Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2026 HSC ASSESSMENT TASK 3

Mathematics
Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using blue or black pen
- Calculators approved by NESA may be used
- A separate reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Student Number and and select your Class teacher on the front of each booklet

(Please tick or highlight)

- ○ Ms Sarofim
- ○ Ms Everingham
- ○ Ms Cai
- ○ Mr Berry
- ○ Dr Vranešević
- ○ Ms Moss
- ○ Ms Lee

Class Teacher:

Student Number:

(To be used by the exam markers only.)

Question No	1-10	11	12	13	14	Total	Total
Mark	/10	/15	/15	/15	/15	/70	/100

Section I: Multiple Choice

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1 Find the value of p so that the vectors $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} p \\ 20 \end{pmatrix}$ are parallel.

A. -8

B. -4

C. 4

D. 50

2 The roots of the polynomial $x^3 + 4x^2 - 3x + 2 = 0$ are α, β and γ. Evaluate

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$

A. $-\frac{2}{3}$

B. $-\frac{3}{4}$

C. $\frac{2}{3}$

D. $\frac{3}{2}$

3 Solve

$$\frac{3}{x+2} > 1$$

A. $(1, \infty)$

B. $(-2, 1)$

C. $(-\infty, -2) \cup (-2, \infty)$

D. $(-\infty, -2) \cup (1, \infty)$

4 Which of the following is an expression for $a\cos x - b\sin x$?

A. $\sqrt{a^2+b^2}\cos\left(x+\tan^{-1}\frac{a}{b}\right)$

B. $\sqrt{a^2+b^2}\cos\left(x-\tan^{-1}\frac{a}{b}\right)$

C. $\sqrt{a^2+b^2}\cos\left(x+\tan^{-1}\frac{b}{a}\right)$

D. $\sqrt{a^2+b^2}\cos\left(x-\tan^{-1}\frac{b}{a}\right)$

5 Find

$$\int \frac{\cos x}{1+\sin^2 x}\,dx$$

A. $\tan^{-1}(\sin x)+C$

B. $-\dfrac{1}{1+\sin x}+C$

C. $\ln|1+\sin^2 x|+C$

D. $2\ln|\sin x|+C$

6 $P(x)$ is a polynomial with a factor of $(x+1)$ and a remainder of 8 when it is divided by $(2x+1)$. What is the remainder when the quotient of $P(x)$ and $(x+1)$ is divided by $(2x+1)$?

A. -16

B. -4

C. 4

D. 16

7 $f(x)$ is a one-to-one function which passes through the point (a, b).
Let $g(x) = f^{-1}(x)$.
What is the gradient of the tangent to $g(x)$ at the point (b, a)?

A. $\dfrac{1}{f'[g(a)]}$

B. $\dfrac{1}{f'[g(b)]}$

C. $\dfrac{1}{g'[f(a)]}$

D. $\dfrac{1}{g'[f(b)]}$

8 What is the constant term in the expansion of $(2 + x)\left(4x + \frac{1}{x^2}\right)^8$?

A. $2^{10}\binom{8}{3}$

B. $2^3\binom{8}{1} + \binom{8}{2}$

C. $2^4\binom{8}{4}$

D. There is no constant term

9 Evaluate the integral

$$\int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} \cos^{-1}(\sin x)\, dx$$

A. 0

B. 3π

C. $\dfrac{3\pi^2}{2}$

D. $\dfrac{9\pi^2}{4}$

10 How many ways can the letters from FLUFFBALL be arranged such that there are no identical letters in adjacent positions?

A. 2040

B. 5670

C. 6480

D. 9960

Section II

60 marks
Attempt Questions 11-14
Allow abut 1 hour 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Use the Question 11 Writing Booklet

(a) Solve $|2x + 5| > 5$ **2**

(b) Two vectors are defined as $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j}$. Find

(i) $\underset{\sim}{u} + \underset{\sim}{v}$ **1**

(ii) $\underset{\sim}{u} \cdot \underset{\sim}{v}$ **1**

(iii) $\left|2\underset{\sim}{u} - 3\underset{\sim}{v}\right|$ **2**

(c) Use the substitution $t = \tan\theta$ to solve $\cos 2\theta + \tan\theta = 1$ for $0 \leq \theta \leq 2\pi$ **3**

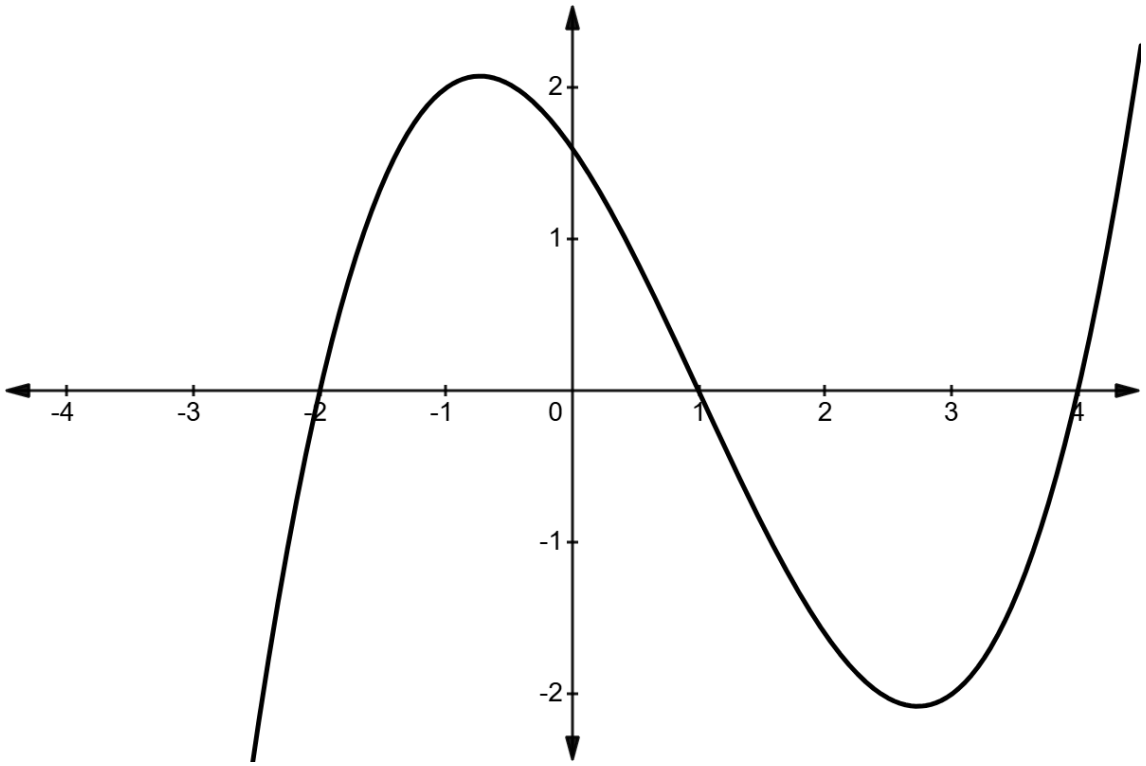
(d) The polynomial $f(x) = x^3 + ax^2 + bx + 9$ has a double root at $x = -3$. Find the values of a and b. **3**

