Student Number: _____ **Teacher Name:** _____

PENRITH SELECTIVE HIGH SCHOOL
MATHEMATICS DEPARTMENT

2026 TERM 3 TRIAL HSC EXAMINATION

YEAR 12 Mathematics Extension 1

General Instructions:

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided with this examination paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- No correction tape or white out allowed.

Total marks: 70

Section I – 10 marks (pages 2–4)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–8)
- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Question / Topic	Multiple Choice	Q.11	Q.12	Q.13	Q.14	Total
Combinatorics	7 **/ 1**		a **/ 2**			**/ 3**
Functions	5 **/ 1**	a **/ 2**		a **/ 3**		**/ 6**
Proof				b **/ 4**		**/ 4**
Trigonometric Functions	1, 6, 9 **/ 3**		b, c **/ 5+2**			**/ 10**
Calculus	2, 3 **/ 2**	b, c, d **/ 2+4+3**	d, e **/ 3+3**	c **/ 4**		**/ 21**
Vectors	8, 10 **/ 2**			d **/ 4**	a, b **/ 5+10**	**/ 21**
Binomial Distribution	4 **/ 1**	e **/ 4**				**/ 5**
Total	**/ 10**	**/ 15**	**/ 15**	**/ 15**	**/ 15**	**/ 70**

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Find the domain and range of $y = \frac{1}{2}\cos^{-1}(3 - x)$.

A. The domain is $0 \leq x \leq \frac{\pi}{2}$, and the range is $2 \leq y \leq 4$.

B. The domain is $-4 \leq x \leq -2$, and the range is $0 \leq y \leq \frac{\pi}{2}$.

C. The domain is $2 \leq x \leq 4$, and the range is $0 \leq y \leq \frac{\pi}{2}$.

D. The domain is $2 \leq x \leq 4$, and the range is $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$.

2 Find $\int \frac{1}{\sqrt{9 - 16x^2}}\,dx$.

A. $\frac{1}{4}\sin^{-1}\frac{16x^2}{9} + C$

B. $\frac{1}{4}\cos^{-1}\frac{16x^2}{9} + C$

C. $\frac{1}{4}\sin^{-1}\frac{3}{4x} + C$

D. $-\frac{1}{4}\cos^{-1}\frac{4x}{3} + C$

3 Which function is a general solution of the differential equation $y'' + 4y = 0$?

A. $y = Ae^{2x} + Be^{-2x}$

B. $y = A\sin x + B\cos x$

C. $y = A\sin 2x + B\cos 2x$

D. $y = A\sin 4x + B\cos 4x$

4 A company manufactures microchips. Typically, about 3% of all microchips produced are defective. In a shipment of 80 microchips, find the probability that at least three microchips are defective.

A. 0.5681

B. 0.9098

C. 0.4319

D. 0.5193

5 The remainder when the polynomial $P(x)$ is divided by $(x+2)(x-5)$ is $3x+1$.
What is the remainder when $P(x)$ is divided by $x+2$?

A. 1

B. -5

C. -9

D. 16

6 Let $t = \tan\frac{x}{2}$. What values of t satisfy the equation $2\cos x + \sin x = -2$?

A. $t = 2$

B. $t = -2$ and $t = \pi$

C. $t = 2$ and $t = \pi$

D. $t = -2$

7 Let $P(x) = 1 - x^2 + x^4 - x^6 + x^8 - x^{10}$.
What is the coefficient of x^7 in the expansion of $P(x-1)$?

A. 0

B. 128

C. 112

D. -128

8 What is the unit vector in the direction of $\overrightarrow{AB}$, where A and B are the points $(-2, 1)$ and $(6, 3)$, respectively?

A. $\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$

B. $8\mathbf{i} + 2\mathbf{j}$

C. $\frac{2}{17}\mathbf{i} + \frac{1}{34}\mathbf{j}$

D. $\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

9 Given that $\tan x = k$, where $\pi < x < \frac{3\pi}{2}$, determine the value of $\sin\frac{x}{2}$.

A. $-\sqrt{\frac{1}{2}\left(1 + \frac{k}{\sqrt{1+k^2}}\right)}$

B. $\sqrt{\frac{1}{2}\left(1 + \frac{k}{\sqrt{1+k^2}}\right)}$

C. $-\sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1+k^2}}\right)}$

D. $\sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1+k^2}}\right)}$

10 If $|\underset{\sim}{a} - \underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}|$, where $\underset{\sim}{a} \neq \underset{\sim}{0}$ and $\underset{\sim}{b} \neq \underset{\sim}{0}$, which one of the following must be true?

A. $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

B. $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$

C. $\underset{\sim}{a} = \underset{\sim}{b}$

D. $\underset{\sim}{a} \perp \underset{\sim}{b}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section.

- Answer the questions in the spaces provided.
- Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate Writing Booklet

(a) Solve $|4x - 3| > 11$. **2**

(b) A cup of coffee is heated to 90 °C. The room temperature is maintained at a constant 20 °C, and the coffee cools to 50 °C after 8 minutes. The temperature of the coffee can be modelled by the equation $T = 20 + 70e^{-kt}$, where T is the temperature of the coffee in degrees Celsius, t is the time in minutes since the coffee was placed in the room, and k is a constant. **2**

Determine the time taken for the temperature of the coffee to reach 25 °C.

