North Sydney Girls High School

2026 **HSC TRIAL EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading Time – 10 minutes
- Working Time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – **60 marks** (pages 8 – 15)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

NAME:_____ **TEACHER:**_____

STUDENT NUMBER:

Question	1-10	11	12	13	14	Total
Mark	/10	/15	/15	/15	/15	/70

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1 Given that $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, what is $\overrightarrow{AB}$?

A. $\begin{pmatrix} 5 \\ 11 \end{pmatrix}$

B. $\begin{pmatrix} -5 \\ 11 \end{pmatrix}$

C. $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

D. $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$

2 The polynomial $p(x) = x^3 - 6x^2 + 32$ has one repeated zero of multiplicity 2 at $x = \alpha$.
What is the value of α?

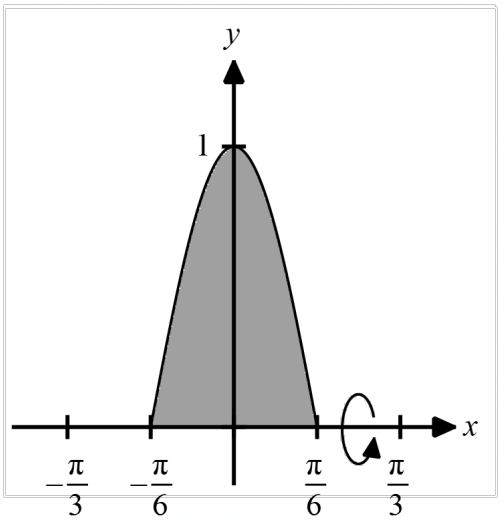
A. -2

B. 0

C. 2

D. 4

3 A solid of revolution is to be formed by rotating the region bounded by the x-axis and the curve $y = \cos 3x$ between $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$ about the x-axis, as shown in the diagram.



Which of the following gives the volume of solid of revolution formed?

A. $\pi \int_0^{\frac{\pi}{6}} 1 + \cos 6x \; dx$

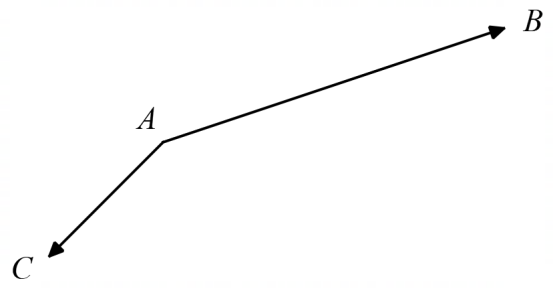
B. $\pi \int_0^{\frac{\pi}{6}} \cos^2 3x \; dx$

C. $\frac{\pi}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3x \; dx$

D. $\frac{\pi}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1 - \cos 6x \; dx$

4 The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ is given by $\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\right)\underset{\sim}{v}$.

The diagram below shows points *A, B* and *C,* where $\overrightarrow{AB} = 3\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{AC} = -\underset{\sim}{i} - \underset{\sim}{j}$.



Which of the following is the projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$?

A. $\frac{2}{5}\left(3\underset{\sim}{i} + \underset{\sim}{j}\right)$

B. $\frac{-2}{5}\left(3\underset{\sim}{i} + \underset{\sim}{j}\right)$

C. $2\left(\underset{\sim}{i} + \underset{\sim}{j}\right)$

D. $2\left(-\underset{\sim}{i} - \underset{\sim}{j}\right)$

5 Five different schools send two prefects each to attend a conference. The group of 10 is to be seated at a circular table. In how many ways can the prefects be seated if prefects from the same school are to be seated together?

A. $4!$

B. $2\times 4!$

C. $2^4\times 4!$

D. $2^5\times 4!$

6 You are given that $f(x)$ is a one-to-one function defined for $x\in(-\infty,\infty)$.
Which of the following functions cannot have an inverse?

A. $f(|x|)$

B. $|f(x)|$

C. $\dfrac{1}{f(x)}$

D. $\sqrt{f(x)}$

7 How many distinct solutions are there to the equation $\sin x+\sin 3x=\sin 2x$ for $0\le x\le 2\pi$?

A. 5

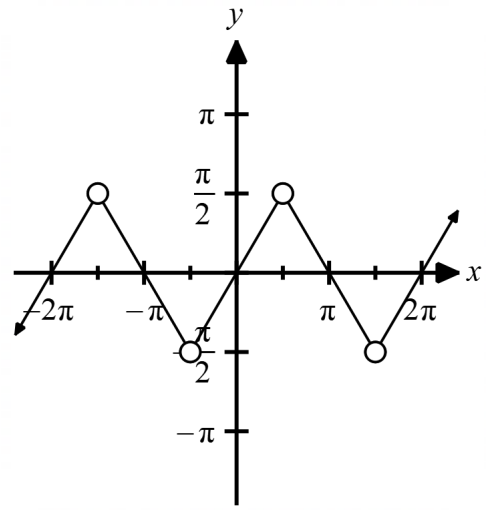
B. 6

C. 7

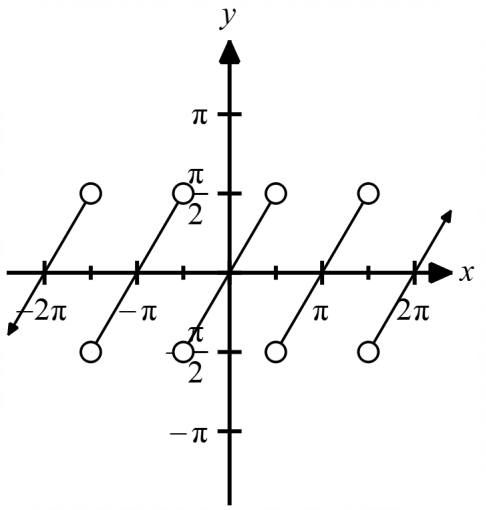
D. 8

8 Which graph represents the function $y = \tan^{-1}(\tan x)$?

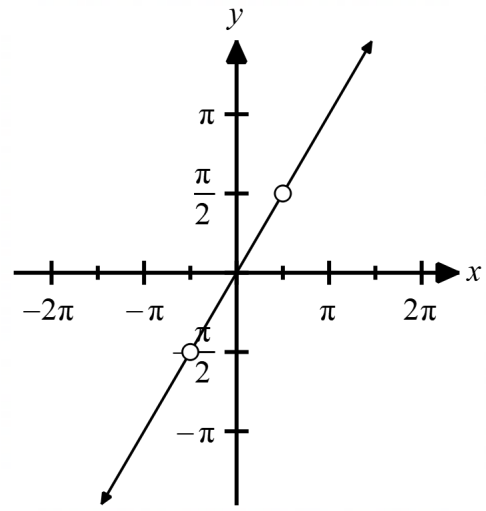
A.



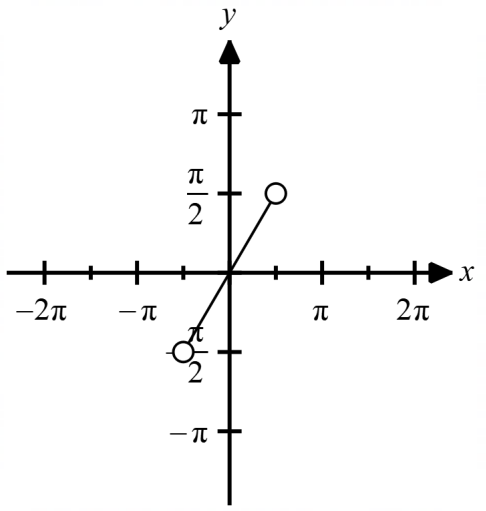
B.



C.



D.



9 It is known that the function $f(x)$ is one-to-one and continuous for all real x.

Let $g(x) = f^{-1}(x)$.

Which of the following statements MUST be true for any value of x where $f'(x)$ and $g'(x)$ are both defined?