(e) Use the substitution $u^2 = x + 1$ to evaluate the integral **3**

$$\int_3^8 \frac{x}{\sqrt{x+1}}\,dx$$

Question 12 (15 Marks) Use the Question 12 Writing Booklet

(a) The graph $y = f(x)$ is shown below



Use the axes provided in your Question 12 Writing Booklet to sketch the following

(i) $y = |f(x)|$ **1**

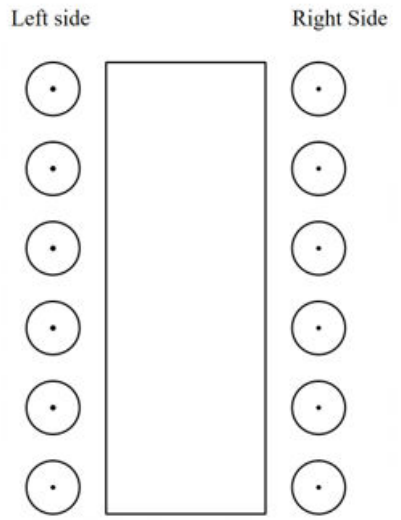
(ii) $y = f(|x|)$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y^2 = f(x)$ **3**

Question 12 continues on page 8

(b) At a tribunal, 12 guests are to be seated on 2 sides of a rectangular table with 6 chairs on each side as shown in the diagram below. **2**



Harry and Kim need to sit on the opposite side to Evrart, Joyce and Titus. In how many ways can they be seated?

(c) (i) Sketch the graph of $y = 2\cos^{-1} x$ clearly showing all intercepts and endpoints **1**

(ii) Find the area bounded by the curve $y = 2\cos^{-1} x$, the x-axis and the y-axis in the first quadrant **2**

(iii) The area in part (ii) is rotated about the y-axis. Find the exact volume of the solid formed **3**

End of Question 12

Question 13 (15 Marks) Use the Question 13 Writing Booklet

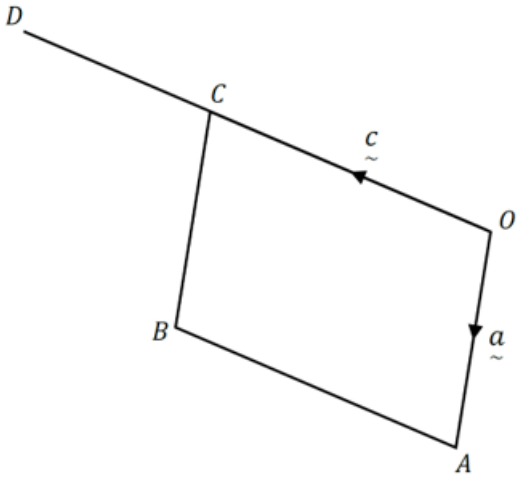
(a) Prove by mathematical induction that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for integers $n \geq 1$. **3**

(b) (i) Solve $x^3 - 4x^2 + 5x - 2 = 0$ for x **2**

(ii) Express $2\sin^2\theta - \sin\theta\sec\theta = 2\sin 2\theta - 2$ in terms of $\tan\theta$ **2**

(iii) Hence or otherwise solve $2\sin^2\theta - \sin\theta\sec\theta = 2\sin 2\theta - 2$ for $0 \leq \theta \leq \pi$ **1**

(c) In the parallelogram $OABC$, $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



X is the midpoint of the interval AC, and OCD is a straight line such that the ratio $OC:CD = k:1$

(i) Find $\overrightarrow{OX}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ **1**

(ii) Write and expression for $\overrightarrow{OD}$ in terms of $\underset{\sim}{c}$ **1**

(iii) Given $\overrightarrow{XD} = 3\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}$, find the value of k **2**

(d) (i) Show that **1**

$$\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$$

(ii) Hence or otherwise show that **2**

$$\binom{n}{0} + \frac{1}{2}\binom{n}{1}x + \frac{1}{3}\binom{n}{2}x^2 + \frac{1}{4}\binom{n}{3}x^3 + \cdots + \frac{1}{n+1}\binom{n}{n}x^n = \frac{(1+x)^{n+1} - 1}{x(n+1)}$$

Question 14 (15 Marks) Use the Question 14 Writing Booklet

(a) A set contains n positive integers greater than 1. The prime factors of each integer in the set are less than 10. Find the minimum value of n such that the set must contain a pair of integers whose product is a perfect square. **2**

(b) A particle is launched from the origin with a speed of U at an angle of θ to the horizontal. It passes through point $P(p, q)$ with a speed of W.

The velocity vector of the particle as it passes through P is perpendicular to the velocity vector of when the particle was launched from the origin.

There is a constant acceleration due to gravity of g.

The equations of motion are given by

$$\underset{\sim}{a} = -g\underset{\sim}{j}$$

$$\underset{\sim}{v} = U\cos\theta\,\underset{\sim}{i} + (U\sin\theta - gt)\underset{\sim}{j}$$

$$\underset{\sim}{r} = Ut\cos\theta\,\underset{\sim}{i} + (Ut\sin\theta - \frac{1}{2}gt^2)\underset{\sim}{j}$$

(do NOT prove these).

