# NORMANHURST BOYS HIGH SCHOOL

# MATHEMATICS EXTENSION 1

2026 Year 12 Course Assessment Task 4 (Trial Examination)
Friday 7th August, 2026

**General instructions**

- Working time – 2 hours.
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.

**SECTION I**

- Mark your answers on the answer grid provided (on page 13)

**SECTION II**

- Answer the questions in the Section II Writing Booklet. Extra writing booklets are available.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

**NESA STUDENT #:** ..... **# BOOKLETS USED:** .....

**Class** (please ✔)

- ◯ 12MAX.2 – Ms Lee
- ◯ 12MAX.3 – Ms Chaussivert
- ◯ 12MAX.4 – Ms Kim
- ◯ 12MXX.1 – Ms Ham
- ◯ 12MXX.2 – Mr Ho

Marker's use only.

| QUESTION | 1-10 | 11 | 12 | 13 | 14 | Total | % |
|---|---|---|---|---|---|---|---|
| MARKS | /10 | /17 | /16 | /14 | /13 | /70 | |

# Section I

**10 marks**
**Attempt Question 1 to 10**
**Allow approximately 15 minutes for this section**

Mark your answers on the answer grid provided (labelled as page 13).

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**Questions** **Marks**

**1.** What is the domain of the function $f(x) = \sin^{-1}\left(\frac{4}{x}\right)$? **1**

(A) $[-4, 4]$

(B) $(-\infty, -4] \cup [4, \infty)$

(C) $\left[-\frac{8}{\pi}, \frac{8}{\pi}\right]$

(D) $\left(-\infty, -\frac{8}{\pi}\right] \cup \left[\frac{8}{\pi}, \infty\right)$

**2.** Consider the function $f(x) = \sqrt{4 - x}$. **1**
Which of the following is the correct inverse function, $f^{-1}(x)$?

(A) $f^{-1}(x) = 4 - x^2$

(B) $f^{-1}(x) = 4 - x^2,\ x \geq 0$

(C) $f^{-1}(x) = 4 - x^2,\ x \leq 0$

(D) $f^{-1}(x) = 4 - x^2,\ x \leq 4$

**3.** Two different soccer teams each have 18 players. Each team is to choose one of its players as captain. How many ways can this be done? **1**

(A) $\binom{36}{2}$

(B) ${}^{36}\mathrm{P}_2$

(C) $\binom{18}{2} \times 2$

(D) $18^2$

**4.** What is the solution to the inequality $3x + 2 < |2x - 1|$? **1**

(A) $x < -\frac{1}{5}$

(B) $x < -3$

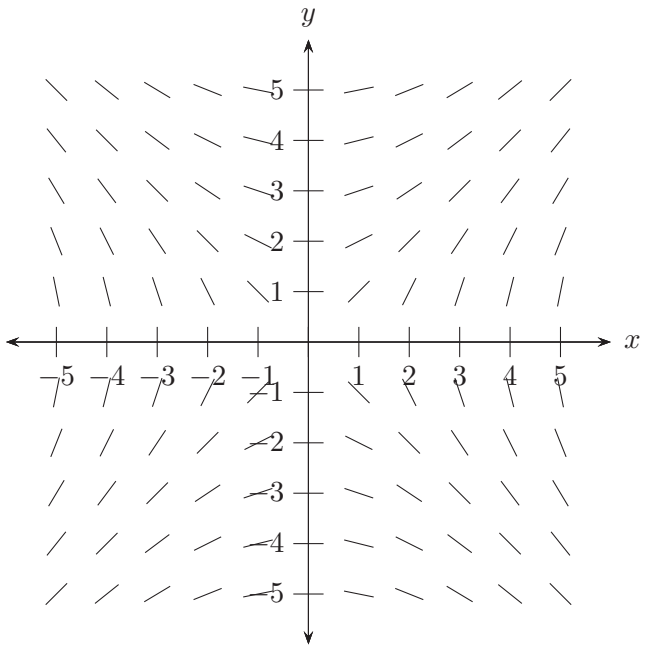
(C) $-3 < x < \frac{1}{5}$

(D) $x < -\frac{1}{5}$ or $x > 3$

**5.** There are twelve red, nine blue, and seven green socks in a drawer. What is the minimum number of socks that need to be selected at random to ensure that you have a pair of socks of the same colour? **1**

(A) 2 (B) 3 (C) 4 (D) 5

**6.** Which differential equation is represented by the following slope field? **1**




(A) $\dfrac{dy}{dx} = -\dfrac{x}{y}$

(B) $\dfrac{dy}{dx} = -\dfrac{y}{x}$

(C) $\dfrac{dy}{dx} = \dfrac{y}{x}$

(D) $\dfrac{dy}{dx} = \dfrac{x}{y}$

**7.** If $|\underset{\sim}{p} + \underset{\sim}{q}| = |\underset{\sim}{p}| + |\underset{\sim}{q}|$ and $\underset{\sim}{p}, \underset{\sim}{q} \neq \underset{\sim}{0}$, which of the following must be true? **1**

(A) $\underset{\sim}{p}$ is parallel to $\underset{\sim}{q}$

(B) $|\underset{\sim}{p}| = |\underset{\sim}{q}|$

(C) $\underset{\sim}{p} = \underset{\sim}{q}$

(D) $\underset{\sim}{p}$ is perpendicular to $\underset{\sim}{q}$

**8.** A curve has the equation: **1**

$$\frac{(y-2)^2}{9} - \frac{(x+1)^2}{4} = 1$$

Which of the following expresses the curve in parametric form?