A. $f'(x) = \dfrac{-1}{g'(x)}$

B. $f'(x) = \dfrac{1}{g'(x)}$

C. $f'(x)g'(x) < 0$

D. $f'(x)g'(x) > 0$

10 Which statement is always true for real numbers a and b where $0 \le a < b \le 1$?

A. $\cos^{-1} a < \cos^{-1} b$

B. $\cos^{-1} a - \sin^{-1} a > \cos^{-1} b - \sin^{-1} b$

C. $\cos^2 a < \cos^2 b$

D. $\sin a \cos a > \sin b \cos b$

End of Section I

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) The parametric equations of a line are given below. **2**

$$x = 2 - 5t$$
$$y = 3t + 1$$

Find the Cartesian equation of this line in the form $y = mx + c$.

(b) Find the inverse function, $f^{-1}(x)$, of the function $f(x) = e^{2x} - 1$. **2**

(c) The graph of $y = f(x)$ is given on the separate Response Sheet for Question 11 (c). **2**

On the graph provided in the response sheet, sketch the graph of $y = \sqrt{f(x)}$, clearly showing all important features.

(d) Consider the polynomial **3**

$$P(x) = x^3 - 2x^2 - 7x - 4.$$

Show that $x + 1$ is a factor of $P(x)$ and hence find $P(x)$ in fully factorised form.

Question 11 continues on page 9

(e) Solve $\frac{2x}{3-x} \leq 4$. **3**

(f) Find the coefficient of x^{10} in the expansion of $\left(\frac{2}{x} - x^3\right)^6$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Use Mathematical induction to prove that $4n^3 - n$ is divisible by 3 for all integers $n \geq 1$. **3**

(b) (i) Express $3\sin x - \sqrt{3}\cos x$ in the form $2\sqrt{3}\sin(x+\alpha)$, where $0 < \alpha < 2\pi$. **2**

(ii) Hence, or otherwise, solve $3\sin x = \sqrt{3}\cos x - \sqrt{3}$, where $0 \leq x < 2\pi$. **2**

(c) Evaluate $\int_0^{\frac{\sqrt{3}}{2}} \frac{8x^2}{\sqrt{1-x^2}}\,dx$ using the substitution $x = \sin\theta$. **4**

(d) Charles is a member of an elite racing group of 11 people. A team of 5 people is to be formed from the group. Use a combinatorial argument in this context to prove that **2**

$$\binom{11}{5} = \binom{10}{4} + \binom{10}{5}.$$

(e) By sketching a graph, or otherwise, solve $(|x|-1)(|x|-2) = 6$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Let $f(x) = \tan^{-1} x - \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$, for $x > -1$.

(i) Show that $f(x) = k$ where k is a real number. **2**

(ii) Hence, find the value of k. **1**

(b) A particle is moving on the curve with equation **4**

$$y = 2\arcsin 3x, \quad -\frac{1}{3} \le x \le \frac{1}{3}.$$

The particle has coordinates (x, y) at time t, where distances are measured in metres and time in seconds.

At $t = T$, the y coordinate of the particle is $\dfrac{1}{3}\pi$ m and the y coordinate is increasing at a rate of 2 m/s.

Find the exact rate at which the x coordinate of the particle is changing at $t = T$.

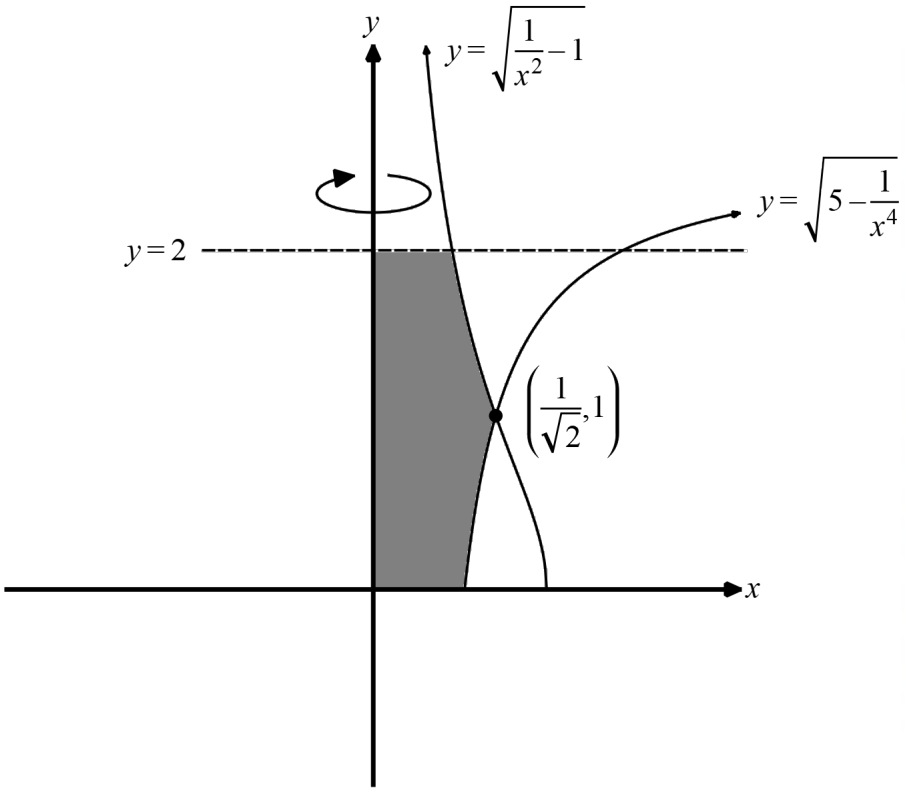
(c) At a particular high school, 17 students, including Xavi and Yuri, are enrolled to study Mathematics Extension 1. There are three Mathematics Extension 1 classes with a maximum class size of 6. Each class is taught by a different teacher.

(i) Will there always be a class that is full? Justify your answer. **1**

(ii) Students are randomly assigned into classes. Find the number of ways the classes can be assigned to teachers such that Xavi and Yuri are in the same class. **2**

Question 13 continues on page 12

(d) A solid trophy is to be made by rotating the shaded region below around the y-axis. **5**



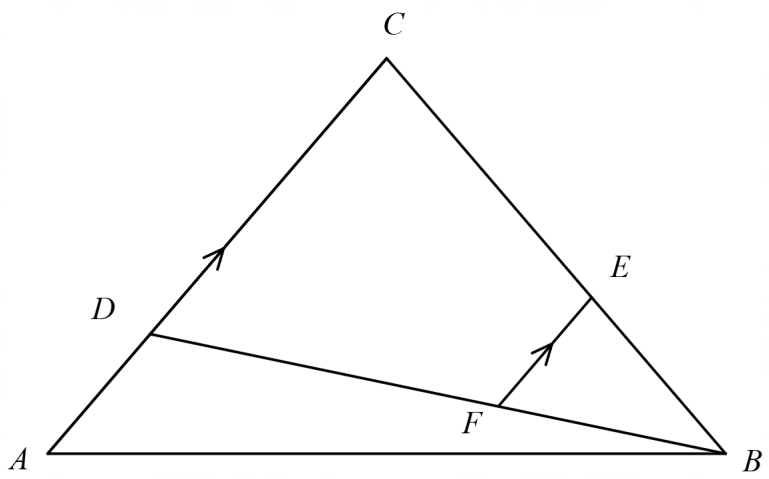
Show that the volume of the trophy is $\frac{\pi^2}{4}\ u^3$.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) In triangle *ABC, D* lies on *AC* such that $\overrightarrow{AD} = \frac{1}{4}\overrightarrow{AC}$, *E* lies on *BC* such that $\overrightarrow{BE} = \frac{1}{3}\overrightarrow{BC}$.

F lies on *BD* such that $FE \parallel AC$.



(i) Show that $\overrightarrow{FE} = \overrightarrow{FB} + \frac{1}{3}\overrightarrow{BC}$. **1**

(ii) Let $\overrightarrow{FE} = k\overrightarrow{DC}$. Use vector methods to show that $\overrightarrow{FE} = \overrightarrow{AD}$. **3**

Question 14 continues on page 14

(b) The polynomial $p(x) = x^3 + bx^2 + cx + d$ has stationary points at $x = \alpha$ and $x = \beta$, where $\alpha \neq \beta$.

