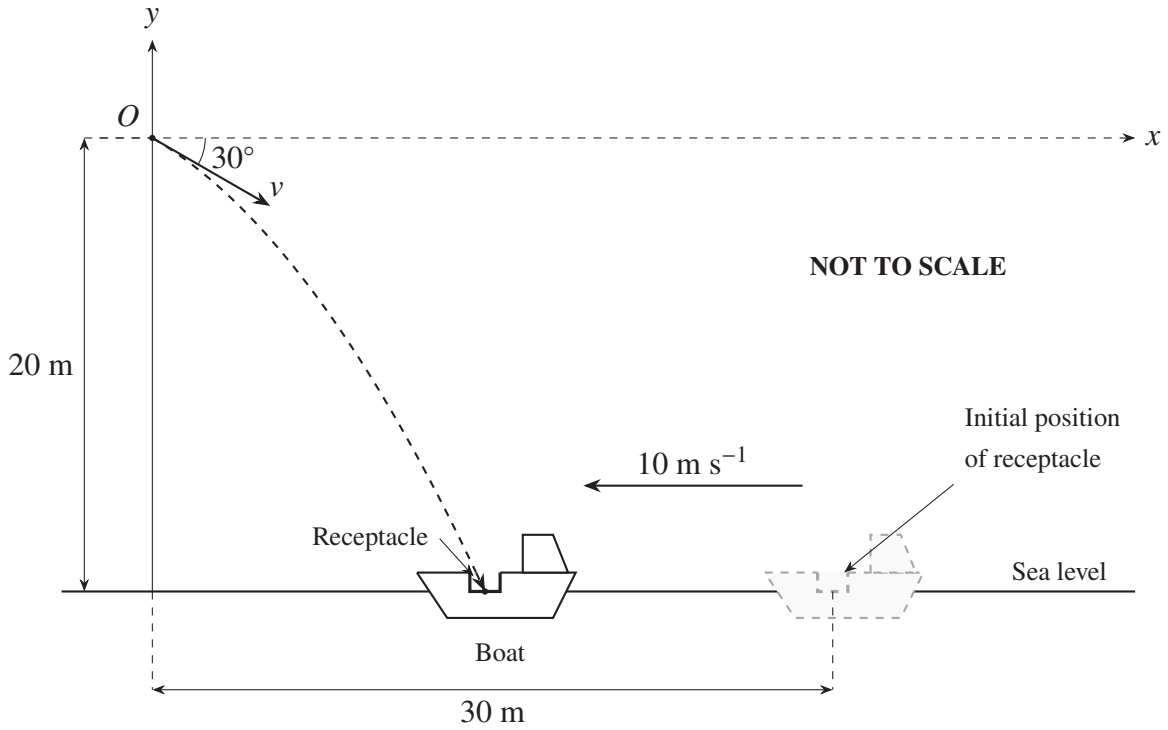
Question 15 continues on page 14

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(c) A canister is projected from a point O, 20 metres above horizontal sea level, with speed v m s^{-1} at an angle of 30° below the horizontal. Air resistance is neglected.

Take O as the origin, with the positive x-axis horizontal in the direction of projection and the positive y-axis vertically upwards. Time t is measured in seconds, and all position vectors are measured in metres. The acceleration of the canister is

$$\ddot{\underset{\sim}{r}}(t) = -9.8\underset{\sim}{j} \text{ m s}^{-2}.$$



(i) Show that the position of the canister at time t is **2**

$$\underset{\sim}{r}(t) = \frac{\sqrt{3}}{2}vt\,\underset{\sim}{i} - \left(\frac{1}{2}vt + 4.9t^2\right)\underset{\sim}{j}.$$

(ii) A receptacle fixed to a boat is modelled as a point at sea level. Initially, the receptacle is 30 metres horizontally from the point at sea level vertically below O. The boat moves horizontally towards this point at a constant speed of 10 m s^{-1}. **2**

Hence, at time $t \geq 0$, the position vector of the receptacle relative to O is

$$\underset{\sim}{r}_B(t) = (30 - 10t)\underset{\sim}{i} - 20\underset{\sim}{j}.$$

Determine, correct to the nearest integer, the speed v required for the canister to land in the receptacle.

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Question 16 (8 marks) Use the Question 16 Writing Booklet

(a) For the following, you may assume the identity

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta},$$

where $\theta \in \mathbb{R}$, $\cos\theta \neq 0$ and $1 - 3\tan^2\theta \neq 0$.

(i) Consider the cubic equation **2**

$$x^3 - \sqrt{3}x^2 - 3x + \frac{1}{\sqrt{3}} = 0.$$

By letting $x = \tan\theta$, solve the equation, leaving your answers in exact form.

(ii) Hence find the value of **1**

$$\tan^2\frac{\pi}{18} + \tan^2\frac{7\pi}{18} + \tan^2\frac{13\pi}{18}.$$

(iii) Hence, or otherwise, find the value of **1**

$$\tan^3\frac{\pi}{18} + \tan^3\frac{7\pi}{18} + \tan^3\frac{13\pi}{18}.$$

(b) The path of a moving particle is given by the position vector **4**

$$\underset{\sim}{r}(t) = \left(5\cos t - 4\cos\frac{5t}{2}\right)\underset{\sim}{i} + \left(5\sin t - 4\sin\frac{5t}{2}\right)\underset{\sim}{j}, \qquad t \geq 0,$$

where t is measured in seconds and position is measured in metres.

Find the first interval of time after $t = 0$ during which the acceleration has a component directly opposing the motion of the particle.

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Question 17 (10 marks) Use the Question 17 Writing Booklet

(a) Let X be a random variable with an arbitrary probability distribution P, finite mean μ, and finite variance σ^2, where $\sigma > 0$. For $k > 0$, define the random variable Y by

$$Y = \begin{cases} 1 & \text{if } |X - \mu| \geq k\sigma \\ 0 & \text{if } |X - \mu| < k\sigma. \end{cases}$$

You may assume that:

- for any random variables A and B, $E(A \pm B) = E(A) \pm E(B)$;
- for any random variable A and constant α, $E(\alpha A) = \alpha E(A)$;
- if C is a non-negative random variable, then $E(C) \geq 0$.

(i) Prove that $E(Y) = P(|X - \mu| \geq k\sigma)$. **1**

(ii) Show that, for all possible values of Y, **1**

$$(X - \mu)^2 - k^2\sigma^2 Y \geq 0.$$

(iii) Using (i) and (ii), prove that $P(|X - \mu| \geq k\sigma) \leq \dfrac{1}{k^2}$. **2**

(iv) Hence determine the minimum proportion of observations that is guaranteed to lie in the interval $(\mu - 2\sigma, \mu + 2\sigma)$ for any probability distribution P. **1**

(b) Let F be a differentiable function defined on the interval $[0, a)$, where $a > 0$, satisfying

$$F(x) = \int_0^x \left(e^t + [F(t)]^2\right) dt.$$

It is given that the range of F is $[0, \infty)$.

(i) By considering $F'(x)$, explain why F is one-to-one and hence invertible. **1**

(ii) Let $G = F^{-1}$. Show that, for $x \geq 0$, **2**

$$G'(x) \leq \frac{1}{1 + x^2}.$$

(iii) Hence deduce that $a \leq \dfrac{\pi}{2}$. **2**

End of paper

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2026 Year 12 Trial HSC

Mathematics Extension 1

Multiple Choice Worked Solutions

Answer key

D	C	B	C	B	D	A	D	A	B

Question 1

Answer: D

For $X \sim \text{Bin}(n, p)$,

$$E(X) = np = 12, \qquad \text{Var}(X) = np(1-p) = 8.$$

Since $np = 12$,

$$12(1-p) = 8 \implies 1 - p = \frac{2}{3} \implies p = \frac{1}{3}.$$

Therefore

$$n = \frac{12}{p} = \frac{12}{1/3} = 36.$$

Hence $\boxed{n = 36}$, so the correct option is $\boxed{D}$.

Question 2

Answer: C

There are four colours. To avoid obtaining four blocks of the same colour for as long as possible, we could draw

$$3 + 3 + 3 + 3 = 12$$

blocks: three of each colour. The next block must make one colour occur at least four times. Thus the minimum guaranteed number is

$$12 + 1 = \boxed{13},$$

so the correct option is $\boxed{C}$.