(A) $\begin{cases} x = 2\sec\theta - 1 \\ y = 3\tan\theta + 2 \end{cases}$

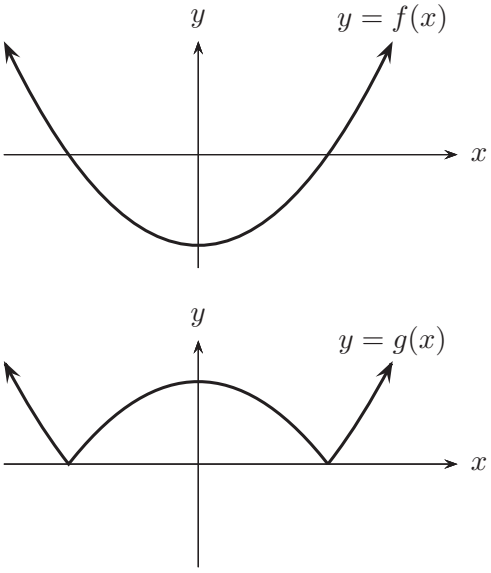
(B) $\begin{cases} x = 2\sin\theta - 1 \\ y = 3\cos\theta + 2 \end{cases}$

(C) $\begin{cases} x = 2\tan\theta - 1 \\ y = 3\sec\theta + 2 \end{cases}$

(D) $\begin{cases} x = 4\sec\theta - 1 \\ y = 3\tan\theta + 2 \end{cases}$

**Examination continues overleaf...**

**9.** The graph of $y = f(x)$ is sketched below. **1**




Which of the following transformations of the graph of $y = f(x)$ results in the graph of $y = g(x)$?

(A) $|y| = f(x)$

(B) $y = f(|x|)$

(C) $|y| = f(|x|)$

(D) $y = |f(|x|)|$

**10.** Which of the following is equal to $\tan^{-1} x - \tan^{-1} \frac{1}{x}$? **1**

(A) $\tan^{-1}\left(\dfrac{x^2-2}{2x}\right)$

(B) $\tan^{-1}\left(\dfrac{x^2-2}{x}\right)$

(C) $\tan^{-1}\left(\dfrac{x^2-1}{2x}\right)$

(D) $\tan^{-1}\left(\dfrac{x^2-1}{x}\right)$

# Section II

**60 marks**
**Attempt Questions 11 to 14**
**Allow approximately 1 hour and 45 minutes for this section.**

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (17 marks) **Marks**

(a) i. Write the expression $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$. **2**

ii. Hence, or otherwise, solve $\sqrt{3}\sin x + \cos x = \sqrt{3}$ for $0 \leq x \leq 2\pi$. **2**

(b) Solve for $x$: **3**

$$\frac{1}{x-1} \geq 3$$

(c) The polynomial $x^3 - 2x^2 + kx + 24 = 0$ has roots $\alpha, \beta$, and $\gamma$.

i. Find the value of $\alpha + \beta + \gamma$. **1**

ii. Find the value of $\alpha\beta\gamma$. **1**

iii. It is known that two of the roots are equal in magnitude but opposite in sign. Find the third root and hence find the value of $k$. **2**

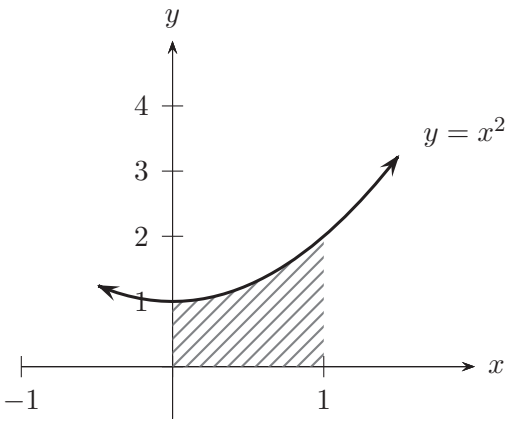
(d) Eight people are to be seated around a circular table. In how many ways can they be seated if:

i. There are no restrictions? **1**

ii. The two tallest people must sit next to each other? **1**

iii. The two tallest people must not sit next to each other? **1**

(e) The region bounded by the graph $y = x^2 + 1$ and the $x$ axis between $x = 0$ and $x = 1$ is rotated around the $x$ axis to form a solid. **3**




Calculate the exact volume of this solid.

**Question 12** (16 marks) **Marks**

(a) It is given that $\theta = \cos^{-1}\left(\frac{12}{13}\right) + \tan^{-1}\left(\frac{3}{4}\right)$. **3**

Find the exact value of $\sin\theta$ without the use of a calculator.

(b) i. Show that $\dfrac{d}{dx}\left(\tan^3 x\right) = 3\sec^4 x - 3\sec^2 x$ **2**

ii. Hence, or otherwise, find $\displaystyle\int \sec^4 x \, dx$ **2**

(c) Using the substitution $u = \tan x$, find the exact value of **3**

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 + \tan^2 x} \, dx$$

(d) i. Use the binomial theorem to find the constant term in the expansion of **2**

$$\left(2x^3 - \frac{1}{x}\right)^{12}$$

ii. For what values of $n$ does $\left(2x^3 - \dfrac{1}{x}\right)^n$ have a non-zero constant term? **1**