(i) Show that $b^2 > 3c$. **2**

(ii) You are given that $m^3 - n^3 = (m-n)(m^2 + mn + n^2)$. (Do NOT prove this) **3**

Show that $\dfrac{p(\alpha) - p(\beta)}{\alpha - \beta} = \dfrac{2}{9}(3c - b^2)$.

(iii) Hence, or otherwise, find the relationship between α and β such that $p(\alpha) > p(\beta)$. **1**

Question 14 continues on page 15

(c) Particle A is projected from the origin with initial speed V m s^{-1} at an angle θ with the horizontal plane. The position vector of particle A, t seconds after it is projected, is given by

$$\vec{r}_A(t) = \begin{pmatrix} Vt\cos\theta \\ Vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}. \quad \text{(Do NOT prove this)}$$

(i) Show that the velocity vector of particle A is given by $\vec{v}_A(t) = \begin{pmatrix} V\cos\theta \\ V\sin\theta - gt \end{pmatrix}$. **1**

After T seconds, particle B is projected from the origin with the same initial speed and angle as particle A. The position vector of particle B is given by

$$\vec{r}_B(t) = \begin{pmatrix} V(t-T)\cos\theta \\ V(t-T)\sin\theta - \frac{1}{2}g(t-T)^2 \end{pmatrix}, \; t \geq T. \quad \text{(Do NOT prove this)}$$

(ii) Show that the particles will never travel in perpendicular directions at any time t if $T < \dfrac{2V\cos\theta}{g}$. **4**

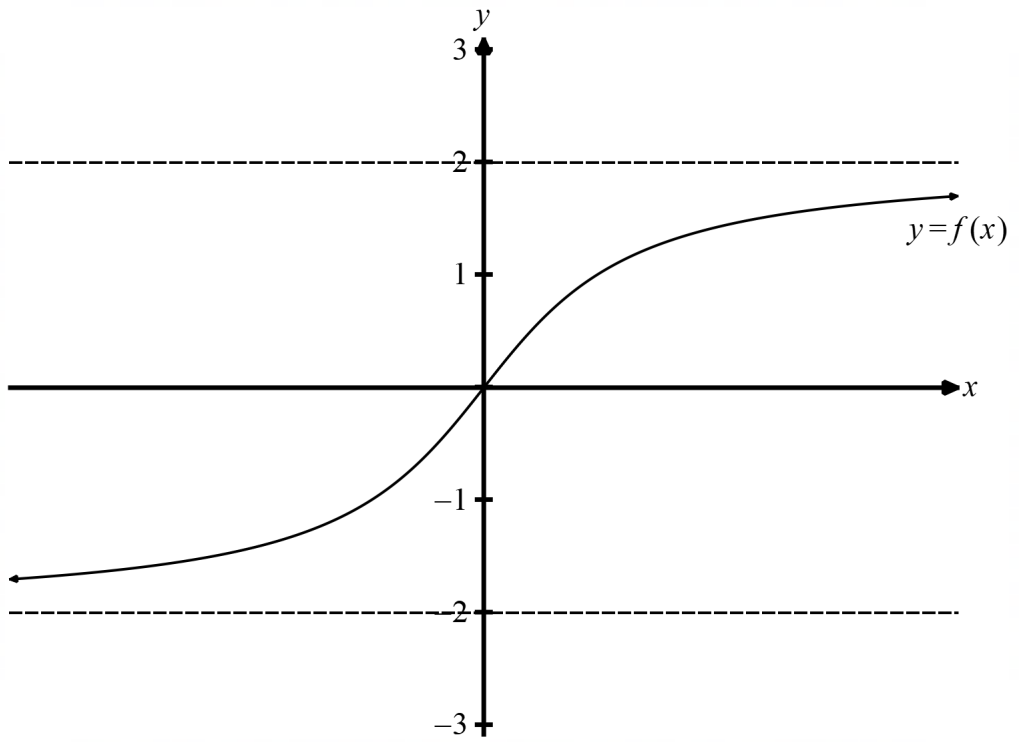
End of paper

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Student Number:_____

Question 11 (c) – Response Sheet

11 (c) The graph of $y = f(x)$ is shown below.



On the same grid above, sketch the graph of $y = \sqrt{f(x)}$, clearly showing all important features. **2**

Place this sheet inside your Question 11 answer booklet.

North Sydney Girls High School

2026 **HSC TRIAL EXAMINATION**

Mathematics Extension 1

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- Attempt Questions 1 – 10
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NAME:_____ **TEACHER:**_____

STUDENT NUMBER:

Question	1-10	11	12	13	14	Total
Mark	/10	/15	/15	/15	/15	/70

Section I

10 marks
Attempt Questions 1-10
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Use the multiple choice answer sheet for Questions 1-10.

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1 Given that $\overrightarrow{OA}=\begin{pmatrix}2\\-7\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}-3\\4\end{pmatrix}$, what is $\overrightarrow{AB}$?

A. $\begin{pmatrix}5\\11\end{pmatrix}$

B. $\begin{pmatrix}-5\\11\end{pmatrix}$

C. $\begin{pmatrix}1\\3\end{pmatrix}$

D. $\begin{pmatrix}-1\\-3\end{pmatrix}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-3\\4\end{pmatrix}-\begin{pmatrix}2\\-7\end{pmatrix}=\begin{pmatrix}-5\\11\end{pmatrix}$$

2 The polynomial $p(x)=x^3-6x^2+32$ has one repeated zero of multiplicity 2 at $x=\alpha$.
What is the value of α?

A. -2

B. 0

C. 2

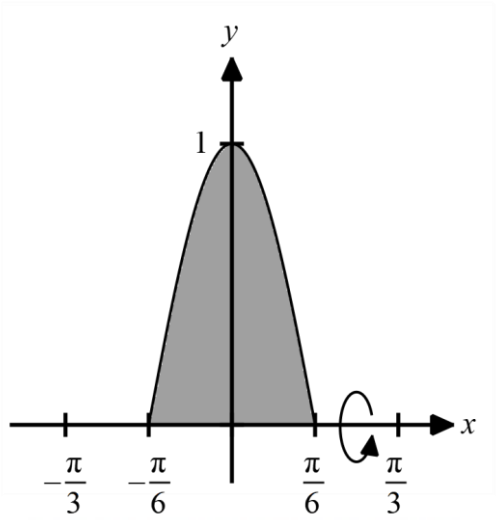
D. 4

$$p(\alpha)=0 \text{ and } p'(\alpha)=0$$

$$p'(x)=3x^2-12x=3x(x-4)$$
$$\therefore p'(4)=0 \text{ and } p'(0)=0$$

$$p(4)=4^3-6(4)^2+32=64-96+32=0$$
$$p(0)=32\neq 0$$

3 A solid of revolution is to be formed by rotating the region bounded by the x-axis and the curve $y = \cos 3x$ between $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$ about the x-axis, as shown in the diagram.



Which of the following gives the volume of solid of revolution formed?

A. $\pi \int_0^{\frac{\pi}{6}} 1 + \cos 6x \; dx$

B. $\pi \int_0^{\frac{\pi}{6}} \cos^2 3x \; dx$

C. $\frac{\pi}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3x \; dx$

D. $\frac{\pi}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1 - \cos 6x \; dx$

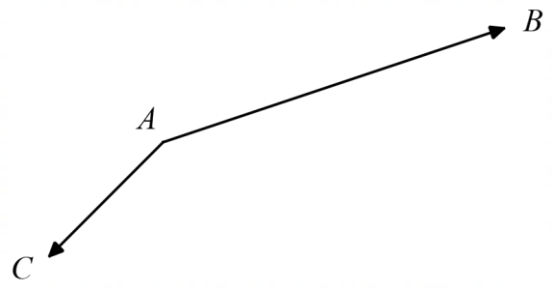
$$V = \pi \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3x \; dx \quad = 2\pi \int_0^{\frac{\pi}{6}} \cos^2 3x \; dx \quad \left(\because y = \cos^2 3x \text{ is even}\right)$$

$$= 2\pi \int_0^{\frac{\pi}{6}} \frac{1}{2}\left(1 + \cos 6x\right) \; dx$$

$$= \pi \int_0^{\frac{\pi}{6}} 1 + \cos 6x \; dx$$

4 The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ is given by $\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\right)\underset{\sim}{v}$.