Question 3

Answer: B

We have

$$p(x-1) = \sum_{k=0}^{12} (x-1)^k.$$

Only the terms with $k \geq 9$ can contribute to the coefficient of x^9. In $(x-1)^k$, this coefficient is

$$\binom{k}{9}(-1)^{k-9}.$$

Therefore

$$\begin{aligned}
[x^9]p(x-1) &= \binom{9}{9} - \binom{10}{9} + \binom{11}{9} - \binom{12}{9} \\
&= 1 - 10 + 55 - 220 \\
&= \boxed{-174}.
\end{aligned}$$

Hence the correct option is $\boxed{B}$.

Question 4

Answer: C

The principal range of $\sin^{-1}$ is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

For $\pi/2 \leq \theta \leq \pi$, the angle θ lies in Quadrant II, while

$$0 \leq \pi - \theta \leq \frac{\pi}{2} \quad \text{and} \quad \sin(\pi - \theta) = \sin\theta.$$

Therefore the angle in the principal range having sine $\sin\theta$ is $\pi - \theta$:

$$\boxed{\sin^{-1}(\sin\theta) = \pi - \theta}.$$

Hence the correct option is $\boxed{C}$.

Question 5

Answer: B

Since the vectors are perpendicular, their dot product is zero:

$$\begin{aligned}
\begin{pmatrix} k+2 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} k-1 \\ 2 \end{pmatrix} &= 0, \\
(k+2)(k-1) - 18 &= 0, \\
k^2 + k - 20 &= 0, \\
(k+5)(k-4) &= 0.
\end{aligned}$$

Thus $k = -5$ or $k = 4$. Of these, only -5 appears among the listed numerical options. Hence the correct option is $\boxed{B}$.

Question 6

Answer: D

If $(x - a)$ has multiplicity 2, then a is a common root of P and P'. Now

$$P'(x) = 3x^2 + 2x - 5 = (3x + 5)(x - 1).$$

The possible repeated roots are therefore $-5/3$ and 1. Checking $x = 1$,

$$P(1) = 1 + 1 - 5 + 3 = 0.$$

Indeed,

$$P(x) = x^3 + x^2 - 5x + 3 = (x - 1)^2(x + 3).$$

Hence the repeated factor is $(x - 1)$, so $\boxed{a = 1}$. The correct option is $\boxed{D}$.

Question 7

Answer: A

The slope field has the following features:

- the slope is zero along $y = 0$;
- slopes are positive for $y > 0$ and negative for $y < 0$, so the numerator must involve y;
- for a fixed nonzero y, the slopes become less steep as $|x|$ increases, so the denominator must increase with x^2.

These features match

$$\boxed{\frac{dy}{dx} = \frac{y}{1+x^2}}.$$

Hence the correct option is $\boxed{A}$.

Question 8

Answer: D

The graph on the right has two branches symmetric about the x-axis. It exists only where the original graph has $f(x) \geq 0$. The equation

$$y^2 = f(x)$$

is equivalent to

$$y = \pm\sqrt{f(x)},$$

which produces exactly these two symmetric branches wherever $f(x) \geq 0$.

The equation $y^2 = |f(x)|$ would also produce branches where $f(x) < 0$, which are absent from the graph on the right. Therefore the correct equation is

$$\boxed{y^2 = f(x)},$$

so the correct option is $\boxed{D}$.

Question 9

Answer: A

Define

$$g(x) = \cos^{-1} x - \frac{\pi}{2}.$$

Using $\cos^{-1}(-x) = \pi - \cos^{-1} x$,

$$\begin{aligned} g(-x) &= \cos^{-1}(-x) - \frac{\pi}{2} \\ &= \pi - \cos^{-1} x - \frac{\pi}{2} \\ &= -\left(\cos^{-1} x - \frac{\pi}{2}\right) \\ &= -g(x). \end{aligned}$$

Thus g is odd. Since x^2 is even, $x^2 g(x)$ is odd. Its integral over the symmetric interval $[-c, c]$ is therefore zero:

$$\boxed{\int_{-c}^{c} x^2 \left(\cos^{-1} x - \frac{\pi}{2}\right) dx = 0}.$$

Hence the correct option is $\boxed{A}$.

Question 10

Answer: B

The first solid has volume

$$V = \pi \int_0^{\pi/2} \sin^2 x \, dx.$$

The second solid has radius $a\sin(\omega x - \delta)$, so its volume is

$$V_2 = \pi \int_{\delta/\omega}^{(\delta+\pi/2)/\omega} a^2 \sin^2(\omega x - \delta) \, dx.$$

Let

$$u = \omega x - \delta, \qquad du = \omega \, dx, \qquad dx = \frac{du}{\omega}.$$

The new limits are $u = 0$ and $u = \pi/2$. Hence

$$\begin{aligned} V_2 &= \frac{a^2}{\omega}\pi \int_0^{\pi/2} \sin^2 u \, du \\ &= \boxed{\frac{a^2}{\omega}V}. \end{aligned}$$

Therefore the correct option is $\boxed{B}$.

End of Multiple Choice Solutions

Solutions

Question 11

(a) By first completing the square on the quadratic, evaluate **3**

$$\int_{-1}^{3} \frac{1}{\sqrt{7+2x-x^2}}\,dx.$$

Completing the square: 1 mark

$$\begin{aligned} 7+2x-x^2 &= -(x^2-2x-7) \\ &= -\left[(x-1)^2-1-7\right] \\ &= 8-(x-1)^2 \end{aligned}$$

Some students lost a mark here for incorrectly completing the square (sign errors or incorrect constant).

Integration: 1 mark

$$\int_{-1}^{3} \frac{1}{\sqrt{8-(x-1)^2}}\,dx = \left[\sin^{-1}\left(\frac{x-1}{2\sqrt{2}}\right)\right]_{-1}^{3}$$

Substituting the bounds: *Students lost marks here for substituting the bounds in incorrectly.*

$$\begin{aligned} &= \sin^{-1}\left(\frac{3-1}{2\sqrt{2}}\right) - \sin^{-1}\left(\frac{-1-1}{2\sqrt{2}}\right) \\ &= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \end{aligned}$$

$\boxed{= \dfrac{\pi}{2}}$ **1 mark**

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(b) Differentiate **2**

$$y = \cos^{-1}\left(\frac{1}{\sqrt{1+4x^2}}\right), \qquad x > 0.$$

Chain rule application: *Students lost marks for incorrect application of the chain rule (particularly differentiating the inner function* $(1 + 4x^2)^{-1/2}$*).*

Let $u = \dfrac{1}{\sqrt{1+4x^2}} = (1+4x^2)^{-1/2}$, so

$$\frac{du}{dx} = -\tfrac{1}{2}(1+4x^2)^{-3/2}(8x) = -4x(1+4x^2)^{-3/2}.$$

Then

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}.$$

Since

$$1-u^2 = 1 - \frac{1}{1+4x^2} = \frac{4x^2}{1+4x^2} \quad \Rightarrow \quad \sqrt{1-u^2} = \frac{2x}{\sqrt{1+4x^2}} \quad (x > 0),$$

$$\frac{dy}{dx} = -\frac{\sqrt{1+4x^2}}{2x} \times \left(-4x(1+4x^2)^{-3/2}\right) \textbf{ 1 mark}$$

$$= \frac{4x\sqrt{1+4x^2}}{2x(1+4x^2)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{2}{1+4x^2}} \qquad \textbf{1 mark}$$

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(c) In triangle ABC, $AB = 10$ cm, $AC = 12$ cm, and $\angle BAC$ is increasing at a rate of 0.1 rad s^{-1}. At what rate is the area of $\triangle ABC$ increasing when $\angle BAC = \dfrac{\pi}{3}$? **2**

Identifying and stating the chain rule: *Students dropped $\frac{1}{2}$ from the area of a triangle formula.*

$$A = \frac{1}{2}(AB)(AC)\sin\theta = \frac{1}{2}(10)(12)\sin\theta = 60\sin\theta$$

$$\frac{dA}{dt} = \frac{dA}{d\theta}\cdot\frac{d\theta}{dt} = 60\cos\theta\cdot\frac{d\theta}{dt} \qquad \textbf{1 mark}$$

Substituting $\theta = \dfrac{\pi}{3}$ and $\dfrac{d\theta}{dt} = 0.1$:

$$\frac{dA}{dt} = 60\cos\left(\frac{\pi}{3}\right)(0.1) = 60\left(\frac{1}{2}\right)(0.1)$$

$\boxed{\dfrac{dA}{dt} = 3 \text{ cm}^2\text{s}^{-1}}$ **1 mark**

(d) How many distinct arrangements of the letters of the word **KUGEL-BLITZ** have K and Z adjacent to each other? **1**

Mostly well done; however some students did not notice that the letter "L" is repeated twice.