(i) Show that the time when the particle passes through point P is given by **2**

$$t = \frac{U}{g\sin\theta}$$

(ii) Show that the launch speed of the particle is $U = \sqrt{pg\tan\theta}$ **1**

(iii) Prove that the product of U and W is independent of q and θ **3**

(c) The hands of an analogue clock are OA and OB, where $OA = 8$ cm and and $OB = 15$ cm, O is the origin and $t \geq 0$ is the number of minutes past midnight. **3**

There is an angle of θ between the hands at the origin

How fast, in cm/min, is the distance between the tips of the hands changing at 3:00? Give your answer as an exact value.

Question 14 continues on page 11

(d) The polynomial, with real coefficients, $P(x) = x^4 + ax^3 + bx^2 + ax + c$ has exactly three distinct real roots at $\tan\theta$, $\tan 2\theta$ and $\tan 3\theta$. **4**

Find the smallest positive value for θ.

End of paper

Q1. / $\frac{2}{-5} = \frac{p}{20}$

$-5p = 40$

$p = -8$

$\therefore$ (A)

Q2. / $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-3/1}{-2/1}$

$= \frac{3}{2}$

$\therefore$ (D)

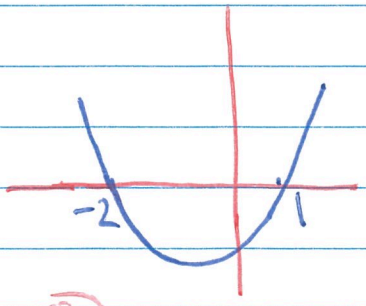
Q3. / $\frac{3}{(x+2)} > 1$

$3(x+2) > (x+2)^2$

$(x+2)^2 - 3(x+2) < 0$

$(x+2)[(x+2) - 3] < 0$

$(x+2)(x-1) < 0$



$\therefore$ (B)

Q4./ $r\cos(x+\theta) = a\cos x - b\sin x$

$r\cos x\cos\theta - r\sin x\sin\theta = a\cos x - b\sin x$

$\therefore\ r\cos\theta = a$ ①, $r\sin\theta = b$ ②

② ÷ ①

$\frac{r\sin\theta}{r\cos\theta} = \frac{b}{a}$

$\tan\theta = \frac{b}{a}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

① ^{2} + ② ^{2}

$(r\cos\theta)^2 + (r\sin\theta)^2 = a^2 + b^2$

$r^2(\cos^2\theta + \sin^2\theta) = a^2 + b^2$

$\therefore\ r = \sqrt{a^2+b^2}$

$\therefore\ a\cos x - b\sin x = \sqrt{a^2+b^2}\cos\left(x + \tan^{-1}\left(\frac{b}{a}\right)\right)$

$\therefore$ C

Q5./ $\int \frac{\cos x}{1+\sin x}\,dx = \tan^{-1}(\sin x) + C$

($f'(x)$ over $[f(x)]^2$)

$\therefore$ A

Q6./ $P(x) = (x+1)Q(x)$

$P(-1/2) = 8$

$\therefore\ 8 = (-1/2 + 1)Q(-1/2)$

$Q(-1/2) = \frac{8}{(1/2)}$

$= 16$

$\therefore$ D

Q7./ $f[g(x)] = x$

$g'(x) f'(g(x)) = 1$

$\therefore g'(x) = \frac{1}{f'[g(x)]}$

$g'(b) = \frac{1}{f'[g(b)]}$

$\therefore$ (B)

Q8./ $t_{k,r} = \binom{3}{k} 2^{3-k} x^k \binom{6}{r} x^{6-r} (x^{-2})^r$

$= \binom{3}{k}\binom{6}{r} 2^{3-k} x^{6+k-3r}$

$\therefore k - 3r = -6$

$3r - k = 6$

$3r = 6 + k$

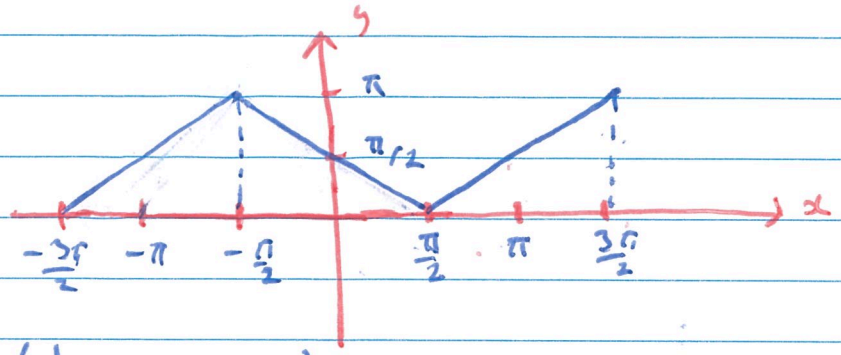
k	r	
0	2	✓
1	7/3	✗
2	8/3	✗
3	3	✓

$\therefore$ constant $= \binom{3}{0}\binom{6}{2} 2^{3-0} + \binom{3}{3}\binom{6}{3} 2^{3-3}$

$= 8\binom{6}{2} + \binom{6}{3}$

$\therefore$ (A)

Q9./ $\int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} \cos^{-1}\sin x\,dx = \int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} \cos^{-1}\left[\cos\left(\frac{\pi}{2}-x\right)\right]dx$



$3 \times \left(\frac{1}{2} \times \pi \times \pi\right)$

base ↑ ↑ height

$= \frac{3\pi^2}{2}$

∴ (C)

Q10./

$3!\left[\binom{4}{1}\binom{5}{1} + \binom{4}{1}\binom{3}{1}\binom{6}{2} + \binom{4}{3}\binom{7}{3}\right]$

↑ arrange UBA

↑ FLFLF ↑ L ↑ FLF ↑ F ↑ LL ↑ FFF ↑ LLL

$= 2040$

∴ (A)

Detailed explanation to Q10

Begin by arranging the non repeated letters.
3! ways

_ U _ B _ A _

notice there are 4 'gaps'

Case 1 (FFF all in the same gap)

_ F L F L F _ U _ B _ A _

4 × 5 (2 Ls must go between Fs)

4: choose gap

5: choose position for extra L

Case 2 (FF together, F apart)

_ F L F _ U _ F _ B _ A _

4 × 3 × $\binom{6}{2}$ (one L must go between Fs)