(e) Given that $\sin x = a$, where $x \in \left(\dfrac{3\pi}{2}, 2\pi\right)$, show that **3**

$$\cos\left(\frac{x}{2}\right) = -\frac{\sqrt{1 + \sqrt{1 - a^2}}}{\sqrt{2}}$$

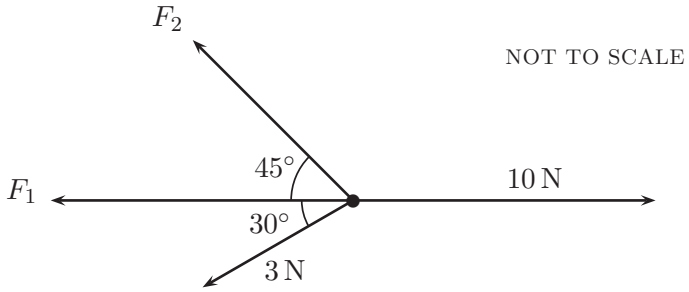
**Examination continues overleaf...**

**Question 13** (14 marks) **Marks**

(a) The diagram below shows a stationary particle acted on by four forces. **3**

Two of the forces have magnitudes 3 N and 10 N, as shown below. Force $F_1$ acts horizontally to the left, and the angle between $F_1$ and $F_2$ is 45°.

The magnitudes of all forces are measured in newtons.




Find the magnitudes of $F_1$ and $F_2$.

(b) A tank initially contains 300 grams of salt that is dissolved in 50 L of water. A solution containing 15 grams of salt per litre of water is poured into the tank at a rate of 2 litres per minute and the mixture in the tank is kept well stirred.

At the same time, 2 litres of the mixture flows out of the tank per minute.

i. Show that **1**

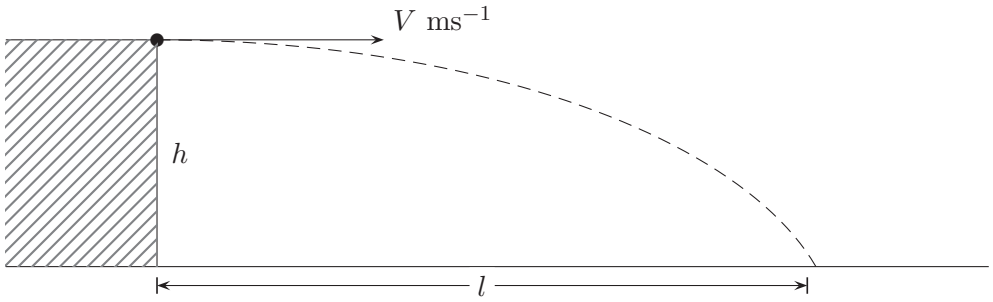
$$\frac{dm}{dt} = 30 - \frac{m}{25}$$

where $m$ represents the mass, in grams, of salt in the tank at time $t$ minutes.

ii. Find the expression of $m$ in terms of $t$. **3**

iii. Find the eventual concentration of salt in the tank. **1**

(c) A projectile is fired horizontally from the top of a cliff $h$ metres high at an initial speed of $V$ ms$^{-1}$.




The projectile strikes the water $l$ metres from the base of the cliff.
Let $g$ be the acceleration due to gravity and assume air resistance is negligible.

i. Show the projectile hits the water when $t = \sqrt{\dfrac{2h}{g}}$. **3**

ii. If $l$ equals twice the height of the cliff, at what angle does the projectile hit the water? **2**

iii. Show the speed at which the projectile hits the water is $2\sqrt{hg}$ ms$^{-1}$. **1**

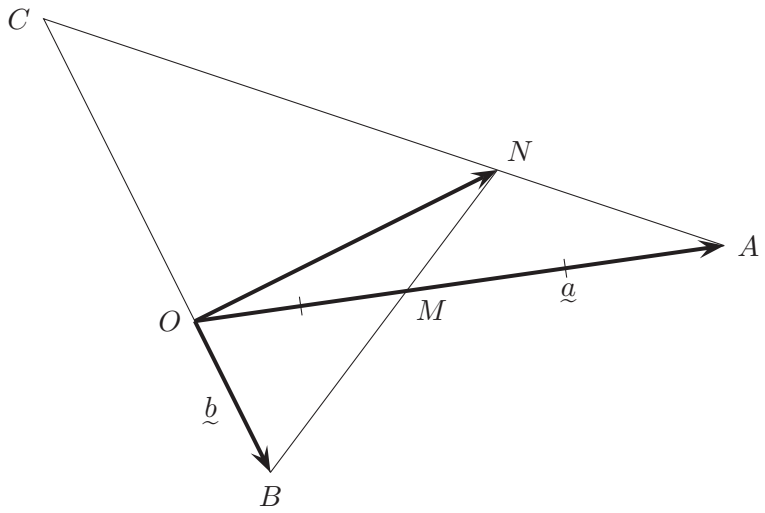
**Examination continues overleaf...**

**Question 14** (13 marks) **Marks**

(a) Prove by mathematical induction that, for $n \geq 1$, **4**

$$\sin\theta + \sin(3\theta) + ... + \sin\Big((2n-1)\ \theta\Big) = \frac{\sin^2(n\theta)}{\sin\theta}$$