KUGELBLITZ has 10 letters, with L repeated twice.

Treat K and Z as a single unit (which can occur as KZ or ZK), giving 9 units to arrange, still with L repeated twice:

$$2 \times \frac{9!}{2!}$$

$\boxed{=9! = 362\,880}$

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Question 12

(a) Using

$$\sin 4\theta = 2\sin 2\theta \cos 2\theta,$$

we have

$$\begin{aligned}\frac{2\sin 2\theta - \sin 4\theta}{2\sin 2\theta + \sin 4\theta} &= \frac{2\sin 2\theta - 2\sin 2\theta\cos 2\theta}{2\sin 2\theta + 2\sin 2\theta\cos 2\theta} \\ &= \frac{2\sin 2\theta(1-\cos 2\theta)}{2\sin 2\theta(1+\cos 2\theta)} \\ &= \frac{1-\cos 2\theta}{1+\cos 2\theta},\end{aligned}$$

1 mark for getting to simplified form before substituting

where cancellation is valid since $\sin 2\theta \neq 0$.

Let

$$t = \tan\theta.$$

Then

$$\cos 2\theta = \frac{1-t^2}{1+t^2}.$$

Hence

$$\begin{aligned}\frac{1-\cos 2\theta}{1+\cos 2\theta} &= \frac{1-\dfrac{1-t^2}{1+t^2}}{1+\dfrac{1-t^2}{1+t^2}} \\ &= \frac{\dfrac{2t^2}{1+t^2}}{\dfrac{2}{1+t^2}} \\ &= t^2 \\ &= \boxed{\tan^2\theta}.\end{aligned}$$

1 mark to getting final proof required without skipping steps.

(b) Let $u = 2 - \sqrt{x}$, so that

$$\sqrt{x} = 2 - u, \qquad x = (2-u)^2, \qquad dx = -2(2-u)\,du.$$

Therefore

$$\int \sqrt{2-\sqrt{x}}\,dx = \int u^{1/2}[-2(2-u)],du$$

1 mark for correctly rewriting integrand in terms of u only

$$= -2\int \left(2u^{1/2} - u^{3/2}\right),du$$

$$= -\frac{8}{3}u^{3/2} + \frac{4}{5}u^{5/2} + C$$

$$= \boxed{-\frac{8}{3}(2-\sqrt{x})^{3/2} + \frac{4}{5}(2-\sqrt{x})^{5/2} + C}$$

1 mark for final answer

or

$$= \boxed{-\frac{8}{3}\sqrt{(2-\sqrt{x})^3} + \frac{4}{5}\sqrt{(2-\sqrt{x})^5} + C}$$

or

$$= \boxed{-\frac{8}{3}\left(\sqrt{2-\sqrt{x}}\right)^3 + \frac{4}{5}\left(\sqrt{2-\sqrt{x}}\right)^5 + C}$$

It was VERY disappointing to see the number of students forgetting to substitute back into x after integrating.

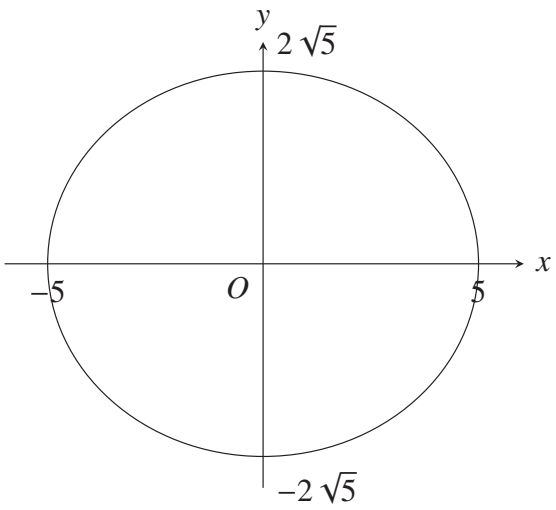
(c) From

$$\frac{x^2}{25} + \frac{y^2}{20} = 1,$$

the semi-axes are 5 in the x-direction and $2\sqrt{5}$ in the y-direction.

(i) The ellipse is obtained from the unit circle by the dilation

$$(x, y) \longmapsto (5x, 2\sqrt{5}\, y).$$



1 mark, also a bit disappointed at the number of students who got the x and y intercepts mixed up.

(ii) From the equation of the ellipse,

$$y^2 = 20\left(1 - \frac{x^2}{25}\right).$$

Therefore

$$\begin{aligned} V &= \pi \int_{-5}^{5} y^2\, dx \\ &= 20\pi \int_{-5}^{5} \left(1 - \frac{x^2}{25}\right), dx \qquad \text{1 mark} \\ &= 20\pi \left[x - \frac{x^3}{75}\right]_{-5}^{5} \\ &= \boxed{\frac{400\pi}{3} \text{ units}^3}. \qquad \text{1 mark} \end{aligned}$$

(iii) At position x, the area of the cross-section perpendicular to the x-axis is

$$A(x) = \pi y^2 = 20\pi\left(1 - \frac{x^2}{25}\right).$$

This is maximised when $x = 0$. Hence

$$\boxed{A_{\max} = 20\pi \text{ units}^2}. \qquad \text{1 mark}$$

This question was a VERY good indicator of students who had learnt volume by understanding how the formula for a solid of revolution is deduced by summing up an infinite number of circular discs (hence cross-section is circular), against those who route memorised the formula.

Question 13

(a) (i) Using Pascal's identity,

$$\binom{m+1}{r+1} = \binom{m}{r} + \binom{m}{r+1}.$$

Therefore

$$\binom{m+1}{r+1} - \binom{m}{r+1} = \boxed{\binom{m}{r}}.$$

(ii) Using part (i) with $r = 3$,

$$\begin{aligned}
&\binom{n-3}{3} + \binom{n-2}{3} + \binom{n-1}{3} + \binom{n}{3} \\
&= \left[\binom{n-2}{4} - \binom{n-3}{4}\right] + \left[\binom{n-1}{4} - \binom{n-2}{4}\right] \\
&\quad + \left[\binom{n}{4} - \binom{n-1}{4}\right] + \left[\binom{n+1}{4} - \binom{n}{4}\right].
\end{aligned}$$

The intermediate terms cancel, giving

$$\boxed{\binom{n+1}{4} - \binom{n-3}{4}}.$$

(b) Since

$$\overrightarrow{BP} = 2\left(\underset{\sim}{a} + \underset{\sim}{b}\right),$$

we have

$$\begin{aligned}
\overrightarrow{OP} &= \overrightarrow{OB} + \overrightarrow{BP} \\
&= \underset{\sim}{b} + 2\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \\
&= 2\underset{\sim}{a} + 3\underset{\sim}{b}.
\end{aligned}$$

(i) Since

$$\overrightarrow{AQ} = \lambda\overrightarrow{AB},$$

we have

$$\begin{aligned}
\overrightarrow{OQ} &= \overrightarrow{OA} + \overrightarrow{AQ} \\
&= \underset{\sim}{a} + \lambda\left(\underset{\sim}{b} - \underset{\sim}{a}\right) \\
&= \boxed{(1-\lambda)\underset{\sim}{a} + \lambda\underset{\sim}{b}}.
\end{aligned}$$

(ii) Also,

$$\overrightarrow{OQ} = \mu\overrightarrow{OP} = 2\mu\underset{\sim}{a} + 3\mu\underset{\sim}{b}.$$

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Equating coefficients of $\underline{a}$ and $\underline{b}$,

$$1 - \lambda = 2\mu, \qquad \lambda = 3\mu.$$

Substituting

$$\mu = \frac{\lambda}{3}$$

into the first equation gives

$$1 - \lambda = \frac{2\lambda}{3}.$$

Hence

$$3 - 3\lambda = 2\lambda,$$

so

$$\lambda = \frac{3}{5}.$$

Therefore

$$\mu = \frac{\lambda}{3} = \frac{1}{5}.$$

Thus

$$\boxed{\lambda = \frac{3}{5}, \qquad \mu = \frac{1}{5}}.$$

(c) Let $P(n)$ denote the statement

$$\sum_{r=1}^{n} r^2(r+1) = \frac{n(n+1)(n+2)(3n+1)}{12}.$$

For $n = 1$, the left-hand side is

$$1^2 \times 2 = 2,$$

while the right-hand side is

$$\frac{1 \cdot 2 \cdot 3 \cdot 4}{12} = 2.$$

Thus $P(1)$ is true.