The diagram below shows points *A*, *B* and *C*, where $\overrightarrow{AB} = 3\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{AC} = -\underset{\sim}{i} - \underset{\sim}{j}$.



Which of the following is the projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$?

A. $\frac{2}{5}\left(3\underset{\sim}{i} + \underset{\sim}{j}\right)$

B. $\frac{-2}{5}\left(3\underset{\sim}{i} + \underset{\sim}{j}\right)$

C. $2\left(\underset{\sim}{i} + \underset{\sim}{j}\right)$

D. $2\left(-\underset{\sim}{i} - \underset{\sim}{j}\right)$

Quick check: projection vector is parallel to $\overrightarrow{AC}$ but in opposite direction to $\overrightarrow{AC}$. Only C can be correct.

$$\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\right)\underset{\sim}{v} = \left(\frac{3\times(-1)+1\times(-1)}{\left(\sqrt{(-1)^2+(-1)^2}\right)^2}\right)\overrightarrow{AC} = \frac{-4}{2}\left(-\underset{\sim}{i} - \underset{\sim}{j}\right) = 2\left(\underset{\sim}{i} + \underset{\sim}{j}\right)$$

5 Five different schools send two prefects each to attend a conference. The group of 10 is to be seated at a circular table. In how many ways can the prefects be seated if prefects from the same school are to be seated together?

A. $4!$

B. $2\times 4!$

C. $2^4\times 4!$

D. $2^5\times 4!$

Group the pairs of prefects from the same school. There are five groups to arrange in a circular table in $4!$ ways. Each pair within the group can be arranged in $2!=2$ ways for a total of 2^5 ways. Therefore, total number of arrangements is $2^5\times 4!$.

6 You are given that $f(x)$ is a one-to-one function defined for $x\in(-\infty,\infty)$.
Which of the following functions cannot have an inverse?

A. $f(|x|)$

B. $|f(x)|$

C. $\dfrac{1}{f(x)}$

D. $\sqrt{f(x)}$

$f(|x|)$ is always even. Therefore, it cannot have an inverse (its inverse will not be a function)

7 How many distinct solutions are there to the equation $\sin x + \sin 3x = \sin 2x$ for $0 \le x \le 2\pi$?

A. 5

B. 6

C. 7

D. 8

Using sum-to-product result

$$\frac{1}{2}(\sin x + \sin 3x) = \frac{1}{2}(\sin(2x + x) + \sin(2x - x)) = \sin 2x \cos x$$

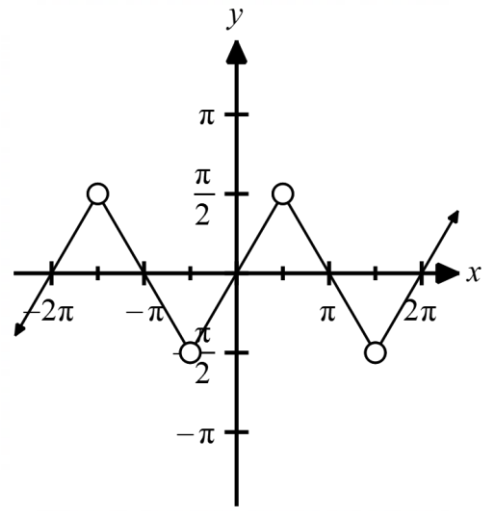
$$\sin x + \sin 3x = \sin 2x$$
$$2 \sin 2x \cos x = \sin 2x$$
$$\sin 2x(2 \cos x - 1) = 0$$

$$\sin 2x = 0 \qquad \cos x = \frac{1}{2}$$

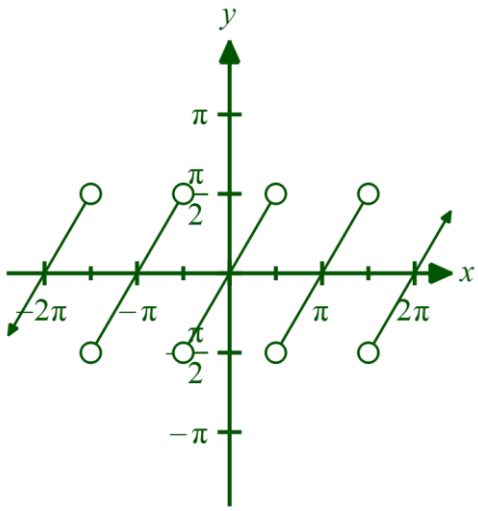
$\sin 2x = 0$ has 5 solutions for $0 \le x \le 2\pi$ and $\cos x = \frac{1}{2}$ has 2 solutions for $0 \le x \le 2\pi$, and the solutions are unique. Therefore, there are 7 solutions in total.

8 Which graph represents the function $y = \tan^{-1}(\tan x)$?

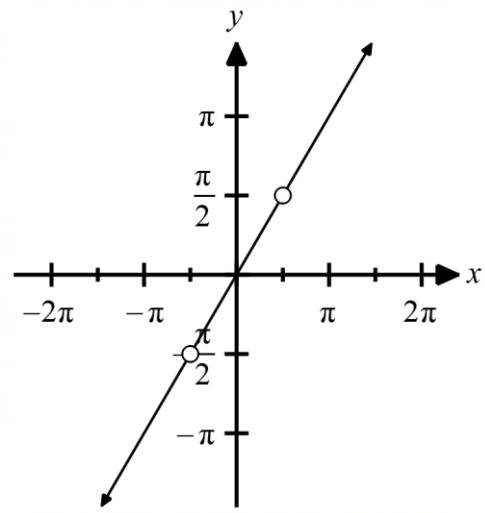
A.



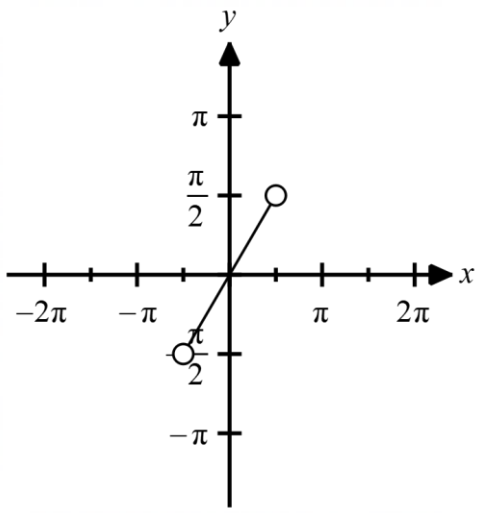
B.



C.



D.



9 It is known that the function $f(x)$ is one-to-one and continuous for all real x.

Let $g(x)=f^{-1}(x)$.

Which of the following statements MUST be true for any value of x where $f'(x)$ and $g'(x)$ are both defined?