4: choose FF gap

3: choose F gap

$\binom{6}{2}$: choose position for two extra Ls

Case 3 (All Fs apart)

_ F _ U _ F _ B _ F _ A _

$^4C_3 \times {}^7C_3$

$^4C_3$: choose gaps for Fs

$^7C_3$: choose gaps for Ls

Question 11

(a) $|2x+5| > 5$

$\therefore \quad 2x+5 > 5 \quad \text{or} \quad 2x+5 < -5$

$2x > 0 \quad \text{or} \quad 2x < -10$

$x > 0 \quad \text{or} \quad x < -5$

(b)(i) $2\underline{i} + 3\underline{j} + \underline{i} - 2\underline{j} = 3\underline{i} + \underline{j}$

(ii) $(2)(1) + (3)(-2) = 2 - 6$

$= -4$

(iii) $4\underline{i} + 6\underline{j} - (3\underline{i} - 6\underline{j}) = \underline{i} + 12\underline{j}$

$|2\underline{u} - 3\underline{v}| = \sqrt{1^2 + 12^2}$

$= \sqrt{145}$

(c) $\cos 2\theta + \tan\theta = 1$

$t = \tan\theta$

$\frac{1-t^2}{1+t^2} + t = 1$

$\therefore \quad 1 - t^2 + t(1+t^2) = 1 + t^2$

$1 - t^2 + t + t^3 = 1 + t^2$

$t^3 - 2t + t = 0$

$t(t^2 - 2t + 1) = 0$

$t(t-1)^2 = 0$

$\therefore \quad t = 0 \quad \text{or} \quad t = 1$

$\tan\theta = 0 \quad \text{or} \quad \tan\theta = 1$

$\therefore \quad \theta = 0, \pi, 2\pi, \pi/4 \text{ or } 5\pi/4$

(d) $f(-3) = (-3)^3 + a(-3)^2 + b(-3) + 9$

$= -27 + 9a - 3b + 9$

$= 9a - 3b - 18$

$= 0$

$\therefore 3a - b - 6 = 0$

$b = 3a - 6 \quad \ldots ①$

$f'(x) = 3x^2 + 2ax + b$

$f'(-3) = 0$

$\therefore 3(-3)^2 + 2a(-3) + b = 0$

$27 - 6a + b = 0 \quad \ldots ②$

① → ②

$27 - 6a + 3a - 6 = 0$

$21 - 3a = 0$

$3a = 21$

$a = 7$

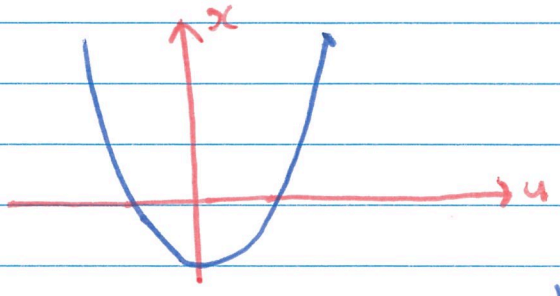
sub → ①

$b = 3(7) - 6$

$b = 15$

(e) $u^2 = x + 1$

$x = u^2 - 1$



restrict to $u \geq 0$ so function is one-one

when $x = 3$ $u^2 = 4$, $u = 2$

when $x = 8$ $u^2 = 9$, $u = 3$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$$\therefore \int_3^8 \frac{x}{\sqrt{x+1}}\,dx = \int_2^3 \frac{u^2-1}{u} 2u\,du$$

$$= \int_2^3 (2u^2 - 2)\,du$$

$$= \left[\frac{2u^3}{3} - 2u\right]_2^3$$

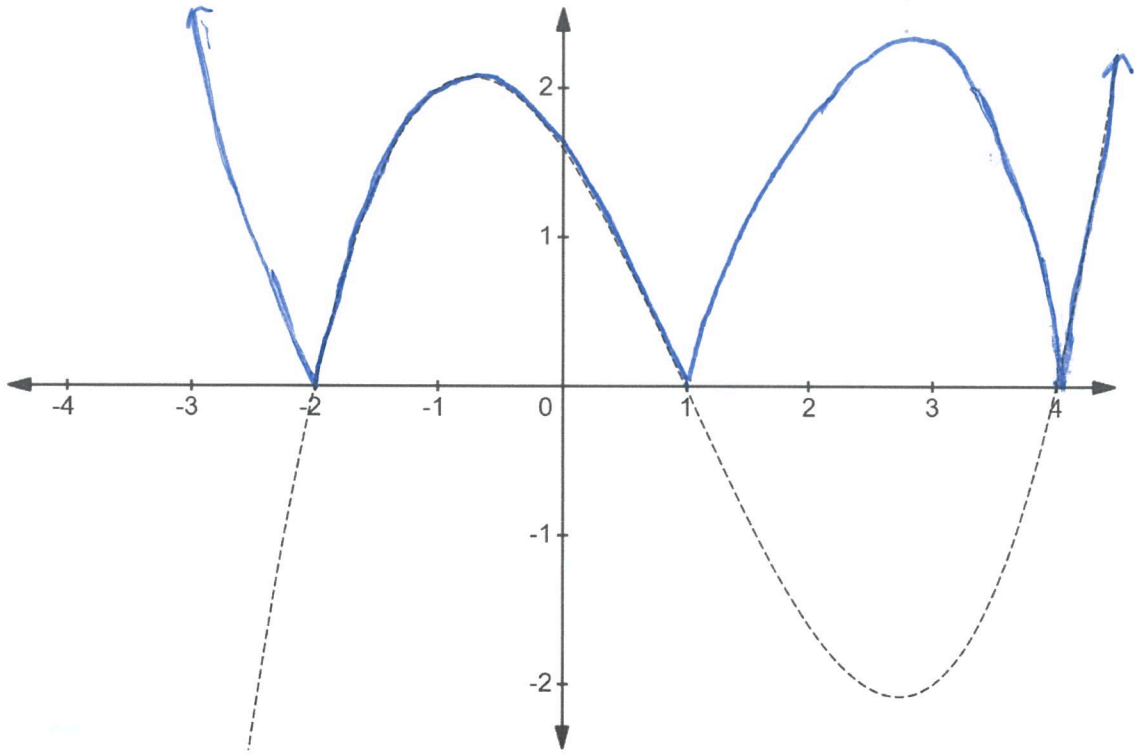
$$= \left(\frac{2\times 3^3}{3} - 2\times 3\right) - \left(\frac{2\times 2^3}{3} - 2^2\right)$$