Assume that $P(k)$ is true for some integer $k \geq 1$. That is,

$$\sum_{r=1}^{k} r^2(r+1) = \frac{k(k+1)(k+2)(3k+1)}{12}.$$

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Then

$$\begin{aligned}\sum_{r=1}^{k+1} r^2(r+1) &= \sum_{r=1}^{k} r^2(r+1) + (k+1)^2(k+2) \\ &= \frac{k(k+1)(k+2)(3k+1)}{12} + (k+1)^2(k+2) \quad \text{by assumptions} \checkmark \\ &= \frac{(k+1)(k+2)}{12}\left[k(3k+1) + 12(k+1)\right] \\ &= \frac{(k+1)(k+2)}{12}\left(3k^2 + 13k + 12\right) \\ &= \frac{(k+1)(k+2)(k+3)(3k+4)}{12} \\ &= \frac{(k+1)\big((k+1)+1\big)\big((k+1)+2\big)\big(3(k+1)+1\big)}{12}. \checkmark\end{aligned}$$

Hence $P(k+1)$ is true.

Therefore, by mathematical induction,

$$\sum_{r=1}^{n} r^2(r+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

for all integers $n \geq 1$.

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Mathematics Extension 1 2026: Task 4	Marker's Comments
Solutions: Question 14	

Question 14(a) *[2 marks]* For $-1 \leq x \leq 1$, show that

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}(x).$$

Solution

Let

$$\theta = \tan^{-1}(x).$$

Since $-1 \leq x \leq 1$,

$$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, \qquad \text{so} \qquad -\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2}.$$

Also,

$$\sin(2\theta) = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2x}{1+x^2}. \quad \checkmark$$

Because 2θ lies in the principal range of $\sin^{-1}$,

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}(x). \quad \checkmark$$

Alternative solution (using the derivative):

Define

$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right), \qquad g(x) = 2\tan^{-1}(x).$$

For $-1 < x < 1$,

$$\begin{aligned} f'(x) - g'(x) &= \frac{\dfrac{2(1-x^2)}{(1+x^2)^2}}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}} - \frac{2}{1+x^2} \\ &= \frac{\dfrac{2(1-x^2)}{(1+x^2)^2}}{\dfrac{\sqrt{(1-x^2)^2}}{1+x^2}} - \frac{2}{1+x^2} \\ &= \frac{\dfrac{2(1-x^2)}{(1+x^2)^2}}{\dfrac{|1-x^2|}{1+x^2}} - \frac{2}{1+x^2} \\ &= \frac{\dfrac{2(1-x^2)}{(1+x^2)^2}}{\dfrac{1-x^2}{1+x^2}} - \frac{2}{1+x^2} \qquad \ldots(*) \\ &= \frac{2}{1+x^2} - \frac{2}{1+x^2} \\ &= 0. \quad \checkmark \end{aligned}$$

Note, at $(*)$, $|1-x^2| = 1-x^2 > 0$ since $-1 < x < 1$.

Hence

$$f(x) - g(x) \equiv C \qquad \text{for } -1 < x < 1.$$

Taking $x = 0$,

$$C = f(0) - g(0) = 0.$$

Mark allocation

First mark
Awarded for correctly establishing

$$\sin\left(2\tan^{-1}x\right) = \frac{2x}{1+x^2},$$

or an equivalent valid connection, using a substitution, a double-angle identity or a derivative argument.

Second mark
Awarded for correctly completing the proof over the full interval $[-1, 1]$. For the trigonometric method, this required establishing

$$-1 \leq x \leq 1 \implies -\frac{\pi}{4} \leq \tan^{-1}x \leq \frac{\pi}{4},$$

and hence $-\frac{\pi}{2} \leq 2\tan^{-1}x \leq \frac{\pi}{2}$, so that

$$\sin^{-1}\left(\sin(2\tan^{-1}x)\right) = 2\tan^{-1}x.$$

An alternative valid method, such as a derivative argument, could also earn this mark provided it established the identity over the full interval. Any endpoints excluded by the chosen method also had to be checked.

A correct trigonometric connection without the principal-range justification received the first mark only. No mark was awarded if no valid mathematical connection between the two sides was established.

Clear transcription slips were overlooked when the preceding working showed the correct calculation.

Common issues

- The most common issue was failing to establish the principal-value range. Proving that two angles have equal sines or equal tangents does not by itself prove that the arguments ('angles') are equal.
- Merely stating *because* $-1 \leq x \leq 1$ did not supply the required range argument. The range of $2\tan^{-1}x$ had to be derived explicitly.
- Some responses did not check the endpoints $x = \pm 1$. This was particularly common in tangent and derivative methods that were valid only on $(-1, 1)$. Tangent methods ran into the problem of having a denominator where $x = \pm 1$ had to be excluded.
- Some responses used an unsupported inverse or half-angle step, such as asserting $x = \tan(\theta/2)$ or $\tan(A/2) = x$ without selecting and justifying the correct branch.
- Some responses covered only part of the interval. In particular, assuming a first-quadrant angle handled only $x \geq 0$ and did not justify the negative case.
- A recurring incorrect identity was to identify

$$\frac{2\tan\theta}{1+\tan^2\theta}$$

as $\tan 2\theta$. This expression is $\sin 2\theta$.

Therefore

$$f(x) = g(x) \qquad \text{for } -1 < x < 1.$$

It remains to check the endpoints.
For $x = 1$,

$$LHS = f(1) = \sin^{-1}(1) = \frac{\pi}{2} = 2\tan^{-1}(1) = g(1),$$

and for $x = -1$,

$$f(-1) = \sin^{-1}(-1) = -\frac{\pi}{2} = 2\tan^{-1}(-1) = g(-1).$$

Hence

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}(x) \qquad \text{for } -1 \le x \le 1. \quad \checkmark$$

- Square-root and sign errors occurred, especially replacing $\sqrt{(1-x^2)^2}$ by $x^2 - 1$. On $[-1, 1]$, the correct value is

$$|1 - x^2| = 1 - x^2.$$

- A graph suggesting that the two functions coincide was not sufficient to establish the identity.

Question 14(b) *[3 marks]* A hospital emergency department records patient arrivals over a day divided into $n = 86\,400$ one-second intervals. The number of arrivals in each interval is modelled as an independent Bernoulli random variable, with probability

$$p = \frac{1}{710}$$

of one arrival in any interval. Let $\widehat{p}$ be the proportion of intervals containing an arrival.

Using a normal approximation to the sampling distribution of $\widehat{p}$, find the smallest integer m such that the probability that more than m patients arrive during the day is less than 5%. Do not use a continuity correction. Interpolation is not required.

Solution 1: Working with the total number of arrivals

We are told to use the normal approximation to the sampling distribution of $\widehat{p}$, so there is no requirement to test whether $np, nq \ge 5$ or 10.

Let

$$q = 1 - p = \frac{709}{710}.$$

The sampling distribution of the sample proportion is

$$\widehat{p} \sim N\left(p, \frac{pq}{n}\right).$$

Let $X = n\widehat{p}$ be the total number of arrivals. It follows that

$$X \sim N(np, npq).$$

Hence

$$\left.\begin{aligned} \mu_X &= np = \frac{86\,400}{710} \approx 121.6901, \\ \sigma_X &= \sqrt{npq} \approx 11.0236 \end{aligned}\right\} \quad \checkmark$$

We require

$$P(X > m) < 0.05.$$

For use of the z-score table, taking the complementary probability,

$$P(X \le m) > 1 - 0.05 = 0.95.$$

After standardising X, this means

$$P\left(Z \le \frac{m - \mu_X}{\sigma_X}\right) > 0.95.$$

From the standard normal table,

$$\Phi(z) = 0.95 \implies z \approx 1.65. \quad \checkmark$$

Mark allocation

First mark
Awarded for obtaining the correct distribution parameters. This could be done using the proportion model

$$\mu_{\widehat{p}} = \frac{1}{710}, \qquad \sigma_{\widehat{p}} = \sqrt{\frac{pq}{86\,400}},$$

or the equivalent count model

$$\mu_X = 86\,400p, \qquad \sigma_X = \sqrt{86\,400pq}.$$

Second mark
Awarded for correctly setting up the upper-tail probability as

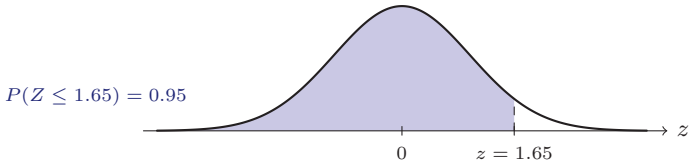
$$P\left(\widehat{p} > \frac{m}{86\,400}\right) < 0.05$$

or $P(X > m) < 0.05$, and then using the complementary probability 0.95 to obtain the appropriate upper-tail critical z-value. Since interpolation was not required, the nearest value from the supplied table was expected.