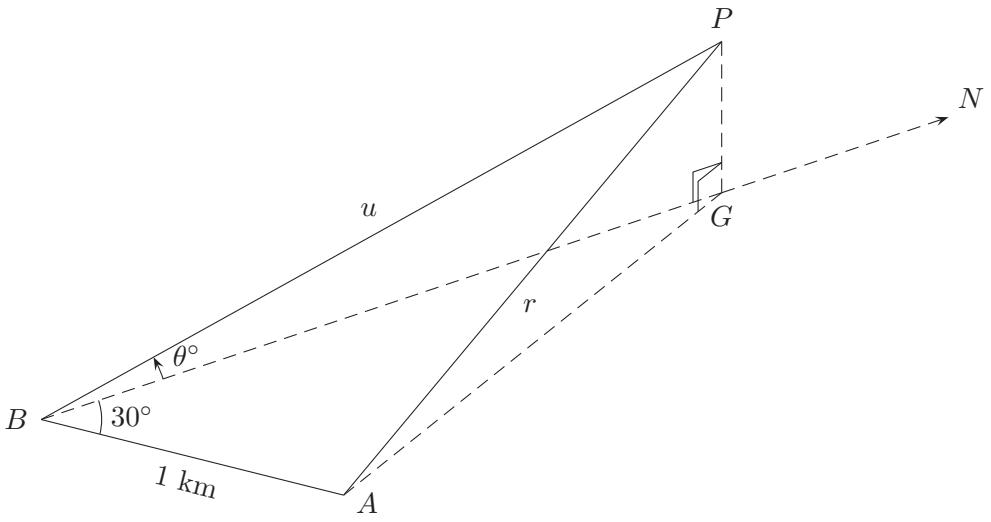
(b) In the diagram below, $O$ divides $BC$ in the ratio $1:3$ and $M$ is the midpoint of $OA$. **4**




If $A$ and $B$ have position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ from $O$ respectively, and $BM$ is produced to meet $AC$ at $N$, find the position vector of $N$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(c) A plane $P$ takes off from a point $B$. It flies due north at a constant angle $\theta$ to the horizontal. An observer is located at $A$, 1 km from $B$, at a bearing of 030° from $B$.

Let $u$ kilometres be the distance from $B$ to the plane and let $r$ kilometres be the distance from the observer to the plane. The point $G$ is on the ground directly below the plane.




i. Show that $r = \sqrt{1 + u^2 - \sqrt{3}u\cos\theta}$ **3**

ii. The plane is travelling at a constant speed of 300 kilometres per hour. At what rate, in terms of $\theta$, is the distance of the plane from the observer changing 4 minutes after take off? **2**

**End of paper.**

Normanhurst Boys High School

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Student Number

2026 Year 12 Course Assessment Task 4

# Mathematics Extension 1

## Section II Writing Booklet

**Instructions**

- Use this writing booklet for Section II ONLY
- Follow the instructions inside the booklet as to where you should write your answers
- Write the number of this booklet and the total number of booklets that you have used for this section (eg: **1** of **3**)

**Answer Question 11 on pages 2–5.**

| 1 | 1 |
|---|---|

Question Number

Multiple choice : (1) B (2) B (3) D (4) A (5) C
(6) D (7) A (8) C (9) D (10) C

11(a) i) $\sqrt{3}\sin x + \cos x \equiv R\sin(x+\alpha)$

$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore R\cos\alpha = \sqrt{3}$ (1) & $R\sin\alpha = 1$ (2)

(2) ÷ (1) $\tan\alpha = \frac{1}{\sqrt{3}} \therefore \alpha = \frac{\pi}{6}$ ✓

sub $\alpha = \frac{\pi}{6}$ into (2) : $R\sin\left(\frac{\pi}{6}\right) = 1$

$\frac{R}{2} = 1$

$R = 2$ ✓

$\therefore \sqrt{3}\sin x + \cos x = 2\sin\left(x + \frac{\pi}{6}\right)$

ii) $2\sin\left(x+\frac{\pi}{6}\right) = \sqrt{3}$

$\sin\left(x+\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{2\pi}{3}$ ✓ ....

$x = \frac{\pi}{6}, \frac{\pi}{2}$ ✓ $\left(0 \le x \le 2\pi\right)$



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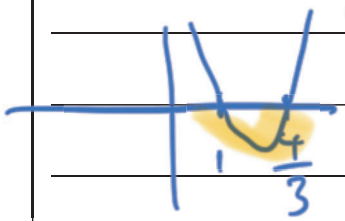
b) $(x-1)^2 \times \frac{1}{x-1} \geq 3 \times (x-1)^2$ ✓ $x \neq 1$

$x - 1 \geq 3(x-1)^2$

$3(x-1)^2 - (x-1) \leq 0$

$(x-1)[3(x-1)-1] \leq 0$

$(x-1)(3x-4) \leq 0$ ✓




$\therefore 1 < x \leq \frac{4}{3}$ . ✓

c) i) $\alpha + \beta + \gamma = -\frac{b}{a} = 2$ ✓

ii) $\alpha\beta\gamma = -\frac{d}{a} = -24$ ✓

iii) let the roots be $\alpha, -\alpha, \beta$

$\alpha - \alpha + \beta = 2$

$\beta = 2$ $\therefore$ the third root is 2. ✓

$\alpha \times -\alpha \times 2 = -24$

$\alpha^2 = 12$

$\alpha = 2\sqrt{3}$

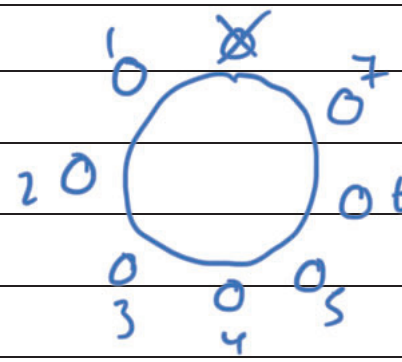
$-\alpha^2 + \alpha\beta - \alpha\beta = k$