$$= 12 - \frac{4}{3}$$

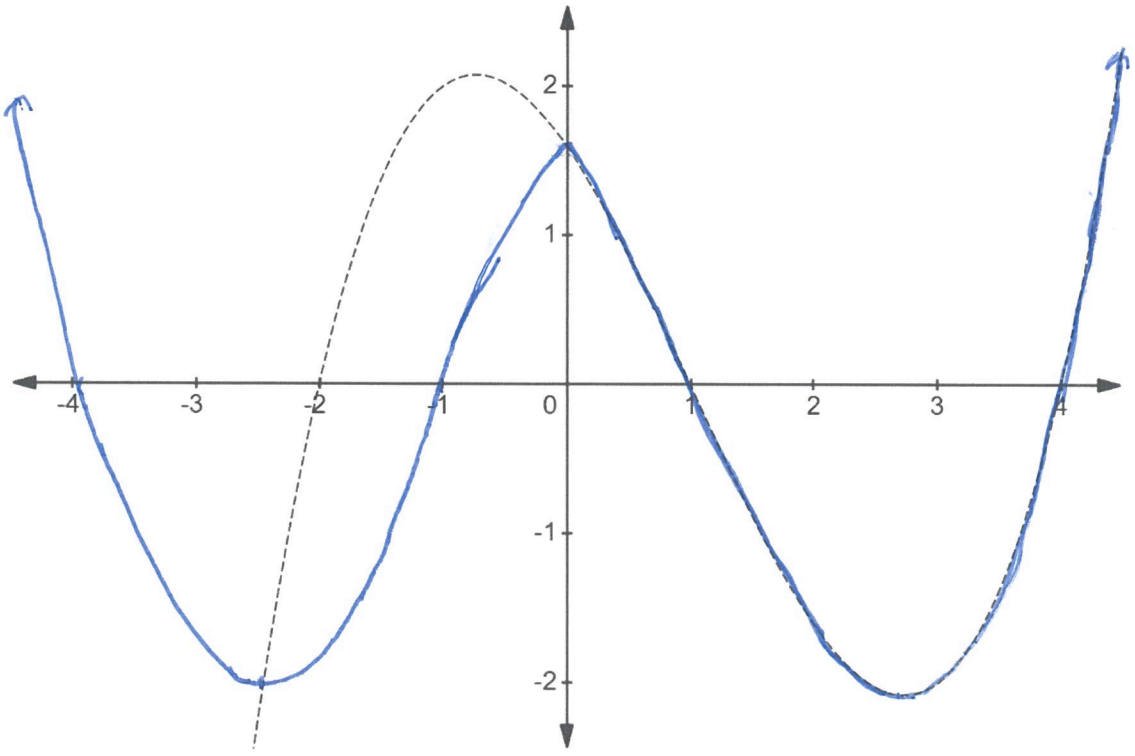
$$= \frac{32}{3} \text{ or } 10\frac{2}{3}$$

Question 12

(a) (i)



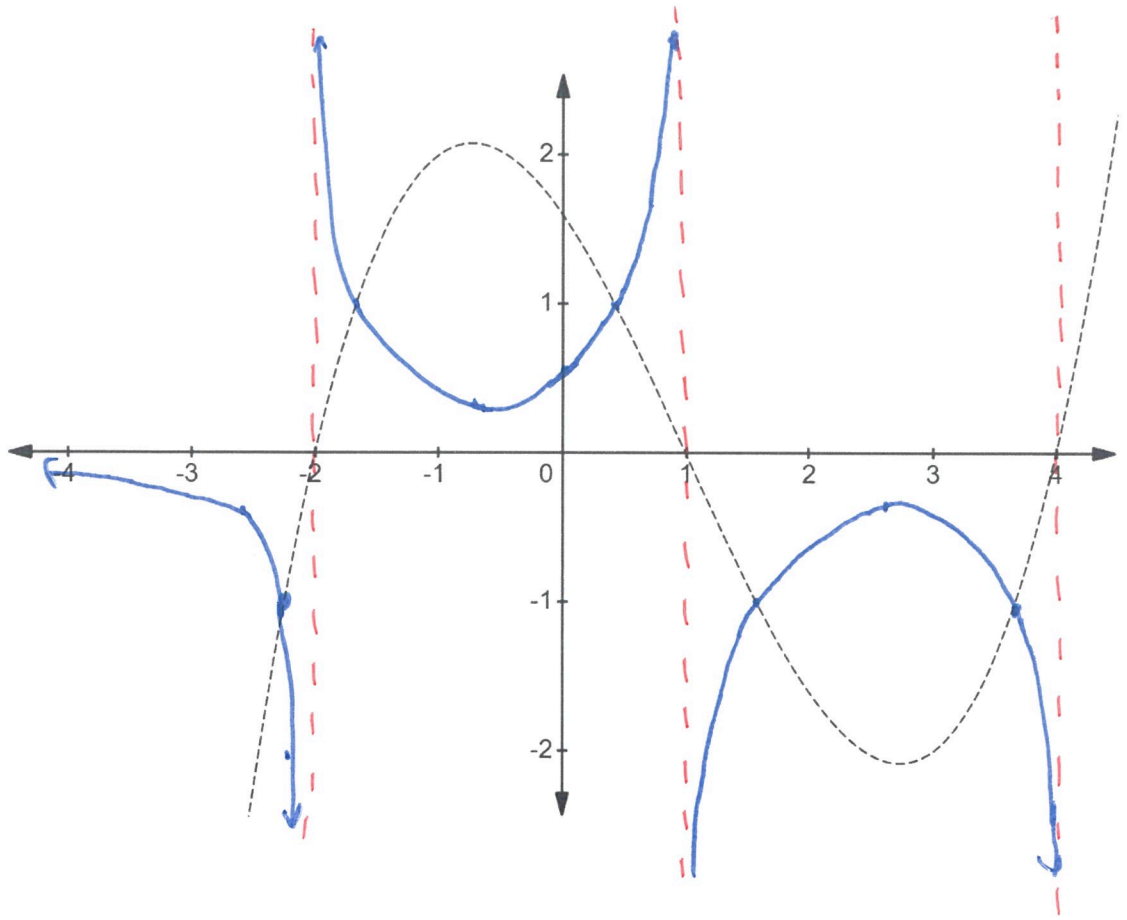
(ii)