Third mark
Awarded for correctly solving for the cutoff and expressing it as a patient count, obtaining a cutoff of approximately 139.88.

The final integer choice was not treated as a separate mark. Accordingly, final answers of 139 or 140 were accepted for marking purposes (even though 140 is the mathematically correct answer), provided the preceding method and cutoff calculation were correct. The upper-tail setup and critical z-value belong to the second mark, while solving and converting to a patient count belong to the third mark.

Error carried forward was allowed for this mark when one earlier value, such as the value of z, was incorrect but the subsequent algebra and conversion were correct. A second



Therefore, we require m such that

$$\frac{m - \mu_X}{\sigma_X} > 1.65.$$

Hence

$$\begin{aligned} m &> \mu_X + 1.65\sigma_X \\ &\approx 121.6901 + 1.65(11.0236) \\ &\approx 139.88. \end{aligned}$$

Since m must be an integer, the smallest possible value is

$$m = 140. \quad \checkmark$$

Indeed,

$$P(X > 139) \approx 0.0582, \qquad P(X > 140) \approx 0.0484,$$

confirming that 140 is the smallest integer satisfying the condition.

Solution 2: Working entirely with the sample proportion

Using

$$\widehat{p} \sim N\left(p, \frac{pq}{n}\right), \qquad p = \frac{1}{710}, \qquad q = \frac{709}{710},$$

we have

$$\left.\begin{aligned} \mu_{\widehat{p}} &= p = \frac{1}{710}, \\ \sigma_{\widehat{p}} &= \sqrt{\frac{pq}{n}} = \sqrt{\frac{\left(\frac{1}{710}\right)\left(\frac{709}{710}\right)}{86\,400}} \approx 0.0001276 \end{aligned}\right\} \quad \checkmark$$

More than m patients arrive precisely when

$$\widehat{p} > \frac{m}{n}.$$

Hence we require

$$P\left(\widehat{p} > \frac{m}{n}\right) < 0.05.$$

Without a continuity correction, standardising $\widehat{p}$ gives

$$\frac{\frac{m}{n} - p}{\sqrt{pq/n}} > z_{0.95} \approx 1.65. \quad \checkmark$$

Therefore

$$\begin{aligned} \frac{m}{n} &> p + 1.65\sqrt{\frac{pq}{n}}, \\ m &> n\left(p + 1.65\sqrt{\frac{pq}{n}}\right) \\ &\approx 139.88. \end{aligned}$$

Thus

$$m = 140. \quad \checkmark$$

As a check,

$$P\left(\widehat{p} > \frac{139}{86\,400}\right) \approx 0.0582, \qquad P\left(\widehat{p} > \frac{140}{86\,400}\right) \approx 0.0484.$$

independent error prevented error carried forward.

Common issues

- The most common issue was incorrect handling of the upper 5% tail and its critical value. Incorrect z-values included 2 (probably assuming 95% of the area about the mean is the same as the requirement for 95% of the area from the lower tail upward), 2.58, 3.32, 0.8289 and 0.05 instead of 1.65. Some responses used a negative z-value, used the lower tail, or changed 0.95 back to 0.05.
- Some responses mixed the count distribution with the sample-proportion distribution. Examples included combining the count mean np with the proportion standard deviation $\sqrt{pq/n}$, or changing m from a patient count into a proportion without explanation.
- Some responses obtained a critical proportion of approximately 0.001618 but did not convert it into a patient count by multiplying by 86 400. In some cases, the value was incorrectly multiplied by 710.
- Incorrect variances or standard deviations included using $np/(1-p)$, taking $n = 1$, omitting the factor $1/\sqrt{86\,400}$, inverting part of the standard deviation, or using an incorrect numerical value.
- Some responses used the wrong inequality direction, complement or sign. Examples included writing $(m - \mu)/\sigma < 0.05$, using $z = -1.65$, or finding the lower 5% cutoff instead of the upper 5% cutoff.
- Some responses made unsupported probability statements, such as equating a probability with a negative number or with a z-score. A probability must lie between 0 and 1.
- Minor rounding and final-integer errors were generally less serious and did not necessarily cost a conceptual mark.
- Arithmetic and algebraic errors occurred after otherwise correct setups.

Question 14(c) A virus population $V(t) > 0$ in a laboratory culture is modelled by

$$\frac{dV}{dt} = rV\left(1 - \frac{V}{k}\right) - c, \qquad V(0) > 0,$$

where r, k, c are positive,

$$0 < c < \frac{rk}{4},$$

and c is the constant removal rate due to antiviral treatment. If $V(t)$ reaches zero, the virus is considered eliminated and the model terminates. Do not solve the differential equation.

(i) *[1 mark]* Find the equilibrium virus populations.

Solution

At an equilibrium, $dV/dt = 0$. Hence

$$rV\left(1 - \frac{V}{k}\right) - c = 0.$$

Equivalently,

$$V^2 - kV + \frac{ck}{r} = 0.$$

Therefore the equilibrium populations are

$$V_{\pm} = \frac{k}{2}\left(1 \pm \sqrt{1 - \frac{4c}{rk}}\right). \quad \checkmark$$

The following was not required to be shown, but for completion: since $0 < c < rk/4$, we have $0 < 4c/(rk) < 1$, so $0 < \sqrt{1 - 4c/(rk)} < 1$. Therefore $1 \pm \sqrt{1 - 4c/(rk)} > 0$, and since $k/2 > 0$, both $V_- > 0$ and $V_+ > 0$.

The condition $0 < c < rk/4$ ensures that these are two distinct positive equilibria, with

$$0 < V_- < \frac{k}{2} < V_+.$$

(ii) *[2 marks]* Hence determine the range of initial virus populations for which the virus is eventually eliminated. Justify your answer.

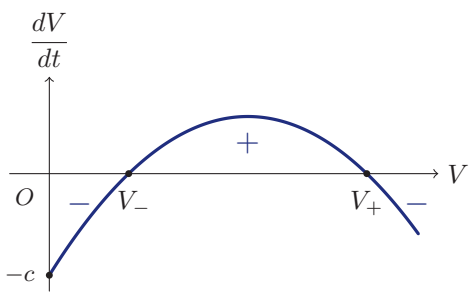
Solution

For the population to decrease, we must have $\dfrac{dV}{dt} < 0$ over the interval required for $V \to 0$.

Consider that

$$\frac{dV}{dt} = -\frac{r}{k}V^2 + rV - c.$$

The coefficient of V^2 in the quadratic is negative, so the graph of $\dfrac{dV}{dt}$ versus V is concave (down). Its two positive zeros are the equilibria V_- and V_+ found in part (i), while $\left.\dfrac{dV}{dt}\right|_{V=0} = -c < 0$.



Mark allocation

Part (i): one mark
Awarded for correctly setting $dV/dt = 0$ and obtaining both equilibrium values V_- and V_+.

Part (ii): first mark
Awarded for recognising that elimination of virus requires

$$\frac{dV}{dt} < 0,$$

and making a legitimate analysis to determine the values of V for which this occurs.

Part (ii): second mark
Awarded for correctly determining and interpreting the relevant region, and concluding that elimination occurs only when

$$0 < V(0) < V_-.$$

Although $\dfrac{dV}{dt} < 0$ also for $V > V_+$, this does not give elimination: if $V(0) > V_+$, then V decreases towards the stable equilibrium V_+, rather than towards zero. A response that correctly identified both regions in which $\dfrac{dV}{dt} < 0$, but incorrectly claimed that both produced elimination, therefore received the first mark only: 1/2 for part (ii).

Error carried forward was allowed where incorrect equilibrium values from part (i) were used consistently in an otherwise valid sign analysis.