A. $f'(x)=\dfrac{-1}{g'(x)}$

B. $f'(x)=\dfrac{1}{g'(x)}$

C. $f'(x)g'(x)<0$

D. $f'(x)g'(x)>0$

$f(x)$ and $g(x)$ are either both monotonically increasing or both monotonically decreasing.
Therefore:
- $f(x)>0$ and $g(x)>0$
- $f(x)<0$ and $g(x)<0$

Both result in $f'(x)g'(x)>0$

10 Which statement is always true for real numbers a and b where $0\le a<b\le 1$?

A. $\cos^{-1}a<\cos^{-1}b$

B. $\cos^{-1}a-\sin^{-1}a>\cos^{-1}b-\sin^{-1}b$

C. $\cos^{2}a<\cos^{2}b$

D. $\sin a\cos a>\sin b\cos b$

Consider

$$f(x)=\cos^{-1}x-\sin^{-1}x=\cos^{-1}x-\left(\frac{\pi}{2}-\cos^{-1}x\right)=2\cos^{-1}x-\frac{\pi}{2}$$

$f(x)$ is a decreasing function for $0\le x\le 1$ as $2\cos^{-1}x$ is a decreasing function.

$$\therefore \cos^{-1}a-\sin^{-1}a>\cos^{-1}b-\sin^{-1}b$$

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) The parametric equations of a line are given below. **2**

$$x = 2 - 5t$$
$$y = 3t + 1$$

Find the Cartesian equation of this line in the form $y = mx + c$.

$$x = 2 - 5t$$
$$t = \frac{2 - x}{5}$$

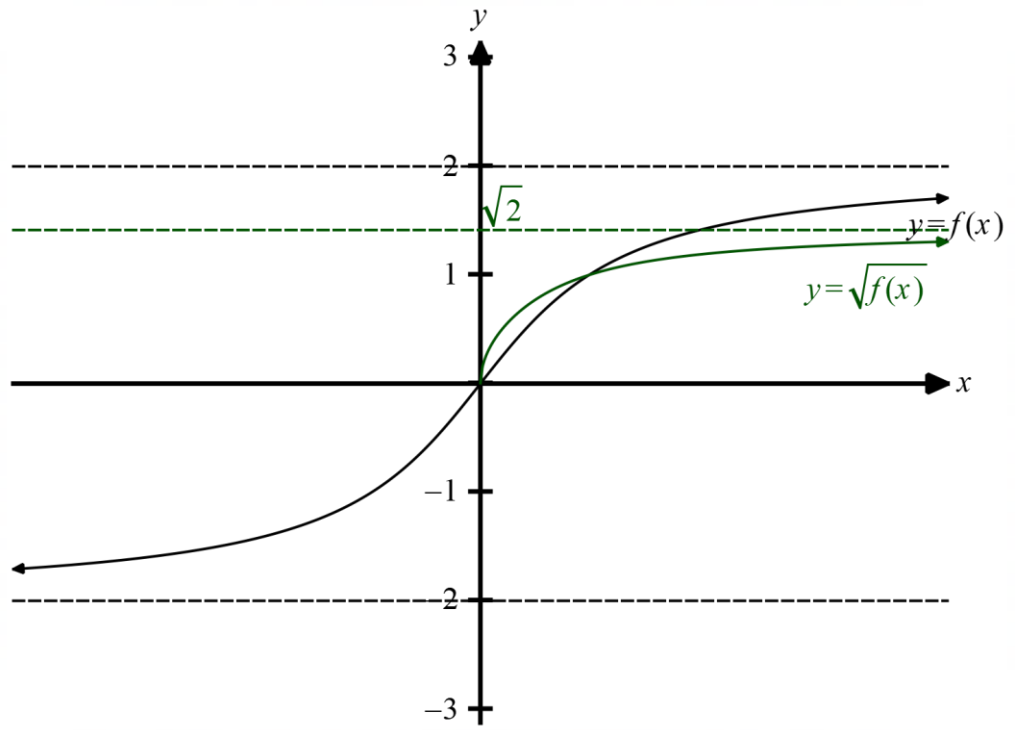
$$y = 3t + 1$$
$$y = 3\left(\frac{2 - x}{5}\right) + 1$$
$$y = \frac{-3x}{5} + \frac{6}{5} + 1$$
$$y = \frac{-3x}{5} + \frac{11}{5}$$

(b) Find the inverse function, $f^{-1}(x)$, of the function $f(x) = e^{2x} - 1$. **2**

$$x = e^{2y} - 1$$
$$e^{2y} = x + 1$$
$$2y = \ln(x + 1)$$
$$y = \frac{1}{2}\ln(x + 1)$$
$$f^{-1}(x) = \frac{1}{2}\ln(x + 1)$$

(c) The graph of $y = f(x)$ is given on the separate Response Sheet for Question 11 (c). **2**

On the graph provided in the response sheet, sketch the graph of $y = \sqrt{f(x)}$, clearly showing all important features.



(d) Consider the polynomial **3**

$$P(x) = x^3 - 2x^2 - 7x - 4 .$$

Show that $x+1$ is a factor of $P(x)$ and hence find $P(x)$ in fully factorised form.

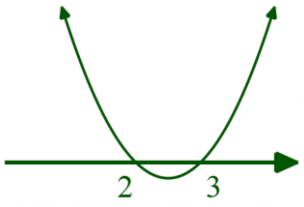
$$P(-1) = (-1)^3 - 2(-1)^2 - 7(-1) - 4 = -1 - 2 + 7 - 4 = 0$$

$$\begin{array}{r} x^2 - 3x - 4 \\ (x+1)\overline{)\,x^3 - 2x^2 - 7x - 4} \\ \underline{x^3 + x^2} \\ -3x^2 - 7x - 4 \\ \underline{-3x^2 - 3x} \\ -4x - 4 \\ \underline{-4x - 4} \end{array}$$

$$P(x) = (x+1)(x^2 - 3x - 4) = (x+1)(x+1)(x-4) = (x+1)^2(x-4)$$

(e) Solve $\dfrac{2x}{3-x} \le 4$. **3**

$$\frac{2x}{3-x} \times (3-x)^2 \le 4 \times (3-x)^2$$
$$2x(3-x) \le 4(3-x)^2$$
$$4(3-x)^2 - 2x(3-x) \ge 0$$
$$(3-x)\left[4(3-x) - 2x\right] \ge 0$$
$$(3-x)(12-6x) \ge 0$$
$$6(x-3)(x-2) \ge 0$$



$\therefore x \le 2,\ x > 3\ \ (\because x \ne 3)$

(f) Find the coefficient of x^{10} in the expansion of $\left(\dfrac{2}{x} - x^3\right)^6$. **3**

The expansion will contain terms in the form of:

$$\binom{6}{r}\left(\frac{2}{x}\right)^{6-r}\left(-x^3\right)^r$$
$$= \binom{6}{r} 2^{6-r} x^{r-6} (-1)^r x^{3r}$$
$$= \binom{6}{r} 2^{6-r} (-1)^r x^{4r-6}$$

To obtain x^{10}:

$$4r - 6 = 10$$
$$r = 4$$

coefficient of x^{10}:

$$\binom{6}{4} 2^{6-4} (-1)^4 = 60$$

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Use Mathematical induction to prove that $4n^3 - n$ is divisible by 3 for all integers $n \geq 1$. **3**

When $n = 1$

$$4(1)^3 - 1 = 3 = 3 \times 1$$

$\therefore$ statement is true for $n = 1$

Assume the statement is true for $n = k$

$$4k^3 - k = 3A \text{ where } A \in \mathbb{Z}$$

When $n = k + 1$

$$\begin{aligned}
&4(k+1)^3 - (k+1) \\
&= 4(k^3 + 3k^2 + 3k + 1) - k - 1 \\
&= (4k^3 - k) + 3 \times 4k^2 + 3 \times 4k + 4 - 1 \\
&= 3A + 3(4k^2 + 4k + 1) \\
&= 3(A + 4k^2 + 4k + 1) \\
&= 3B \quad (\text{where } B = A + 4k^2 + 4k + 1 \in \mathbb{Z}, \ \because A, k \in \mathbb{Z})
\end{aligned}$$

$\therefore$ statement is true for $n = k + 1$ if it is true for $n = k$.

Hence the statement is true for all integers $n \geq 1$ by Mathematical Induction.