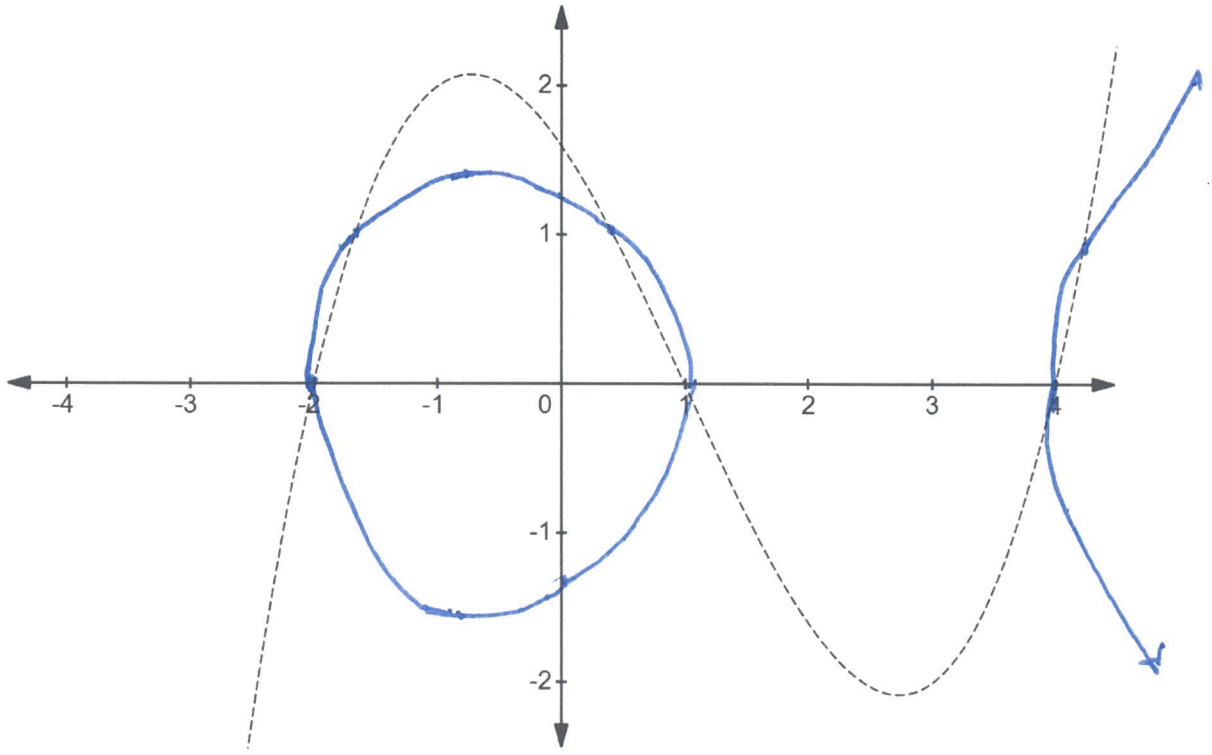
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(iii)



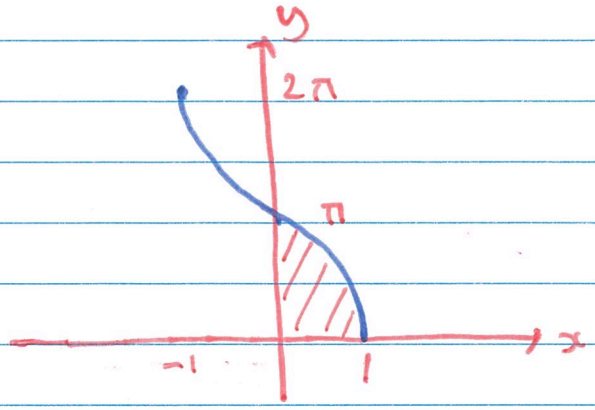
(iv)



(b) $12 \times 5 \times {}^{6}P_3 \times 7! = 36288000$

Harry, Kim, Errant Boyce Titus, the rest

(c) (i)



(ii) $y = 2\cos^{-1}x$

$\cos^{-1}(x) = \frac{y}{2}$

$x = \cos\left(\frac{y}{2}\right)$

$\therefore \int_0^{\pi} \cos\left(\frac{y}{2}\right) dy = \left[2\sin\frac{y}{2}\right]_0^{\pi}$

$= 2\sin\left(\frac{\pi}{2}\right) - \left(2\sin\left(\frac{0}{2}\right)\right)$

$= 2(1)$

$= 2 \text{ units}^2$

(iii) $\pi \int_0^{\pi} \cos^2\left(\frac{y}{2}\right) dy$

$= \pi \int_0^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos y\right) dy$

$= \pi \left[\frac{y}{2} + \frac{1}{2}\sin y\right]_0^{\pi}$

$= \pi \left[\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - \left(\frac{0}{2} + \frac{1}{2}\sin 0\right)\right]$

$= \frac{\pi^2}{2} \text{ units}^3$

Question 13

(a) Base Case ($n = 1$)

$$1^3 + (1+1)^3 + (2+1)^3 = 1 + 8 + 27$$
$$= 36$$
$$= 9(4)$$

$\therefore$ true for $n = 1$

Assume true for $n = k$

Assume $k^3 + (k+1)^3 + (k+2)^3 = 9P, \quad k, P \in \mathbb{Z}$

$\therefore (k+1)^3 + (k+2)^3 = 9P - k^3 \quad **$

Prove true for $n = k+1$

RTP: $(k+1)^3 + (k+2)^3 + (k+3)^3 = 9M, \quad m \in \mathbb{Z}$

$$LHS = (k+1)^3 + (k+2)^3 + (k+3)^3$$
$$= 9P - k^3 + (k+3)^3 \quad (\text{from } **)$$
$$= 9P - k^3 + k^3 + \binom{3}{1}k^2(3) + \binom{3}{2}k(3)^2 + 3^3$$
$$= 9P + 9k^2 + 27k + 27$$
$$= 9(P + k^2 + 3k + 3)$$
$$= 9M \quad (M = P + k^2 + 3k + 3)$$

$\therefore$ true for $n = k+1$

$\therefore$ as the statement is true for $n = 1$ and will be true for $n = k+1$ if it is true for $n = k$, it is true for all positive integers.