$\therefore -(2\sqrt{3})^2 = k$

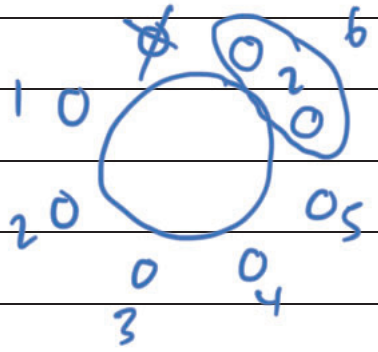
$k = -12$ ✓



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d) i)  $7! = 5040$ ways ✓

ii) $6! \times 2 = 1440$ ways ✓



iii) $5040 - 1440$
$= 3600$ ways. ✓

e) $V = \pi \int_0^1 (x^2+1)^2 \, dx$ ✓ $= \pi \int_0^1 x^4 + 2x^2 + 1 \, dx$

$= \pi \left[ \frac{x^5}{5} + \frac{2x^3}{3} + x \right]_0^1$ ✓

$= \pi \left( \frac{1}{5} + \frac{2}{3} + 1 - 0 \right)$

$\frac{28\pi}{15}$ units$^3$ ✓



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If you require more space to Answer Question 11, you may ask for an extra writing booklet.
If you have used an extra writing booklet for Question 11, tick here. ☐



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**End of Question 11**

**Go to page 7 for Question 12**



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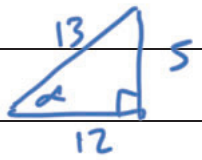
**Answer Question 12 on pages 7–10.**

| 1 | 2 |
|---|---|

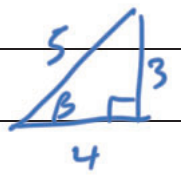
Question Number

12 a) let $\alpha = \cos^{-1}\left(\frac{12}{13}\right)$ and $\beta = \tan^{-1}\left(\frac{3}{4}\right)$ ✓

$\cos\alpha = \frac{12}{13}$




$\tan\beta = \frac{3}{4}$




$\sin\theta = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$= \frac{5}{13}\times\frac{4}{5} + \frac{3}{5}\times\frac{12}{13} = \frac{56}{65}$ ✓

b) i) $\frac{d}{dx}\left[(\tan x)^3\right] = 3(\tan x)^2\sec^2 x$ ✓

$= 3\tan^2 x\sec^2 x$

$= 3(\sec^2 x - 1)\sec^2 x$

$= 3\sec^4 x - 3\sec^2 x$ ✓

ii) $3\int \sec^4 x - \sec^2 x\,dx = \tan^3 x + a$ (part c) ✓

$\int \sec^4 x\,dx - \int \sec^2 x\,dx = \frac{1}{3}\tan^3 x + a$

$\int \sec^4 x\,dx - \tan x + b = \frac{1}{3}\tan^3 x + a$

$\therefore \int \sec^4 x\,dx = \frac{1}{3}\tan^3 x + \tan x + C$ ✓



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c) $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3+\tan^2 x}\,dx$ , when $x=\frac{\pi}{4}$, $u=1$; $x=0$, $u=0$

$= \int_0^1 \frac{\sec^2 x}{3+u^2} \times \frac{du}{\sec^2 x}$ $\quad u=\tan x$, $\frac{du}{dx}=\sec^2 x$

$= \int_0^1 \frac{1}{3+u^2}\,du$ ✓ $\quad \therefore dx = \frac{du}{\sec^2 x}$

$= \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\right]_0^1$ ✓

$= \frac{1}{\sqrt{3}}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}(0)\right)$

$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}$ ✓

d) i) $(2x^3 - x^{-1})^{12}$

$= \sum_{k=0}^{12} \binom{12}{k} (2x^3)^{12-k} (-x^{-1})^k$

$= \sum_{k=0}^{12} \binom{12}{k} 2^{12-k} (-1)^k x^{36-3k} x^{-k}$

typical term: $\binom{12}{k} 2^{12-k} (-1)^k x^{36-4k}$ ✓

independent term: $36-4k=0 \quad \therefore k=9$

$\binom{12}{9} 2^3 (-1)^9 x^0 = -1760.$ ✓



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d) ii) general term of $(2x^3 - x^{-1})^n$ :

$$\binom{n}{k}(2x^3)^{n-k}(-x^{-1})^k$$

$$= \binom{n}{k} 2^{n-k}(-1)^k x^{3n-3k} x^{-k}$$

$$= \binom{n}{k} 2^{n-k}(-1)^k x^{3n-4k}$$

constant term : $3n - 4k = 0$

$$k = \frac{3}{4}n$$

since n and k must be integers, n must be a multiple of 4. ✓

eg $\cos(x) = \cos\left(\frac{x}{2} + \frac{x}{2}\right)$

$= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$

$= 2\cos^2\left(\frac{x}{2}\right) - 1$

$\therefore \sqrt{1-a^2} = 2\cos^2\left(\frac{x}{2}\right) - 1$

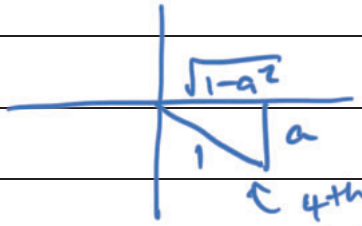
$2\cos^2\left(\frac{x}{2}\right) = \sqrt{1-a^2} + 1$

$\cos^2\left(\frac{x}{2}\right) = \frac{1+\sqrt{1-a^2}}{2}$

$\cos\left(\frac{x}{2}\right) = \pm\frac{\sqrt{1+\sqrt{1-a^2}}}{\sqrt{2}}$ ✓

$\therefore \cos\left(\frac{x}{2}\right) = -\frac{\sqrt{1+\sqrt{1-a^2}}}{\sqrt{2}}$

$\sin x = a$




4th quad. ∴ cos +ve

$\therefore \cos x = \sqrt{1-a^2}$ ✓

if $x \in \left(\frac{3\pi}{2}, 2\pi\right)$

$\frac{x}{2} \in \left(\frac{3\pi}{4}, \pi\right)$

$\therefore \cos\left(\frac{\pi}{2}\right)$ is negative ✓



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**End of Question 12**

**Go to page 11 for Question 13**



Office Use Only - Do NOT write anything, or make any marks below this line.