We see that the population decreases when the graph is below the V-axis and increases when it is above the axis. Hence

$$\begin{aligned} 0 < V < V_- : &\quad \frac{dV}{dt} < 0, \\ V_- < V < V_+ : &\quad \frac{dV}{dt} > 0, \\ V > V_+ : &\quad \frac{dV}{dt} < 0. \quad \checkmark \end{aligned}$$

If $0 < V(0) < V_-$, the population decreases, and there is no equilibrium between 0 and V_- at which it can settle. It therefore continues towards zero. Furthermore, as $V \to 0$,

$$\frac{dV}{dt} \to -c < 0.$$

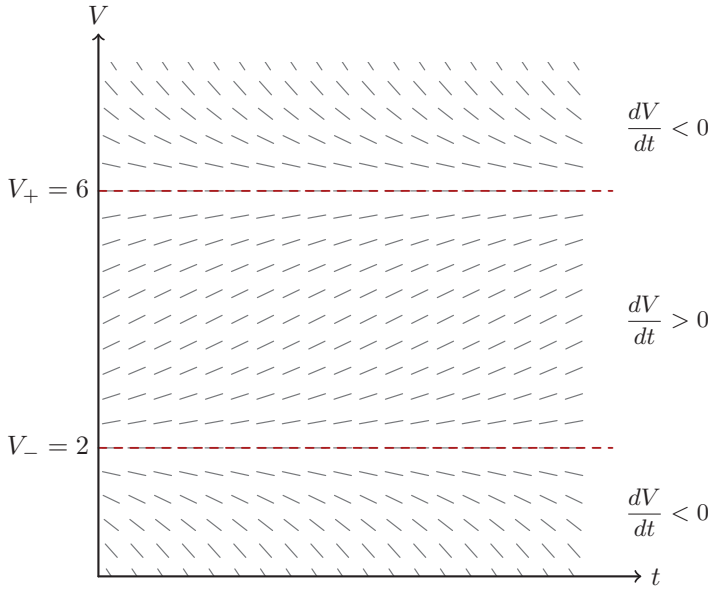
Thus the magnitude of the rate of decrease does not approach zero as the population approaches zero, so V reaches zero in finite time and the model terminates.

If $V(0) = V_-$ or $V(0) = V_+$, the population remains at that equilibrium. If $V_- < V(0) < V_+$, it increases towards V_+; if $V(0) > V_+$, it decreases towards V_+. Therefore no initial population at or above V_- is eliminated.

Hence the required range is

$$0 < V(0) < \frac{k}{2}\left(1 - \sqrt{1 - \frac{4c}{rk}}\right). \quad \checkmark$$

Example slope field for illustration (not required as part of solution):



Common issues

- The most serious error was including $V(0) > V_+$ in the elimination range. A negative derivative above V_+ means that V decreases towards V_+; it does not imply elimination.
- Many responses drew a convex (concave-up) parabola. Since the leading coefficient in
$$\frac{dV}{dt} = -\frac{r}{k}V^2 + rV - c$$
is $-r/k < 0$, the graph must be concave (concave down). Reversing the concavity led to the false conclusion that $V_- < V(0) < V_+$ gives elimination. In the correct model, a solution in this interval approaches V_+.

  Even under the mistaken reversed sign diagram, a decreasing solution would approach V_-, not pass through it to zero. Neither equilibrium is zero, so this qualitative check should have exposed the error.
- Some responses failed to show that $\frac{dV}{dt} < 0$ throughout $0 < V < V_-$. Merely stating that the virus decreases did not provide the required interval-based sign analysis.
- Checking only one value, particularly
$$\left.\frac{dV}{dt}\right|_{V=0} = -c,$$
does not establish the behaviour throughout the required interval and did not earn a part-(ii) mark by itself.
- Some responses treated $V = 0$ as an equilibrium. In fact,
$$\left.\frac{dV}{dt}\right|_{V=0} = -c \neq 0,$$
so zero is not an equilibrium of the stated differential equation.
- Some responses gave the correct initial-value range without adequately explaining why it produces elimination. The correct range alone was not sufficient for the second mark.
- The discriminant determines the number of equilibria, but it does not determine whether a particular solution is eliminated.
- Some sign analyses were incomplete or unsupported. A sketch or a claim that an expression was negative had to be connected to the intervals determined by V_- and V_+.
- Errors when finding the equilibria included mistakes when multiplying through by k, applying the quadratic formula, or simplifying the discriminant.
- The equilibrium endpoints were sometimes ignored. If $V(0) = V_-$ or $V(0) = V_+$, the solution remains at that equilibrium and is not eliminated.

Question 15

a) $\sin 3x + \sin x = \sqrt{3}(\cos 3x + \cos x)$

$\sin(2x+x) + \sin(2x-x) = 2\sin 2x\cos x$

$\cos(2x-(-x)) + \cos(2x+(-x)) = 2\cos 2x\cos(-x)$

$= 2\cos 2x\cos x$

$\therefore 2\sin 2x\cos x = \sqrt{3}(2\cos 2x\cos x)$ (1)

$2\sin 2x\cos x - 2\sqrt{3}(\cos 2x\cos x) = 0$

$2\cos x(\sin 2x - \sqrt{3}\cos 2x) = 0$

$\therefore 2\cos x = 0$ or $\sin 2x - \sqrt{3}\cos 3x = 0$ (1)

$\cos x = 0$ $\sin 2x = \sqrt{3}\cos 3x$

$x = \frac{\pi}{2}$ $\tan 2x = \sqrt{3}$

$2x = \frac{\pi}{3}, \frac{4\pi}{3}$

$x = \frac{\pi}{6}, \frac{2\pi}{3}$

$\therefore x = \frac{\pi}{2}, \frac{\pi}{6}, \frac{2\pi}{3}$ (1) Done well

b) $x + 2y\sqrt{x^2+1}\frac{dy}{dx} = 0$

$2y\sqrt{x^2+1}\frac{dy}{dx} = -x$

$\int 2y\,dy = \frac{1}{2}\int \frac{-2x}{\sqrt{x^2+1}}\,dx$ (1)

$y^2 = -\frac{1}{2} \times 2(x^2+1)^{\frac{1}{2}} + C$

when $y=-2$, $x=0$

$(-2)^2 = -(0^2+1)^{\frac{1}{2}} + C$

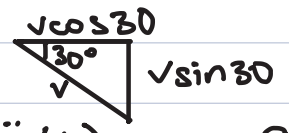
$\therefore C = 5$

$\therefore y^2 = 5 - \sqrt{x^2+1}$ (1)

$\therefore y = -\sqrt{5 - \sqrt{(x^2+1)}}$ (1)

as when $x=0$, $y=-2$

A lot of students left it as y^2 or $y = \pm\ldots$.

c) i) 

$\ddot{\underset{\sim}{r}}(t) = -9.8\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = -9.8t\,\underset{\sim}{j} + \underset{\sim}{c_1}$

when $t = 0$,

$\dot{\underset{\sim}{r}}(0) = v\cos 30\,\underset{\sim}{i} - v\sin 30\,\underset{\sim}{j}$ (1)

$\therefore \dot{\underset{\sim}{r}}(0) = 0 + \underset{\sim}{c_1}$

$\therefore \underset{\sim}{c_1} = v\cos 30\,\underset{\sim}{i} - v\sin 30\,\underset{\sim}{j}$

$\therefore \dot{\underset{\sim}{r}}(t) = v\cos 30\,\underset{\sim}{i} - (9.8t + v\sin 30)\,\underset{\sim}{j}$

$= \frac{\sqrt{3}}{2}v\,\underset{\sim}{i} - \left(9.8t + \frac{v}{2}\right)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \frac{\sqrt{3}}{2}vt\,\underset{\sim}{i} - \left(4.9t^2 + \frac{v}{2}t\right)\underset{\sim}{j} + \underset{\sim}{c_2}$

when $t=0$, $\underset{\sim}{r}(0) = \underset{\sim}{0}$ (1)

$\therefore \underset{\sim}{r}(0) = \underset{\sim}{c_2}$

$\therefore \underset{\sim}{r}(t) = \frac{\sqrt{3}}{2}vt\,\underset{\sim}{i} - \left(4.9t^2 + \frac{v}{2}t\right)\underset{\sim}{j}$

Need to show what the velocity and displacement is initially.