Give your answer correct to 1 decimal place.

(c) A solid is formed by rotating the region bounded by the curves $y = 4 - x^2$ and $y = 2 - x$ about the x-axis. Find the exact value of the volume of the solid. **4**

(d) (i) Complete the square for $x^2 - 6x + 13$, expressing the answer in the form $[f(x)]^2 + a$, where a is a constant and $f(x)$ is a linear function of x. **1**

(ii) Hence, find $\int \frac{1}{x^2 - 6x + 13}\, dx$. **2**

(e) A standard pack of 52 playing cards consists of 13 hearts, 13 diamonds, 13 clubs and 13 spades. Fifteen cards are drawn with replacement from the pack.

(i) Let X be the number of diamonds drawn. Find the mean and standard deviation of X, giving your answers correct to 2 decimal places, where applicable. **2**

(ii) Determine the probability that X is within one standard deviation of the mean. Give your answer correct to 3 decimal places. **2**

Question 12 (15 marks) Use a separate Writing Booklet

(a) There are 5 men and 6 women to choose from. In how many ways can 3 men and 5 women be selected and arranged in a row? **2**

(b) Consider the function $f(x) = \sqrt{3}\cos x + 3\sin x$ for $0 \leq x \leq 2\pi$.

(i) By first expanding $\cos(x-\alpha)$, express $f(x)$ in the form $R\cos(x-\alpha)$. **3**

(ii) Hence, solve $\sqrt{3}\cos x + 3\sin x = 1$, correct to 2 decimal places. **2**

(c) Express $\sin\left(2\cos^{-1}\frac{3}{4}\right)$ in the form $a\sqrt{b}$, where a and b are rational numbers. **2**

(d) A drone is flying horizontally at a constant height of 120 m above the ground. It moves horizontally away from a person at a constant speed of 6 m/s. **3**

Determine the rate at which the distance between the drone and the person is increasing when the drone is 150 m from the person.

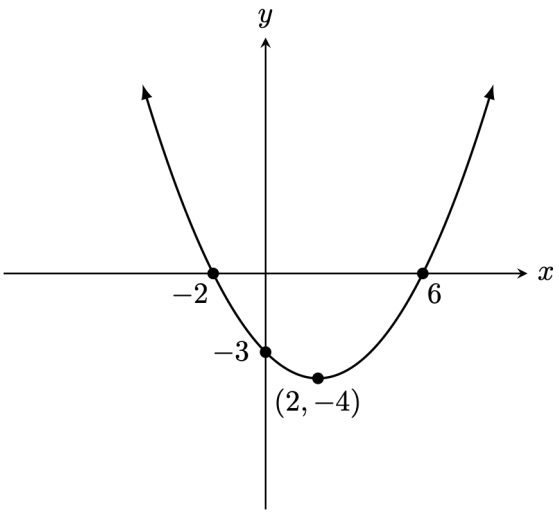
(e) Using the substitution $u = \sqrt{6x-1}$, show that **3**

$$\int \frac{x}{\sqrt{6x-1}}\,dx = \frac{1}{27}(3x+1)\sqrt{6x-1} + C.$$

Question 13 (15 marks) Use a separate Writing Booklet

(a) The graph of $y = f(x)$ is shown below. **3**

Sketch the graph of $y = |f(|x|)|$, clearly showing all key features.



(b) Prove by mathematical induction that $4(3^n) + 8^n$ is divisible by 5 for all integers $n \geq 1$. **4**

(c) Consider the differential equation $\frac{dy}{dx} = \frac{y(1-x)}{x}$.

(i) State any constant solutions. **1**

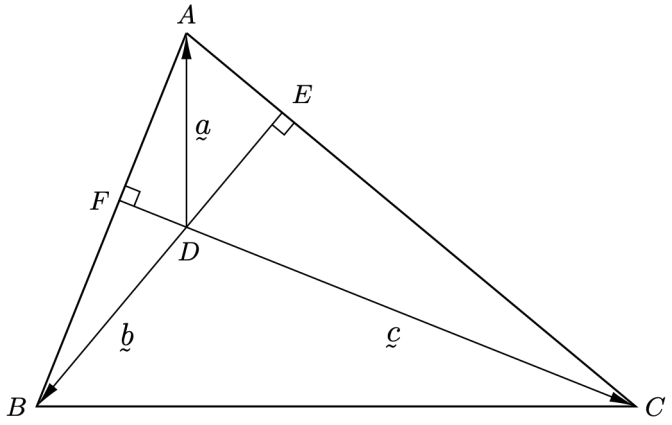
(ii) Find the general solution of the differential equation by separating the variables. **3**

(d) Find all possible values of k given that the length of the projection of $\mathbf{u}$ onto $\mathbf{v}$ is $\frac{108}{17}$, **4**

where $\mathbf{u} = k\mathbf{i} + 9\mathbf{j}$ and $\mathbf{v} = 15\mathbf{i} - 8\mathbf{j}$.