(b)(i) $(1)^3 - 4(1)^2 + 5(1) - 2 = 0$ $\therefore (x-1)$ is a factor

$(2)^3 - 4(2)^2 + 5(2) - 2 = 0$ $\therefore (x-2)$ is a factor

$$
\begin{array}{r|l}
 & x - 1 \\
\hline
x^2 - 3x + 2 & x^3 - 4x^2 + 5x - 2 \\
 & \underline{x^3 - 3x^2 + 2x} \quad \downarrow \\
 & -x^2 + 3x - 2 \\
 & \underline{-x^2 + 3x - 2} \\
 & 0
\end{array}
$$

$\therefore \quad x^3 - 4x^2 + 5x - 2 = (x-1)^2(x-2)$

$\therefore \quad x = 1 \quad$ or $\quad x = 2$

(ii) $2\sin^2\theta - \sin\theta\sec\theta = 2\sin 2\theta - 2$

$2\sin^2\theta - \dfrac{\sin\theta}{\cos\theta} = 4\sin\theta\cos\theta - 2 \qquad (\div \cos^2\theta)$

$2\dfrac{\sin^2\theta}{\cos^2\theta} - \dfrac{\sin\theta}{\cos^2\theta} = \dfrac{4\sin\theta\cos\theta}{\cos^2\theta} - \dfrac{2}{\cos^2\theta}$

$\therefore \quad 2\tan^2\theta - \tan\theta\sec^2\theta = 4\tan\theta - 2\sec^2\theta$

$2\tan^2\theta - \tan\theta(1+\tan^2\theta) = 4\tan\theta - 2(1+\tan^2\theta)$

(iii) $2\tan^2\theta - \tan\theta - \tan^3\theta = 4\tan\theta - 2 - 2\tan^2\theta$

$\therefore \quad \tan^3\theta - 4\tan^2\theta + 5\tan\theta - 2 = 0$

$\therefore \quad (\tan\theta - 1)^2(\tan\theta - 2) = 0$

$\therefore \quad \tan\theta = 1 \quad$ or $\quad \tan\theta = 2$

$\theta = \pi/4 \qquad \theta = \tan^{-1}(2)$

(c)(i) $\overrightarrow{OX} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$

$= \underline{a} + \frac{1}{2}(\underline{c} - \underline{a})$

$= \underline{a} + \frac{1}{2}\underline{c} - \frac{1}{2}\underline{a}$

$= \frac{1}{2}\underline{a} + \frac{1}{2}\underline{c}$

(ii) $\frac{|\overrightarrow{OD}|}{|\overrightarrow{OC}|} = \frac{k+1}{k}$

$|\overrightarrow{OD}| = |\overrightarrow{OC}|\left(\frac{k+1}{k}\right)$

$\therefore \overrightarrow{OD} = \underline{c}\left(1 + \frac{1}{k}\right)$

(iii) $\overrightarrow{XD} = \overrightarrow{OD} - \overrightarrow{OX}$

$3\underline{c} - \frac{1}{2}\underline{a} = \underline{c}\left(1 + \frac{1}{k}\right) - \frac{1}{2}(\underline{a} + \underline{c})$

$3\underline{c} - \frac{1}{2}\underline{a} = \underline{c} + \frac{1}{k}\underline{c} - \frac{1}{2}\underline{a} - \frac{1}{2}\underline{c}$

$3\underline{c} = \frac{1}{2}\underline{c} + \frac{1}{k}\underline{c}$

$\frac{5}{2}\underline{c} = \frac{1}{k}\underline{c}$

$\frac{2}{5}\underline{c} = k\underline{c}$

$\therefore k = \frac{2}{5}$

(d)(i) LHS $= \binom{n+1}{k+1}$

$$= \frac{(n+1)!}{(k+1)!\,[(n+1)-(k+1)]!}$$

$$= \frac{(n+1)\,n!}{(k+1)\,k!\,(n-1-k+1)!}$$

$$= \frac{n+1}{k+1} \times \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n+1}{k+1}\binom{n}{k}$$

$$= \text{RHS}$$

(ii) $(1+x)^{n+1} = \binom{n+1}{0} + \binom{n+1}{1}x + \binom{n+2}{2}x^2 + \ldots + \binom{n+1}{n+1}x^{n+1}$

$$= 1 + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \ldots + \binom{n+1}{n+1}x^{n+1}$$

$\therefore (1+x)^{n+1} - 1 = \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \ldots + \binom{n+1}{n+1}x^{n+1}$

$$= \frac{n+1}{1}\binom{n}{0}x + \frac{n+1}{2}\binom{n}{1}x^2 + \ldots + \frac{n+1}{n+1}\binom{n}{n}x^{n+1} \quad (\text{from (i)})$$

$$= x(n+1)\left[\frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1}x + \ldots + \frac{1}{n+1}\binom{n}{n}x^n\right]$$

$$\therefore \frac{(1+x)^{n+1} - 1}{x(n+1)} = \binom{n}{0} + \frac{1}{2}\binom{n}{1}x + \ldots + \frac{1}{n+1}\binom{n}{n}x^n$$

$$\therefore \binom{n}{0} + \frac{1}{2}\binom{n}{1}x + \frac{1}{3}\binom{n}{2}x^2 + \frac{1}{4}\binom{n}{3}x^3 + \ldots + \frac{1}{n+1}\binom{n}{n}x^n = \frac{(1+x)^{n+1} - 1}{x(n+1)}$$

Question 14

(a) let 2 numbers have the form $2^{a_1}3^{a_2}5^{a_3}7^{a_4}$ and $2^{b_1}3^{b_2}5^{b_3}7^{b_4}$

product $= 2^{a_1+b_1}3^{a_2+b_2}5^{a_3+b_3}7^{a_4+b_4}$

a_1+b_1, a_2+b_2, a_3+b_3 & a_4+b_4 must all be even

$\therefore$ a_1 & b_1, a_2 & b_2, a_3 & b_3, a_4 & b_4 must have the same parity

there are $2\times2\times2\times2 = 16$ possible parity permutations for each number

$\therefore$ by pigeonhole principle $16+1 = 17$ numbers are needed.