**13 (a)**

Since the particle is stationary, the net force acting on it in both the horizontal and vertical directions must equal zero.

Let the horizontal direction to the right be the positive $x$ -axis, and the vertically upward direction be the positive $y$ -axis.

---

**Vertical Components ($\sum F_y = 0$):**

Resolving the vertical forces:

$$\sum F_y = F_2 \sin(45°) - 3\sin(30°) = 0$$

Substitute the exact trigonometric values $\sin(45°) = \frac{\sqrt{2}}{2}$ and $\sin(30°) = \frac{1}{2}$:

$$F_2\left(\frac{\sqrt{2}}{2}\right) - 3\left(\frac{1}{2}\right) = 0$$

$$F_2\left(\frac{\sqrt{2}}{2}\right) = \frac{3}{2}$$

$$F_2 = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}\ \text{N} \approx 2.12\ \text{N}$$

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**Horizontal Components ($\sum F_x = 0$):**

Resolving the horizontal forces:

$$\sum F_x = 10 - F_1 - F_2\cos(45°) - 3\cos(30°) = 0$$

Rearrange to solve for $F_1$:

$$F_1 = 10 - F_2\cos(45°) - 3\cos(30°)$$

Substitute $F_2\cos(45°) = \left(\frac{3}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{3}{2}$ and $\cos(30°) = \frac{\sqrt{3}}{2}$:

$$F_1 = 10 - \frac{3}{2} - 3\left(\frac{\sqrt{3}}{2}\right)$$

$$F_1 = \frac{17 - 3\sqrt{3}}{2}\ \text{N} \approx 5.90\ \text{N}$$

**13 (b)**

**i. Show that $\frac{dm}{dt} = 30 - \frac{m}{25}$**

1. **Volume of liquid in the tank:**
   - Inflow rate $= 2$ L/min
   - Outflow rate $= 2$ L/min

Since the inflow rate equals the outflow rate, the volume of solution remains constant at $V = 50$ L.

2. **Rate of salt entering the tank (**Rate In**):**

$$\text{Rate In} = 2 \text{ L/min} \times 15 \text{ g/L} = 30 \text{ g/min}$$

3. **Rate of salt leaving the tank (**Rate Out**):**
   - Concentration in tank at time $t = \frac{m}{50}$ g/L

$$\text{Rate Out} = 2 \text{ L/min} \times \frac{m}{50} \text{ g/L} = \frac{m}{25} \text{ g/min}$$

4. **Net rate of change of mass of salt:**

$$\frac{dm}{dt} = \text{Rate In} - \text{Rate Out}$$

$$\frac{dm}{dt} = 30 - \frac{m}{25}$$

**ii. Find the expression of $m$in terms of $t$**

Rewrite the differential equation:

$$\frac{dm}{dt} = \frac{750 - m}{25}$$

Separate the variables:

$$\frac{1}{750 - m}\, dm = \frac{1}{25}\, dt$$

Integrate both sides:

$$\int \frac{1}{750 - m}\, dm = \int \frac{1}{25}\, dt$$

$$-\ln(750 - m) = \frac{t}{25} + C$$

$$\ln(750 - m) = -\frac{t}{25} - C$$

$$750 - m = Ae^{-\frac{t}{25}} \text{ (where } A = e^{-C})$$

$$m = 750 - Ae^{-\frac{t}{25}}$$

Use the initial condition $m(0) = 300$at $t = 0$:

$$300 = 750 - Ae^{0}$$

$$A = 750 - 300 = 450$$

Thus, the expression for $m$in terms of $t$is:

$$m(t) = 750 - 450e^{-\frac{t}{25}}$$

---

**iii. Find the eventual concentration of salt in the tank**

As $t \to \infty$:

$$e^{-\frac{t}{25}} \to 0$$

$$\lim_{t\to\infty} m(t) = 750 - 450(0) = 750 \text{ g}$$

The eventual concentration is:

$$\text{Concentration} = \frac{\text{Mass}}{\text{Volume}} = \frac{750 \text{ g}}{50 \text{ L}} = 15 \text{ g/L}$$

**13(c) part i.**

**1. Set up the equations of motion:**

Let the base of the cliff directly below the launch point be the origin $(0, 0)$.