(b) (i) Express $3\sin x - \sqrt{3}\cos x$ in the form $2\sqrt{3}\sin(x+\alpha)$, where $0 < \alpha < 2\pi$. **2**

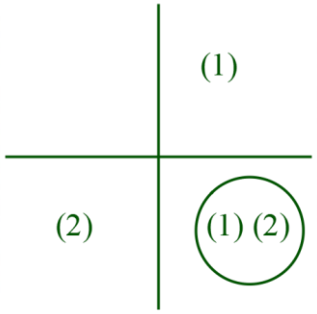
$$3\sin x - \sqrt{3}\cos x = 2\sqrt{3}(\sin x\cos\alpha + \cos x\sin\alpha)$$

$$3\sin x - \sqrt{3}\cos x = (2\sqrt{3}\cos\alpha)\sin x + (2\sqrt{3}\sin\alpha)\cos x$$

By equating coefficients of $\sin x$ and $\cos x$:

$$3 = 2\sqrt{3}\cos\alpha \quad (1) \qquad -\sqrt{3} = 2\sqrt{3}\sin\alpha \quad (2)$$

$$\cos\alpha = \frac{3}{2\sqrt{3}} \quad (1) \qquad \sin\alpha = \frac{-\sqrt{3}}{2\sqrt{3}} \quad (2)$$



$\therefore \alpha$ must be in the 4th quadrant

$$\cos\alpha = \frac{3}{2\sqrt{3}}$$

$$\cos\alpha = \frac{\sqrt{3}}{2}$$

$$\alpha = \frac{11\pi}{6}$$

$$3\sin x - \sqrt{3}\cos x = 2\sqrt{3}\sin\left(x + \frac{11\pi}{6}\right)$$

(ii) Hence, or otherwise, solve $3\sin x = \sqrt{3}\cos x - \sqrt{3}$, where $0 \le x < 2\pi$. **2**

$$3\sin x = \sqrt{3}\cos x - \sqrt{3}$$

$$3\sin x - \sqrt{3}\cos x = -\sqrt{3}$$

$$2\sqrt{3}\sin\left(x + \frac{11\pi}{6}\right) = -\sqrt{3} \quad (\text{from (i)})$$

$$\sin\left(x + \frac{11\pi}{6}\right) = \frac{-1}{2}$$

$$x + \frac{11\pi}{6} = \frac{11\pi}{6}, \frac{19\pi}{6}$$

$$x = 0, \frac{4\pi}{3}$$

(c) Evaluate $\int_0^{\frac{\sqrt{3}}{2}} \frac{8x^2}{\sqrt{1-x^2}}\,dx$ using the substitution $x=\sin\theta$. **4**

$$x=\sin\theta$$
$$\frac{dx}{d\theta}=\cos\theta$$
$$dx=\cos\theta\,d\theta$$

when $x=\frac{\sqrt{3}}{2}$, $\theta=\frac{\pi}{3}$

when $x=0$, $\theta=0$

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{8x^2}{\sqrt{1-x^2}}\,dx=\int_0^{\frac{\pi}{3}} \frac{8\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta=\int_0^{\frac{\pi}{3}} \frac{8\sin^2\theta}{\sqrt{\cos^2\theta}}\cdot\cos\theta\,d\theta=\int_0^{\frac{\pi}{3}} 8\sin^2\theta\,d\theta$$

$$\int_0^{\frac{\pi}{3}} 8\sin^2\theta\,d\theta=4\int_0^{\frac{\pi}{3}} 1-\cos 2\theta\,d\theta=4\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}}=4\left[\left(\frac{\pi}{3}-\frac{1}{2}\sin\left(2\cdot\frac{\pi}{3}\right)\right)-\left(0-\frac{1}{2}\sin(2\cdot 0)\right)\right]$$
$$=4\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$$

(d) Charles is a member of an elite racing group of 11 people. A team of 5 people is to be formed from the group. Use a combinatorial argument in this context to prove that **2**

$$\binom{11}{5}=\binom{10}{4}+\binom{10}{5}.$$

LHS:

Total number of teams that can be formed by picking 5 people from 11: $\binom{11}{5}$

RHS:

Case 1: Charles is on the team

Picking the remaining 4 people on the team from the remaining 10 people: $\binom{10}{4}$

Case 2: Charles is not on the team

Picking the remaining 5 people on the team from the 10 people (excluding Charles): $\binom{10}{5}$

Total: $\binom{10}{4}+\binom{10}{5}$

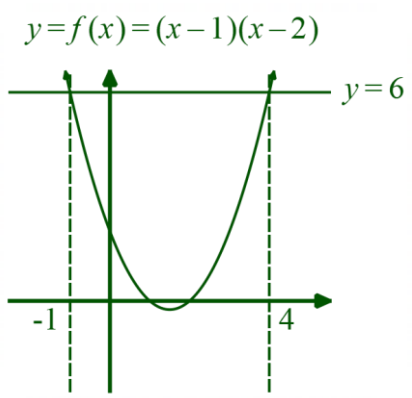
(e) By sketching a graph, or otherwise, solve $\left(|x|-1\right)\left(|x|-2\right)=6$. **2**

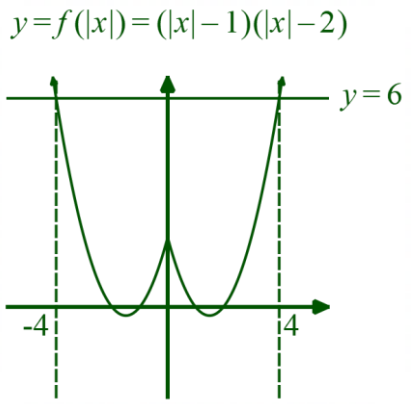
Let $f\left(x\right)=\left(x-1\right)\left(x-2\right)$

Solutions to $f\left(x\right)=6$,

$$\left(x-1\right)\left(x-2\right)=6$$
$$x^2-3x-4=0$$
$$\left(x-4\right)\left(x+1\right)=0$$
$$x=-1,\ 4$$

Consider the graph of $f\left(x\right)=\left(x-1\right)\left(x-2\right)$





$\therefore x=-4,\ 4$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Let $f(x)=\tan^{-1}x-\tan^{-1}\left(\dfrac{x-1}{x+1}\right)$, for $x>-1$.

(i) Show that $f(x)=k$ where k is a real number. **2**

$$f'(x)=\frac{1}{1+x^2}-\frac{1}{1+\left(\dfrac{x-1}{x+1}\right)^2}\times\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$$

$$=\frac{1}{1+x^2}-\frac{2}{(x+1)^2+(x-1)^2}$$

$$=\frac{1}{1+x^2}-\frac{2}{x^2+2x+1+x^2-2x+1}$$

$$=\frac{1}{1+x^2}-\frac{2}{2x^2+2}$$

$$=\frac{1}{1+x^2}-\frac{1}{1+x^2}$$

$$=0$$

$\because f'(x)=0$ then $f(x)$ must be a constant. $\therefore f(x)=k$

(ii) Hence, find the value of k. **1**

$$f(0)=\tan^{-1}0-\tan^{-1}\left(\frac{0-1}{0+1}\right)=0-\tan^{-1}(-1)=0-\left(-\frac{\pi}{4}\right)=\frac{\pi}{4}$$

$$\therefore k=\frac{\pi}{4}$$

(b) A particle is moving on the curve with equation **4**

$$y = 2\arcsin 3x, \quad -\frac{1}{3} \le x \le \frac{1}{3}.$$

The particle has coordinates (x, y) at time t, where distances are measured in metres and time in seconds.

At $t = T$, the y coordinate of the particle is $\frac{1}{3}\pi$ m and the y coordinate is increasing at a rate of 2 m/s.

Find the exact rate at which the x coordinate of the particle is changing at $t = T$.

$$\frac{dy}{dx} = 2\left(\frac{1}{\sqrt{1-(3x)^2}}\right)(3) = \frac{6}{\sqrt{1-9x^2}}$$

when $y = \frac{1}{3}\pi$

$$\frac{1}{3}\pi = 2\arcsin 3x$$

$$\arcsin 3x = \frac{\pi}{6}$$

$$3x = \frac{1}{2}$$

$$x = \frac{1}{6}$$

when $x = \frac{1}{6}$

$$\frac{dy}{dx} = \frac{6}{\sqrt{1-9\left(\frac{1}{6}\right)^2}} = \frac{6}{\sqrt{\frac{3}{4}}} = \frac{12}{\sqrt{3}}$$

$$\frac{dx}{dy} = \frac{\sqrt{3}}{12}$$

At $t = T$

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt} = \frac{\sqrt{3}}{12} \times 2 = \frac{\sqrt{3}}{6} \text{ m/s}$$

(c) At a particular high school, 17 students, including Xavi and Yuri, are enrolled to study Mathematics Extension 1. There are three Mathematics Extension 1 classes with a maximum class size of 6. Each class is taught by a different teacher.