(b)(i) $\underline{v}(0)\cdot\underline{v}(t) = 0$

$(U\cos\theta\,\underline{i} + (U\sin\theta - g(0))\underline{j})\cdot(U\cos\theta\,\underline{i} + (U\sin\theta - gt)\underline{j}) = 0$

$U^2\cos^2\theta + U\sin\theta(U\sin\theta - gt) = 0$

$U^2\cos^2\theta + U^2\sin^2\theta - U\sin\theta gt = 0$

$U^2(\sin^2\theta + \cos^2\theta) = U\sin\theta gt$

$U^2 = U\sin\theta gt$

$\therefore t = \dfrac{U}{g\sin\theta}$

(ii) at P $(r(2) = Ut\cos\theta)$

$p = Ut\cos\theta$

$p = U\left(\dfrac{U}{g\sin\theta}\right)\cos\theta$

$p = \dfrac{U^2\cos\theta}{g\sin\theta}$

$U^2 = \dfrac{pg\sin\theta}{\cos\theta}$

$\therefore U^2 = pg\tan\theta$

$U = \sqrt{pg\tan\theta}$ $\quad (U > 0)$

(iii) $W^2 = (U\cos\theta)^2 + (U\sin\theta - gt)^2$

$= (U\cos\theta)^2 + \left(U\sin\theta - g \times \frac{U}{g\sin\theta}\right)^2$ (from (i))

$= U^2\cos^2\theta + U^2\sin^2\theta - 2U^2 + \frac{U^2}{\sin^2\theta}$

$= U^2\left(\cos^2\theta + \sin^2\theta - 2 + \frac{1}{\sin^2\theta}\right)$

$= U^2\left(1 - 2 + \operatorname{cosec}^2\theta\right)$

$= U^2\left(\operatorname{cosec}^2\theta - 1\right)$

$= U^2\cot^2\theta$

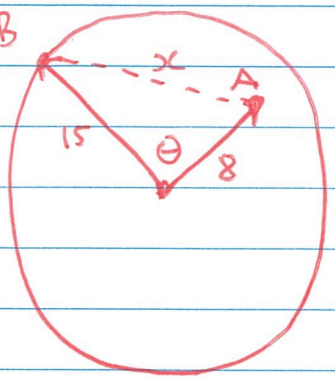
$\therefore W = U\cot\theta$ $(W, U, \cot\theta > 0$ for $0 < \theta < 90°)$

$\therefore UW = U^2\cot\theta$

$= pg\tan\theta\cot\theta$

$= pg$ which is independent of ℓ & θ

(c)



$$x = \sqrt{15^2 + 8^2 - 2 \times 15 \times 8 \times \cos\theta}$$

$$= (289 - 240\cos\theta)^{\frac{1}{2}}$$

$$\frac{dx}{d\theta} = \left(\frac{1}{2}\right)(240\sin\theta)(289 - 240\cos\theta)^{-\frac{1}{2}}$$

$$= \frac{120\sin\theta}{\sqrt{289 - 240\cos\theta}}$$

$$\frac{d\theta}{dt} = \frac{2\pi}{60} - \frac{2\pi}{12 \times 60}$$

$$= \frac{11\pi}{360}$$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$

$$= \frac{120\sin\theta}{\sqrt{289 - 240\cos\theta}} \times \frac{11\pi}{360}$$

when $\theta = \frac{\pi}{2}$

$$= \frac{120\sin\left(\frac{\pi}{2}\right)}{\sqrt{289 - 240\cos\theta}} \times \frac{11\pi}{360}$$

$$= \frac{11\pi}{51} \text{ cm/min}$$

(d) Note that one root must be a double root.

Let the roots be r, r, s & t

$r + r + s + t = r^2s + r^2t + rst + rst$ (both $-\frac{a}{t}$)

$2r + s + t = r^2s + r^2t + 2rst$

$2r - 2rst = r^2s + r^2t - s - t$

$2r(1 - st) = r^2(s + t) - 1(s + t)$

$2r(1 - st) = (s + t)(r^2 - 1)$ (*)

$\therefore \frac{2r}{r^2 - 1} = \frac{s + t}{1 - st}$

$\therefore -\frac{2r}{1 - r^2} = \frac{s + t}{1 - st}$

let $r = \tan 3\theta$

$-\frac{2\tan 3\theta}{1 - \tan^2 3\theta} = \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta}$

$-\tan 6\theta = \tan 3\theta$

$\therefore \tan(-6\theta) = \tan 3\theta$

$\therefore 3\theta - (-6\theta) = 0, \pi, 2\pi \ldots$ (period $= \pi$)

$\therefore 9\theta = \pi$ (smallest value for distinct roots)

$\therefore \theta = \frac{\pi}{9}$

note 1 if we take $r = \tan\theta$ we get $\theta = \frac{\pi}{7}$

$r = \tan 2\theta \Rightarrow \theta = \frac{\pi}{8}$

so all 3 cases should be checked

note 2 to ensure we are not dividing by 0 in (*) we need to check when

$r^2 - 1 = 0$.

$\therefore r^2 = 1$

case 1 $\tan^2\theta = 1$

$\theta = \pi/4$

$\therefore \tan 2\theta = \tan(\pi/2)$ which is undefined

$\therefore$ not a solution

case 2 $\tan^2 2\theta = 1$

$2\theta = \pi/4$

$\theta = \pi/8$ (already found)

case 3 $\tan^2 3\theta = 1$

$3\theta = \pi/4$

$\theta = \pi/12$

Looks good - smaller than $\pi/9$ but then $\tan 2\theta = \pi/6$ & $\tan 3\theta = \pi/4$

subbing into LHS of (*)

$$2\tan(\pi/12)(1 - \tan(\pi/6)\tan(\pi/4)) = 0.2264973\ldots \neq 0$$

$\therefore$ $\theta = \pi/12$ is not a solution

$\theta = \pi/9$ is the answer