$\ddot{\underset{\sim}{r}}(t) = -9.8\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = v\cos 30\,\underset{\sim}{i} - (9.8t + v\sin 30)\,\underset{\sim}{j}$

$r(t) = \frac{\sqrt{3}}{2}vt\,\underset{\sim}{i} - \left(4.9t^2 + \frac{v}{2}t\right)\underset{\sim}{j}$

this is not enough to show your understanding.

ii) $\underset{\sim}{r_B}(t) = \underset{\sim}{r}(t)$

equating components

$\therefore \frac{\sqrt{3}}{2}vt = 30 - 10t$

$\frac{\sqrt{3}}{2}vt + 10t = 30$

$t\left(\frac{\sqrt{3}}{2}v + 10\right) = 30$

$t = \frac{30}{\frac{\sqrt{3}}{2}v + 10}$ sub. into $\underset{\sim}{j}$ component

$$4.9t^2 + \frac{v}{2}t = 20$$

$$4.9\left(\frac{30}{\frac{\sqrt{3}}{2}v + 10}\right)^2 + \frac{v}{2}\left(\frac{30}{\frac{\sqrt{3}}{2}v + 10}\right) = 20$$ ① equation with one pronumeral

$$\frac{4410}{\left(\frac{\sqrt{3}}{2}v + 10\right)^2} + \frac{15v}{\frac{\sqrt{3}}{2}v + 10} = 20$$

$$4410 + 15v\left(\frac{\sqrt{3}}{2}v + 10\right) = 20\left(\frac{\sqrt{3}}{2}v + 10\right)^2$$

$$4410 + \frac{15\sqrt{3}}{2}v^2 + 150v = 20\left(\frac{3}{4}v^2 + 10\sqrt{3}v + 100\right)$$

$$4410 + \frac{15\sqrt{3}}{2}v^2 + 150v = 15v^2 + 200\sqrt{3}v + 2000$$

$$\left(\frac{15\sqrt{3}}{2} - 15\right)v^2 + (150 - 200\sqrt{3})v + 2410 = 0$$

$$v = \frac{-(150 - 200\sqrt{3}) \pm \sqrt{(150 - 200\sqrt{3})^2 - 4\left(\frac{15\sqrt{3}}{2} - 15\right)(2410)}}{2\left(\frac{15\sqrt{3}}{2} - 15\right)}$$

$$= -108.7613\ldots, 11.026277\ldots$$

$$\therefore \quad v = 11.026\ldots$$

$$= 11 \text{ ms}^{-1} \text{ (nearest integer)}$$ ①

OR

$$\frac{\sqrt{3}vt}{2} = 30 - 10t$$

$$\frac{vt}{2} = \frac{30 - 10t}{\sqrt{3}} \quad -①$$

$$4.9t^2 + \frac{vt}{2} = 20 \quad -②$$

Sub. ① into ②

$$4.9t^2 + \frac{30 - 10t}{\sqrt{3}} = 20$$ ① equation with one pronumeral

$$4.9\sqrt{3}t^2 + 30 - 10t - 20\sqrt{3} = 0$$

$$4.9\sqrt{3}t^2 - 10t + 30 - 20\sqrt{3} = 0$$

$$t = \frac{10 \pm \sqrt{(-10)^2 - 4(4.9\sqrt{3})(30 - 20\sqrt{3})}}{2(4.9\sqrt{3})}$$

$$= 1.53460\ldots, -0.3563366\ldots$$

$$v = 11.026277\ldots$$

$$= 11 \text{ ms}^{-1} \text{ (nearest integer)}$$ ①

Suggested Solutions	Marks	Marker's Comments

(a) For the following, you may assume the identity

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta},$$

where $\theta \in \mathbb{R}$, $\cos\theta \neq 0$ and $1 - 3\tan^2\theta \neq 0$.

(i) Consider the cubic equation 2

$$x^3 - \sqrt{3}x^2 - 3x + \frac{1}{\sqrt{3}} = 0.$$

By letting $x = \tan\theta$, solve the equation, leaving your answers in exact form.

Substitute!

$(\tan\theta)^3 - \sqrt{3}(\tan\theta)^2 - 3(\tan\theta) + \frac{1}{\sqrt{3}} = 0$

$3\tan\theta - \tan^3\theta = \frac{1}{\sqrt{3}} - \sqrt{3}\tan^2\theta$

$3\tan\theta - \tan^3\theta = \frac{1}{\sqrt{3}}(1 - 3\tan^2\theta)$

$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \frac{1}{\sqrt{3}}$

(given) $\tan 3\theta = \frac{1}{\sqrt{3}}$ (1) ✓ ✓

$3\theta = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$

$\theta = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$ (1)

Many students gave in general form! Probably learning this from tutoring!! Syllabus has changed!! Keep up to date and check!! 2026-27 - again New Syllabus!!

General form is NOT in syllabus! But still awarded the mark/s. Also want to flag that cycle for tan graphs are different to sin & cos graphs!! So be careful!!

(ii) Hence find the value of 1

$$\tan^2\frac{\pi}{18} + \tan^2\frac{7\pi}{18} + \tan^2\frac{13\pi}{18}.$$

Let $\alpha = \frac{\pi}{18}$, $\beta = \frac{7\pi}{18}$, $\gamma = \frac{13\pi}{18}$

$x^3 - \sqrt{3}x^2 - 3x + \frac{1}{\sqrt{3}} = 0$

$\alpha + \beta + \gamma = \frac{-(-\sqrt{3})}{1} = \sqrt{3}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{-3}{1} = -3$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= (\sqrt{3})^2 - 2(-3)$

$= 9$ (1)

No CFE!

Suggested Solutions	Marks	Marker's Comments

(iii) Hence, or otherwise, find the value of 1

$$\tan^3\frac{\pi}{18}+\tan^3\frac{7\pi}{18}+\tan^3\frac{13\pi}{18}.$$

$x^3-\sqrt{3}x^2-3x+\frac{1}{\sqrt{3}}=0$

$\alpha^3-\sqrt{3}\alpha^2-3\alpha+\frac{1}{\sqrt{3}}=0$ —i

$\beta^3-\sqrt{3}\beta^2-3\beta+\frac{1}{\sqrt{3}}=0$ —ii

$\gamma^3-\sqrt{3}\gamma^2-3\gamma+\frac{1}{\sqrt{3}}=0$ — iii

i + ii + iii

$\alpha^3+\beta^3+\gamma^3-\sqrt{3}\alpha^2-\sqrt{3}\beta^2-\sqrt{3}\gamma^2-3\alpha-3\beta-3\gamma+3\left(\frac{1}{\sqrt{3}}\right)=0$

$\alpha^3+\beta^3+\gamma^3-\sqrt{3}(\alpha^2+\beta^2+\gamma^2)-3(\alpha+\beta+\gamma)+3\left(\frac{1}{\sqrt{3}}\right)=0$

$\alpha^3+\beta^3+\gamma^3=\sqrt{3}(\alpha^2+\beta^2+\gamma^2)+3(\alpha+\beta+\gamma)-3\left(\frac{1}{\sqrt{3}}\right)$

from pt ii) $=\sqrt{3}(9)+3(\sqrt{3})-3\left(\frac{1}{\sqrt{3}}\right)$

$=11\sqrt{3}$ or 19.05 No CFE!

(b) The path of a moving particle is given by the position vector 4

$$\underset{\sim}{r}(t)=\left(5\cos t-4\cos\frac{5t}{2}\right)\underset{\sim}{i}+\left(5\sin t-4\sin\frac{5t}{2}\right)\underset{\sim}{j},\qquad t\geq 0,$$

where t is measured in seconds and position is measured in metres.

Find the first interval of time after $t = 0$ during which the acceleration has a component directly opposing the motion of the particle.

Poorly done:

Breaking down the question.

1) Direction of motion: Always defined by the velocity vector, $v(t)$.
2) Opposing the motion: If acceleration has a component directly opposing motion, ie particle is slowing down.
3) Two vectors oppose each other when the dot product is negative. ie $v(t).a(t)<0$
4) Multiply the 'i' components, and the 'j' components, then solve.