(i) Will there always be a class that is full? Justify your answer. **1**

$$\frac{\text{number of students}}{\text{number of classes}} = \frac{17}{3} = 5.66... > 5$$

$\therefore$ Using the Pigeonhole principle, at least one class has 6 students. There will always be a class that is full.

Or

Distributing 15 students evenly into the 3 classes evenly will give 5 students in each class. The 16^{th} must be distributed to a class and make that class full. Therefore, there'll always be a class that is full

(ii) Students are randomly assigned into classes. Find the number of ways the classes can be assigned to teachers such that Xavi and Yuri are in the same class. **2**

The only possible class sizes are 2 classes of 6 students and 1 class of 5 students.
Consider two cases:
Case 1: Xavi and Yuri are in a class of 6 students

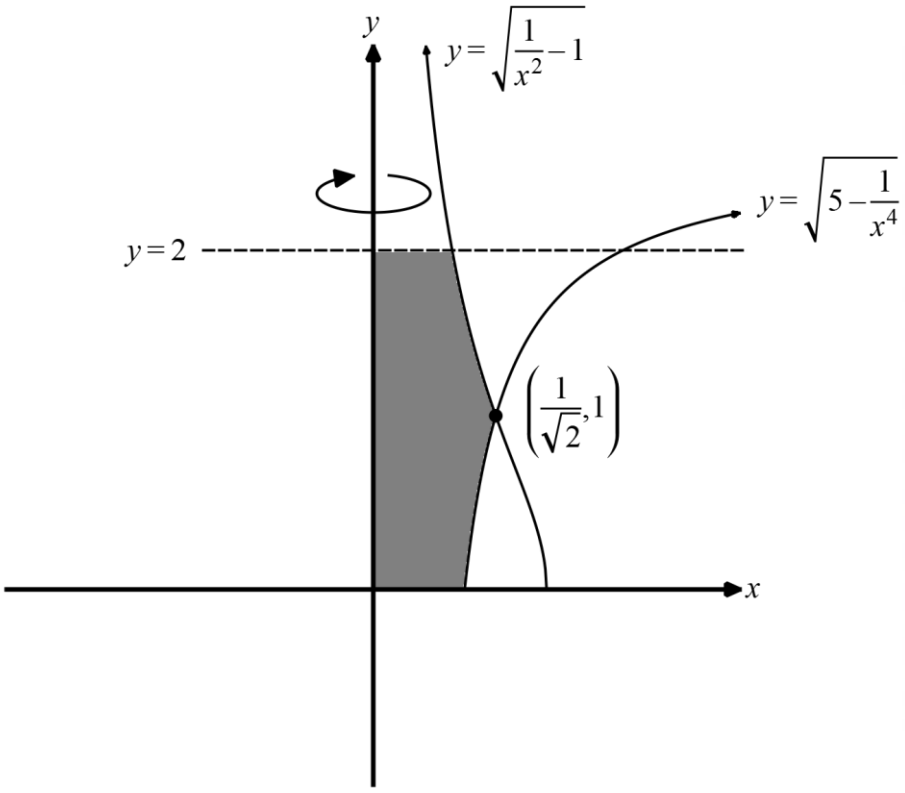
$$^{15}C_4 \times {}^{11}C_6 \times {}^{5}C_5 \times 3! = 3783780$$

Case 2: Xavi and Yuri are in a class of 5 students

$$\frac{^{15}C_3 \times {}^{12}C_6 \times {}^{6}C_6 \times 3!}{2!} = 1261260$$

Total: 5045040

(d) A solid trophy is to be made by rotating the shaded region below around the y-axis. **5**



Show that the volume of the trophy is $\frac{\pi^2}{4}\ u^3$.

$y=\sqrt{5-\frac{1}{x^4}}$

$y^2=5-\frac{1}{x^4}$

$x^4=\frac{1}{5-y^2}$

$x^2=\frac{1}{\sqrt{5-y^2}}$

$y=\sqrt{\frac{1}{x^2}-1}$

$y^2=\frac{1}{x^2}-1$

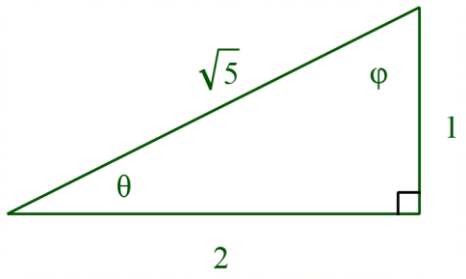
$x^2=\frac{1}{1+y^2}$

$$V_1 = \pi \int_0^1 \frac{1}{\sqrt{5-y^2}}\,dy = \pi\left[\sin^{-1}\left(\frac{y}{\sqrt{5}}\right)\right]_0^1 = \pi \sin^{-1}\frac{1}{\sqrt{5}}$$

$$V_2 = \pi \int_1^2 \frac{1}{1+y^2}\,dy = \pi\left[\tan^{-1} y\right]_1^2 = \pi\left(\tan^{-1} 2 - \tan^{-1} 1\right) = \pi\left(\tan^{-1} 2 - \frac{\pi}{4}\right)$$

$$V_{total} = \pi\left(\sin^{-1}\frac{1}{\sqrt{5}} + \tan^{-1} 2 - \frac{\pi}{4}\right)$$

Consider the following triangle



$$\theta + \varphi = \frac{\pi}{2}$$

$$\sin^{-1}\frac{1}{\sqrt{5}} + \tan^{-1}\frac{2}{1} = \frac{\pi}{2}$$

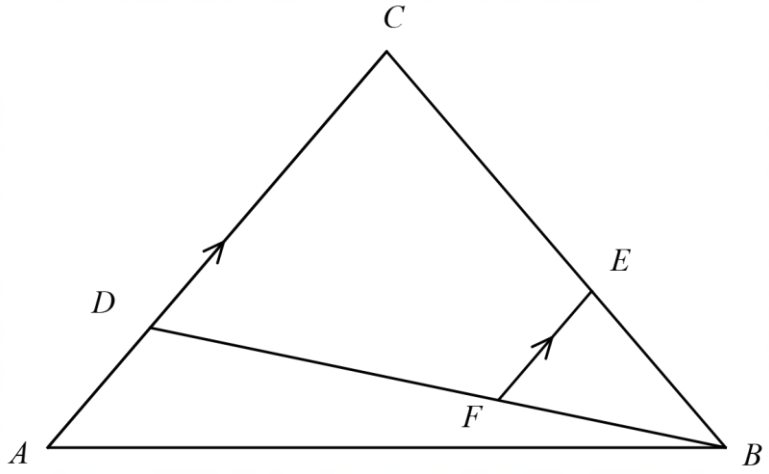
$$\therefore V_{total} = \pi\left(\sin^{-1}\frac{1}{\sqrt{5}} + \tan^{-1} 2 - \frac{\pi}{4}\right) = \pi\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi^2}{4}\ u^3$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) In triangle ABC, D lies on AC such that $\overrightarrow{AD} = \frac{1}{4}\overrightarrow{AC}$, E lies on BC such that $\overrightarrow{BE} = \frac{1}{3}\overrightarrow{BC}$.

F lies on BD such that $FE \parallel AC$.