The initial conditions at $t = 0$are:

- **Initial position:** $x(0) = 0,\ y(0) = h$
- **Initial velocity:** $\dot{x}(0) = V,\ \dot{y}(0) = 0$

Assuming the only force acting on the projectile is gravity (acting downwards) and neglecting air resistance:

$$\ddot{y}(t) = -g$$

---

**2. Find the vertical displacement equation:**

Integrate the vertical acceleration with respect to time $t$to find the vertical velocity:

$$\dot{y}(t) = \int -g\, dt = -gt + C_1$$

Using the initial condition $\dot{y}(0) = 0$:

$$0 = -g(0) + C_1 \implies C_1 = 0$$

$$\dot{y}(t) = -gt$$

Integrate the vertical velocity with respect to time $t$to find the vertical position:

$$y(t) = \int -gt\, dt = -\frac{1}{2}gt^2 + C_2$$

Using the initial condition $y(0) = h$:

$$h = -\frac{1}{2}g(0)^2 + C_2 \implies C_2 = h$$

Thus, the vertical position of the projectile at time $t$is:

$$y(t) = h - \frac{1}{2}gt^2$$

---

**3. Find the time when the projectile hits the water:**

The projectile hits the water when its vertical height is zero ($y = 0$):

$$0 = h - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 = h$$

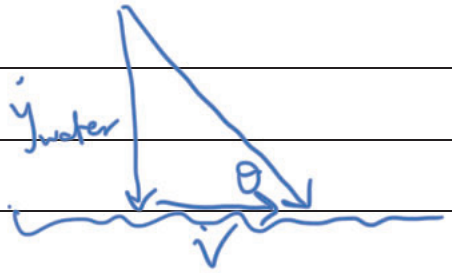
$$t^2 = \frac{2h}{g}$$

Since time $t > 0$:

$$t = \sqrt{\frac{2h}{g}}$$

ii) The particle hits the water when $t = \sqrt{\frac{2h}{g}}$

$\therefore \dot{y}_{water} = -g \times \sqrt{\frac{2h}{g}} = -\sqrt{\frac{g^2 \times 2h}{g}} = -\sqrt{2gh}$ ✓




Now, $x = Vt + C$

Sub $x = 0$, $t = 0 \Rightarrow C = 0$

$\therefore x = Vt$

$\therefore$ when $x = \ell$ : $2h = V \times \sqrt{\frac{2h}{g}}$

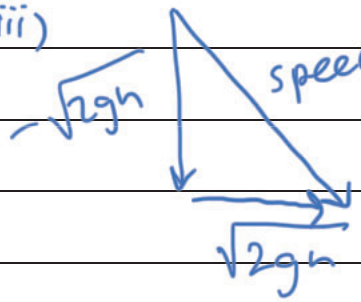
$\therefore V = \sqrt{\frac{g}{2h}} \times 2h$

$= \sqrt{\frac{4gh^2}{2h}} = \sqrt{2gh}$ or ✓

$\therefore \tan\theta = \frac{\sqrt{2gh}}{\sqrt{2gh}} = 1$

$\therefore \theta = 45°$ to the horizontal

iii)




$\text{speed}^2 = (\sqrt{2gh})^2 + (\sqrt{2gh})^2$ ✓

$\text{speed} = \sqrt{2gh + 2gh}$

$= 2\sqrt{gh}$ m/s (speed > 0)

If you require more space to Answer Question 13, you may ask for an extra writing booklet.
If you have used an extra writing booklet for Question 13, tick here. ☐



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**End of Question 13**

**Go to page 17 for Question 14**



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**Answer Question 14 on pages 17–20.**

| 1 | 4 |
|---|---|

Question Number

(14) a) let $P(n)$ be the proposition that

$$\sin\theta + \sin 3\theta + \cdots + \sin\left((2n-1)\theta\right) = \frac{\sin^2(n\theta)}{\sin\theta}$$

Prove $P(1)$ is true

$LHS = \sin\theta \qquad RHS = \frac{\sin^2\theta}{\sin\theta}$

$= \sin\theta$

$LHS = RHS \therefore P(1)$ is true

Assume $P(k)$ is true

$$\sin\theta + \sin 3\theta + \cdots + \left(\sin\left((2k-1)\theta\right)\right) = \frac{\sin^2(k\theta)}{\sin\theta} \quad \checkmark$$

Prove $P(k+1)$ is true

$LHS = \sin\theta + \sin 3\theta + \cdots + \sin\left((2k-1)\theta\right) + \sin\left((2(k+1)-1)\theta\right)$ $\qquad RHS = \frac{\sin^2\left((k+1)\theta\right)}{\sin\theta}$

$= \frac{\sin^2(k\theta)}{\sin\theta} + \sin\left((2k+1)\theta\right)$ (from assumption)

$= \frac{\sin^2(k\theta) + \sin\theta\sin\left((2k+1)\theta\right)}{\sin\theta}$

(using $\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$= \frac{\sin^2(k\theta) + \frac{1}{2}\left[\cos\left(\theta - (2k+1)\theta\right) - \cos\left(\theta + (2k+1)\theta\right)\right]}{\sin\theta}$ $\quad \checkmark$

$= \cos(2k\theta)$

$= \frac{\sin^2(k\theta) + \frac{1}{2}\left[\cos(-2k\theta) - \cos(2k\theta + 2\theta)\right]}{\sin\theta}$



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(using $\cos 2nx = 1 - 2\sin^2(nx)$)

$$= \frac{\sin^2(k\theta) + \frac{1}{2}\left[1 - 2\sin^2(k\theta) - \left(1 - 2\sin^2(k\theta + \theta)\right)\right]}{\sin\theta} \quad \checkmark$$

$$= \frac{\sin^2(k\theta) + \sin^2((k+1)\theta) - \sin^2 k\theta}{\sin\theta}$$

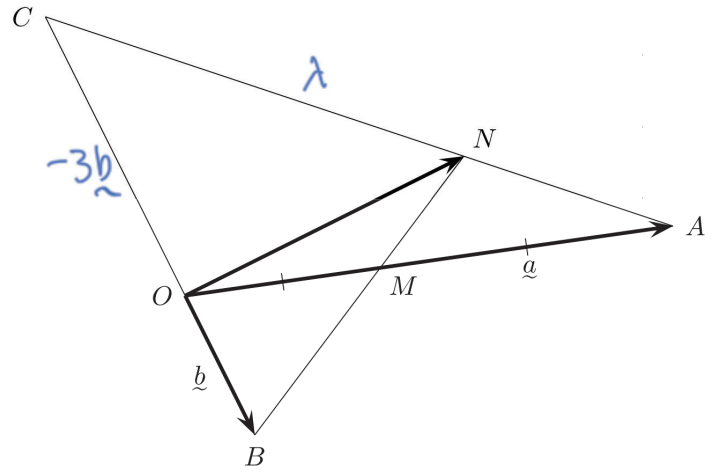
$$= \frac{\sin^2((k+1)\theta)}{\sin\theta}$$