$r(t)=\left(5\cos t-4\cos\frac{5t}{2}\right)\underset{\sim}{i}+\left(5\sin t-4\sin\frac{5t}{2}\right)\underset{\sim}{j}$

$v(t)=\left(-5\sin t-4\left(-\sin\frac{5t}{2}\right)\cdot\frac{5}{2}\right)\underset{\sim}{i}+\left(5\cos t-4\cos\left(\frac{5t}{2}\right)\frac{5}{2}\right)\underset{\sim}{j}$

$=\left(-5\sin t+10\sin\frac{5t}{2}\right)\underset{\sim}{i}+\left(5\cos t-10\cos\frac{5t}{2}\right)\underset{\sim}{j}$

$a(t)=\left(-5\cos t+10\cos\frac{5t}{2}\cdot\frac{5}{2}\right)\underset{\sim}{i}+5(-\sin t)-10\left(-\sin\frac{5t}{2}\cdot\frac{5}{2}\right)\underset{\sim}{j}$

$=\left(-5\cos t+25\cos\frac{5t}{2}\right)\underset{\sim}{i}+\left(-5\sin t+25\sin\frac{5t}{2}\right)\underset{\sim}{j}$

① differentiating correctly to find $v(t)$ & $a(t)$

Say you are driving a car! You are driving with a $v(t)$. You see the traffic light. You have to slow down ie deceleration. ie $v(t)$ opposing $a(t)$... So to stop it has to be $v(t).(a$

MATHEMATICS Extension 1 : Question 16

Suggested Solutions	Marks	Marker's Comments

Acceleration directly opposing motion

$a(t) \cdot v(t) < 0$

Multiply the 'i' components:

$= \left(-5\cos t + 25\cos\frac{5t}{2}\right)\left(-5\sin t + 10\sin\frac{5t}{2}\right)\underline{i}$

$= 25\sin t\cos t - 50\sin\frac{5t}{2}\cos t - 125\sin t\cos\frac{5t}{2} + 250\sin\frac{5t}{2}\cos\frac{5t}{2}$ — i

Multiply the 'j' components:

$= \left(5\sin t + 25\sin\frac{5t}{2}\right)\left(5\cos t - 10\cos\frac{5t}{2}\right)\underline{j}$

$= -25\sin t\cos t + 50\sin t\cos\frac{5t}{2} + 125\sin\frac{5t}{2}\cos t - 250\sin\frac{5t}{2}\cos\frac{5t}{2}$ — ii

i + ii (Highlighted in yellow cancels off)

$-50\sin\frac{5t}{2}\cos t + 125\sin\frac{5t}{2}\cos t - 125\sin t\cos\frac{5t}{2} + 50\sin t\cos\frac{5t}{2}$

$75\sin\frac{5t}{2}\cos t - 75\sin t\cos\frac{5t}{2} < 0$ — iii

$75\left(\sin\frac{5t}{2}\cos t - \sin t\cos\frac{5t}{2}\right) < 0$

$A = \frac{5t}{2} \quad B = t$ (Using compound angle)

$75\left(\sin\left(\frac{5t}{2} - t\right)\right) < 0$ — (1)

$75\sin\frac{3t}{2} < 0$

$\frac{3t}{2} < 0, 180, 360$

$3t < 0, 360, 720$

$t < 0, 120, 240$

After $t = 0$

$120 < t < 240$

$\frac{2\pi}{3} < t < \frac{4\pi}{3}$ — (1) for correct interval

(1) Multiplying i components and j components to show (iii)

Only 2 students got the interval correct!

Another 4 students got lucky! to get 3 marks!

Most students got 1 mark to differentiate correctly for $v(t)$ & $a(t)$!

Question 17

(a) (i) Since $Y = 1$ precisely when

$$|X - \mu| \geq k\sigma,$$

and $Y = 0$ otherwise,

$$\begin{aligned} E(Y) &= 1 \cdot P(|X - \mu| \geq k\sigma) + 0 \cdot P(|X - \mu| < k\sigma) \\ &= \boxed{P(|X - \mu| \geq k\sigma)}. \end{aligned}$$

1 mark if all working shown

(ii) We consider the two possible values of Y.

If $Y = 0$, then

$$(X - \mu)^2 - k^2\sigma^2 Y = (X - \mu)^2 \geq 0.$$

If $Y = 1$, then

1 mark if all working shown

$$|X - \mu| \geq k\sigma.$$

Hence

$$(X - \mu)^2 \geq k^2\sigma^2,$$

and therefore

$$(X - \mu)^2 - k^2\sigma^2 Y = (X - \mu)^2 - k^2\sigma^2 \geq 0.$$

Thus, for all possible values of Y,

$$\boxed{(X - \mu)^2 - k^2\sigma^2 Y \geq 0}.$$

(iii) By part (ii), the random variable

$$(X - \mu)^2 - k^2\sigma^2 Y$$

is non-negative. Therefore

$$E\left((X - \mu)^2 - k^2\sigma^2 Y\right) \geq 0.$$

1 mark or equivalent expression

Using linearity of expectation,

$$E\left((X - \mu)^2\right) - k^2\sigma^2 E(Y) \geq 0.$$

Since

$$E\left((X - \mu)^2\right) = \sigma^2,$$

we obtain

$$\sigma^2 - k^2\sigma^2 E(Y) \geq 0.$$

1 mark

As $\sigma > 0$,

$$1 - k^2 E(Y) \geq 0,$$

and hence

$$E(Y) \leq \frac{1}{k^2}.$$

Using part (i),

$$\boxed{P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}}.$$

(iv) Taking $k = 2$,

$$P(|X - \mu| \geq 2\sigma) \leq \frac{1}{4}.$$

Therefore

$$\begin{aligned} P(\mu - 2\sigma < X < \mu + 2\sigma) &= P(|X - \mu| < 2\sigma) \\ &= 1 - P(|X - \mu| \geq 2\sigma) \\ &\geq 1 - \frac{1}{4} \\ &= \frac{3}{4}. \end{aligned}$$

Hence the minimum proportion guaranteed to lie in

$$(\mu - 2\sigma, \mu + 2\sigma)$$

is

$$\boxed{\frac{3}{4} = 75\%}.$$ **1 mark**

(b) (i) By the Fundamental Theorem of Calculus,

$$F'(x) = e^x + [F(x)]^2.$$

Since

$$e^x > 0$$

and

$$[F(x)]^2 \geq 0,$$

we have

$$F'(x) > 0$$ **1 mark**

for all $x \in [0, a)$.

Therefore F is strictly increasing, so F is one-to-one and hence invertible.

(ii) Since $G = F^{-1}$,

$$F(G(x)) = x.$$

Differentiating,

$$F'(G(x))G'(x) = 1.$$

Hence

$$\begin{aligned} G'(x) &= \frac{1}{F'(G(x))} \\ &= \frac{1}{e^{G(x)} + [F(G(x))]^2} \\ &= \frac{1}{e^{G(x)} + x^2}. \end{aligned}$$ **1 mark**

Also,

$$F(0) = 0.$$

Since F is strictly increasing on $[0, a)$, its inverse satisfies

$$G(x) \geq 0$$

for $x \geq 0$. Therefore

$$e^{G(x)} \geq 1.$$ **1 mark**

Hence

$$e^{G(x)} + x^2 \geq 1 + x^2,$$

so

$$\boxed{G'(x) \leq \frac{1}{1 + x^2}}.$$

(iii) Since $F(0) = 0$, we have

$$G(0) = 0.$$

Therefore, for $x \geq 0$,

$$\begin{aligned} G(x) &= G(x) - G(0) \\ &= \int_0^x G'(t)\, dt \\ &\leq \int_0^x \frac{1}{1 + t^2}\, dt \\ &= \tan^{-1} x. \end{aligned}$$ **1 mark**

Since the range of F is $[0, \infty)$, the domain of G is $[0, \infty)$. Also, as $x \to \infty$, the corresponding values $G(x)$ approach the right-hand endpoint a of the domain of F. Thus

$$\lim_{x \to \infty} G(x) = a.$$

Hence

$$a = \lim_{x\to\infty} G(x) \le \lim_{x\to\infty} \tan^{-1} x = \boxed{\frac{\pi}{2}}.$$ **1 mark**

(a)

(i) Multiplying by 1 and 0 must be shown to get 1 mark.

This part is done well by most students.

(ii) No mark is awarded if the inequality is proved only for $y = 1$.

Not many attempted (iii) and (iv). Some had trouble recognising $E[(X-\mu)^2] = \sigma^2$, consequently couldn't prove the required inequality

(b)

(i) Overall, this part was completed well. However, many students incorrectly applied the Fundamental Theorem of Calculus and stated that $F'(x) = e^t + [F(t)]^2$.

Fundamental Theorem of Calculus in differential form:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

Fundamental Theorem of Calculus in integral form:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Note that $F'(x) \ge 0$ does not necessarily mean that $F(x)$ is strictly increasing. The required condition is $F'(x) > 0$.

(ii) Some students had difficulty finding the derivative of the inverse function

(iii)

It is not recommended to use $\int_0^{\infty} G(x)\,dx$

Instead, use $\int_0^{\infty} G(x)\,dx = \lim_{b\to\infty} \int_0^{b} G(x)\,dx$

$$\begin{aligned}\int_0^{\infty} G(x)\,dx &= \lim_{b\to\infty} [F(x)]_0^b \\ &= \lim_{b\to\infty} \big(F(b) - F(0)\big)\end{aligned}$$

For example, it is incorrect to write $F(\infty) = e^{-\infty} = 0$ as ∞ itself is undefined.