(i) Show that $\overrightarrow{FE} = \overrightarrow{FB} + \frac{1}{3}\overrightarrow{BC}$. **1**

$$\overrightarrow{FE} = \overrightarrow{FB} + \overrightarrow{BE} = \overrightarrow{FB} + \frac{1}{3}\overrightarrow{BC}$$

(ii) Let $\overrightarrow{FE} = k\overrightarrow{DC}$. Use vector methods to show that $\overrightarrow{FE} = \overrightarrow{AD}$. **3**

$$k\overrightarrow{DC} = \overrightarrow{FB} + \frac{1}{3}\overrightarrow{BC}$$

$$k\left(\overrightarrow{DB} + \overrightarrow{BC}\right) = \overrightarrow{FB} + \frac{1}{3}\overrightarrow{BC}$$

$$k\left(\lambda\overrightarrow{FB} + \overrightarrow{BC}\right) = \overrightarrow{FB} + \frac{1}{3}\overrightarrow{BC} \quad \left(\therefore \overrightarrow{DB} \parallel \overrightarrow{FB}\right)$$

$$k\lambda\overrightarrow{FB} + k\overrightarrow{BC} = \overrightarrow{FB} + \frac{1}{3}\overrightarrow{BC}$$

$$\therefore k = \frac{1}{3},\ \overrightarrow{FE} = \frac{1}{3}\overrightarrow{DC},\ \overrightarrow{DC} = 3\overrightarrow{FE}$$

$$\overrightarrow{AD} = \frac{1}{4}\left(\overrightarrow{AD} + \overrightarrow{DC}\right)$$

$$\frac{3}{4}\overrightarrow{AD} = \frac{1}{4}\overrightarrow{DC}$$

$$3\overrightarrow{AD} = \overrightarrow{DC}$$

$$3\overrightarrow{AD} = 3\overrightarrow{FE}$$

$$\overrightarrow{AD} = \overrightarrow{FE}$$

(b) The polynomial $p(x)=x^3+bx^2+cx+d$ has stationary points at $x=\alpha$ and $x=\beta$, where $\alpha \neq \beta$.

(i) Show that $b^2>3c$. **2**

$$p'(x)=3x^2+2bx+c$$

Since there are two stationary points, $p'(x)=0$ must have two solutions, $\therefore \Delta>0$

$$(2b)^2-4(3)(c)>0$$
$$4b^2-12c>0$$
$$4b^2>12c$$
$$b^2>3c$$

(ii) You are given that $m^3-n^3=(m-n)(m^2+mn+n^2)$. (Do NOT prove this) **3**

Show that $\dfrac{p(\alpha)-p(\beta)}{\alpha-\beta}=\dfrac{2}{9}(3c-b^2)$.

$$\begin{aligned}p(\alpha)-p(\beta)&=\alpha^3+b\alpha^2+c\alpha+d-(\beta^3+b\beta^2+c\beta+d)\\&=\alpha^3-\beta^3+b(\alpha^2-\beta^2)+c(\alpha-\beta)\\&=(\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)+b(\alpha-\beta)(\alpha+\beta)+c(\alpha-\beta)\ \text{(using the given result)}\\&=(\alpha-\beta)\left[(\alpha^2+\beta^2+\alpha\beta)+b(\alpha+\beta)+c\right]\end{aligned}$$

$$\begin{aligned}\frac{p(\alpha)-p(\beta)}{\alpha-\beta}&=(\alpha^2+\beta^2+\alpha\beta)+b(\alpha+\beta)+c\\&=(\alpha+\beta)^2-\alpha\beta+b(\alpha+\beta)+c\\&=\left(-\frac{2b}{3}\right)^2-\left(\frac{c}{3}\right)+b\left(-\frac{2b}{3}\right)+c\\&=\frac{4b^2}{9}-\frac{2b^2}{3}+\frac{2c}{3}\\&=\frac{2c}{3}-\frac{2b^2}{9}\\&=\frac{2}{9}(3c-b^2)\end{aligned}$$

(iii) Hence, or otherwise, find the relationship between α and β such that $p(\alpha) > p(\beta)$. **1**

Method 1: Algebraic using (i) and (ii)
From (i)

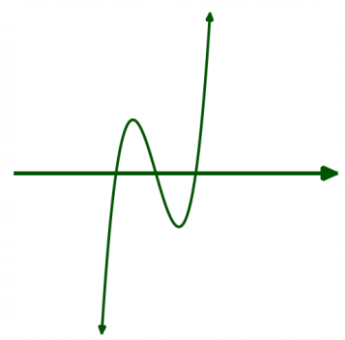
$$b^2 > 3c$$
$$3c - b^2 < 0$$
$$\therefore \frac{2}{9}(3c - b^2) < 0$$

RHS of (ii) is negative.

If $p(\alpha) > p(\beta)$, then $p(\alpha) - p(\beta) > 0$. For LHS of (ii) to also be negative,

$$\alpha - \beta < 0$$
$$\alpha < \beta$$

Method 2: Graphical
$p(x)$ is a monic polynomial of degree 3 with 2 stationary points, its graph must have one maxima and one minima similar to the graph below



If $p(\alpha) > p(\beta)$, then $x = \alpha$ must be the maxima and $x = \beta$ must be the minima.

$$\therefore \alpha < \beta$$

(c) Particle A is projected from the origin with initial speed V m s^{-1} at an angle θ with the horizontal plane. The position vector of particle A, t seconds after it is projected, is given by

$$\vec{r}_A(t) = \begin{pmatrix} Vt\cos\theta \\ Vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}. \quad \text{(Do NOT prove this)}$$

(i) Show that the velocity vector of particle A is given by $\vec{v}_A(t) = \begin{pmatrix} V\cos\theta \\ V\sin\theta - gt \end{pmatrix}$. **1**

$$\frac{d\left(\vec{r}_A(t)\right)}{dt} = \frac{d}{dt}\begin{pmatrix} Vt\cos\theta \\ Vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$

$$\vec{v}_A(t) = \begin{pmatrix} V\cos\theta \\ V\sin\theta - 2\left(\frac{1}{2}gt\right) \end{pmatrix}$$

$$\vec{v}_A(t) = \begin{pmatrix} V\cos\theta \\ V\sin\theta - gt \end{pmatrix}$$

After T seconds, particle B is projected from the origin with the same initial speed and angle as particle A. The position vector of particle B is given by

$$\vec{r}_B(t) = \begin{pmatrix} V(t-T)\cos\theta \\ V(t-T)\sin\theta - \frac{1}{2}g(t-T)^2 \end{pmatrix},\ t \geq T\ .$$ (Do NOT prove this)

(ii) Show that the particles will never travel in perpendicular directions at any time t if $T < \frac{2V\cos\theta}{g}$. **4**

$$\vec{v}_B(t) = \begin{pmatrix} V\cos\theta \\ V\sin\theta - g(t-T) \end{pmatrix}$$

If the particles are travelling at perpendicular directions, $\vec{v}_A(t) \cdot \vec{v}_B(t) = 0$.

$$(V\cos\theta)(V\cos\theta) + (V\sin\theta - gt)(V\sin\theta - g(t-T)) = 0$$
$$V^2\cos^2\theta + V^2\sin^2\theta - gV\sin\theta(t-T) - gVt\sin\theta + g^2t(t-T) = 0$$
$$V^2 - (gV\sin\theta)t + gVT\sin\theta - gVt\sin\theta + g^2t^2 - (g^2T)t = 0$$
$$g^2t^2 - (2gV\sin\theta + g^2T)t + (V^2 + gVT\sin\theta) = 0$$

The particles will never travel in perpendicular directions if there's no solution to the above quadratic equation in t.

$$\Delta < 0$$
$$(2gV\sin\theta + g^2T)^2 - 4g^2(V^2 + gVT\sin\theta) < 0$$
$$4g^2V^2\sin^2\theta + 4g^3VT\sin\theta + g^4T^2 - 4g^2V^2 - 4g^3VT\sin\theta < 0$$
$$g^4T^2 < 4g^2V^2 - 4g^2V^2\sin^2\theta$$
$$g^4T^2 < 4g^2V^2(1 - \sin^2\theta)$$
$$T^2 < \frac{4V^2\cos^2\theta}{g^2}$$
$$T < \frac{2V\cos\theta}{g}$$