$= RHS$

$\therefore P(k+1)$ is true

Hence true for all integers $\geq 1$ by the principle of mathematical induction.

b) trying to find $\overrightarrow{ON}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$






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$\vec{CA} = -(-3\underset{\sim}{b}) + \underset{\sim}{a}$

$= \underset{\sim}{a} + 3\underset{\sim}{b}$

let $\vec{CN} = \lambda \vec{CA}$

$\vec{ON} = -3\underset{\sim}{b} + \lambda(\underset{\sim}{a} + 3\underset{\sim}{b})$

$\therefore \vec{ON} = \lambda \underset{\sim}{a} + (3\lambda - 3)\underset{\sim}{b}$ ①

$\vec{BN} = -4\underset{\sim}{b} + \lambda(\underset{\sim}{a} + 3\underset{\sim}{b})$

$= \lambda \underset{\sim}{a} + (3\lambda - 4)\underset{\sim}{b}$

let $\vec{MN} = \mu \vec{BN}$

$\vec{ON} = \frac{1}{2}\underset{\sim}{a} + \mu(\lambda \underset{\sim}{a} + (3\lambda - 4)\underset{\sim}{b})$

$= (\frac{1}{2} + \mu\lambda)\underset{\sim}{a} + \mu(3\lambda - 4)\underset{\sim}{b}$ ②

equating coefficients of $\underset{\sim}{a}$ & $\underset{\sim}{b}$ in ① & ②

$\frac{1}{2} + \mu\lambda = \lambda$

$\therefore \lambda - \mu\lambda = \frac{1}{2}$

$\lambda = \frac{1}{2(1-\mu)}$ ③

$3\lambda - 3 = \mu(3\lambda - 4)$

$3\lambda - 3 = 3\mu\lambda - 4\mu$

$3\lambda - 3\mu\lambda = 3 - 4\mu$

$\lambda = \frac{3 - 4\mu}{3 - 3\mu}$ ④

sub ③ into ④ : $\frac{1}{2(1-\mu)} = \frac{3-4\mu}{3(1-\mu)}$

$3 - 4\mu = \frac{3}{2}$

$4\mu = \frac{3}{2}$

$\mu = \frac{3}{8}$

$\therefore \lambda = \frac{1}{2(1 - \frac{3}{8})}$

$= \frac{4}{5}$

$\therefore \vec{ON} = \frac{4}{5}\underset{\sim}{a} + (3 \times \frac{4}{5} - 3)\underset{\sim}{b}$

$= \frac{4}{5}\underset{\sim}{a} - \frac{3}{5}\underset{\sim}{b}$



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c) i) in $\Delta PGB$

$\cos\theta = \frac{BG}{u}$ $\therefore BG = u\cos\theta$ (1)

$\sin\theta = \frac{PG}{u}$ $\therefore PG = u\sin\theta$ ✓

in $\Delta PGA$, using pythagoras:

$AG^2 = r^2 - PG^2$

$= r^2 - u^2\sin^2\theta$ (2) ✓

Now using cosine rule in $\Delta ABG$:

$AG^2 = BG^2 + AB^2 - 2 \times BG \times AB \times \cos 30$

sub in (1) and (2)

$r^2 - u^2\sin^2\theta = u^2\cos^2\theta + 1 - 2(u\cos\theta) \times \frac{\sqrt{3}}{2}$

$r^2 = u^2(\sin^2\theta + \cos^2\theta) + 1 - \sqrt{3}u\cos\theta$

$r = \sqrt{u^2 + 1 - \sqrt{3}u\cos\theta}$ ✓

ii) need to find $\frac{dr}{dt}$ when $t = 5$

$\frac{dr}{dt} = \frac{dr}{du} \times \frac{du}{dt}$ ← $\frac{du}{dt}$ is the plane's speed ie. 300 km/h

$r = (u^2 + 1 - \sqrt{3}u\cos\theta)^{\frac{1}{2}}$

$\frac{dr}{du} = \frac{1}{2}(u^2 + 1 - \sqrt{3}u\cos\theta)^{-\frac{1}{2}}(2u - \sqrt{3}\cos\theta)$

$= \frac{2u - \sqrt{3}\cos\theta}{2\sqrt{u^2 + 1 - \sqrt{3}u\cos\theta}}$ ✓

Now when $t = 4$ $u = \frac{4}{60} \times 300 = 20$ km

$\therefore \left.\frac{dr}{dt}\right|_{u=20} = \frac{40 - \sqrt{3}\cos\theta}{2\sqrt{20^2 + 1 - \sqrt{3} \times 20 \times \cos\theta}} \times 300 = \frac{150(40 - \sqrt{3}\cos\theta)}{\sqrt{401 - 20\sqrt{3}\cos\theta}}$ km/h ✓

If you require more space to Answer Question 14, you may ask for an extra writing booklet.
If you have used an extra writing booklet for Question 14, tick here. ☐



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