Question 14 (15 marks) Use a separate Writing Booklet

(a) The ΔABC shown below has altitudes BE and CF, where an altitude is a line segment drawn from a vertex perpendicular to the opposite side. The altitudes intersect at D. Let $\overrightarrow{DA} = \underset{\sim}{a}$, $\overrightarrow{DB} = \underset{\sim}{b}$ and $\overrightarrow{DC} = \underset{\sim}{c}$.



(i) Show that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot \underset{\sim}{c}$. **3**

(ii) Hence, or otherwise, prove that the three altitudes of ΔABC are concurrent; that is, they all pass through a single point. **2**

(b) A ball is projected from the base of a cliff with initial speed u m/s at an angle of θ to the horizontal. At the same time, a second ball is projected horizontally with initial speed w m/s from the top of the same cliff, which is h m high. Take the base of the cliff as the origin $(0, 0)$. The two balls collide.

(i) Show that the position vector of the first ball is **3**

$\mathbf{r_1}(t) = (ut\cos\theta)\mathbf{i} + \left(ut\sin\theta - \frac{g}{2}t^2\right)\mathbf{j}$.

(ii) It is given that $\theta = \tan^{-1} 3$ and the position vector of the second ball is **2**

$\mathbf{r_2}(t) = wt\mathbf{i} + \left(h - \frac{g}{2}t^2\right)\mathbf{j}$.

Show that $\cos\theta = \frac{1}{\sqrt{10}}$, and hence deduce that $u = \sqrt{10}w$.

(iii) Hence, prove that the balls collide at time $t = \frac{h}{3w}$, and that the height of **2**

the balls at the point of collision is $h - \frac{gh^2}{18w^2}$.

(iv) At the instant of collision, the velocity vectors of the two balls are perpendicular. **3**

Show that $h = \frac{3(3 \pm \sqrt{5})w^2}{2g}$.

End of paper

Student Number: Feedback & Solution **Teacher Name:** _____

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Section I

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1 Find the domain and range of $y = \frac{1}{2}\cos^{-1}(3 - x)$.

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B. The domain is $-4 \le x \le -2$, and the range is $0 \le y \le \frac{\pi}{2}$.

C. The domain is $2 \le x \le 4$, and the range is $0 \le y \le \frac{\pi}{2}$.

D. The domain is $2 \le x \le 4$, and the range is $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$.

Domain: $-1 \le 3 - x \le 1$

$-4 \le -x \le -2$

$4 \ge x \ge 2$

$\therefore 2 \le x \le 4$

Range: $0 \le \cos^{-1}(3 - x) \le \pi$

$\therefore 0 \le \frac{1}{2}\cos^{-1}(3 - x) \le \frac{\pi}{2}$

$\therefore 0 \le y \le \frac{\pi}{2}$

Answer: C

2 Find $\int \frac{1}{\sqrt{9 - 16x^2}}\,dx$.

A. $\frac{1}{4}\sin^{-1}\frac{16x^2}{9} + C$

B. $\frac{1}{4}\cos^{-1}\frac{16x^2}{9} + C$

C. $\frac{1}{4}\sin^{-1}\frac{3}{4x} + C$

D. $-\frac{1}{4}\cos^{-1}\frac{4x}{3} + C$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + C = -\cos^{-1}\frac{f(x)}{a} + C$$

$$\int \frac{1}{\sqrt{9 - 16x^2}}\,dx = \int \frac{1}{\sqrt{(3)^2 - (4x)^2}}\,dx = \frac{1}{4}\int \frac{4}{\sqrt{(3)^2 - (4x)^2}}\,dx = -\frac{1}{4}\int \frac{-4}{\sqrt{(3)^2 - (4x)^2}}\,dx$$

$$= -\frac{1}{4}\cos^{-1}\frac{4x}{3} + C$$

Answer: D

3 Which function is a general solution of the differential equation $y'' + 4y = 0$?

A. $y = Ae^{2x} + Be^{-2x}$

B. $y = A\sin x + B\cos x$

C. $y = A\sin 2x + B\cos 2x$

D. $y = A\sin 4x + B\cos 4x$

For option C,
$y = A\sin 2x + B\cos 2x$
$y' = 2A\cos 2x - 2B\sin 2x$
$y'' = -4A\sin 2x - 4B\cos 2x$
$y'' = -4(A\sin 2x + B\cos 2x)$
$y'' = -4y$
$\therefore y'' + 4y = 0$

Answer: C

4 A company manufactures microchips. Typically, about 3% of all microchips produced are defective. In a shipment of 80 microchips, find the probability that at least three microchips are defective.

A. 0.5681

B. 0.9098

C. 0.4319

D. 0.5193

Let X = number of defective microchips.
$P(X \geq 3) = 1 - P(X < 3)$
$= 1 - [P(X = 0) + (P(X = 1) + P(X = 2)]$
$= 1 - [(0.97)^{80} + {}^{80}\mathrm{C}_1(0.03)(0.97)^{79} + {}^{80}\mathrm{C}_2(0.03)^2(0.97)^{78}]$
$= 0.4319$

Answer: C

5 The remainder when the polynomial $P(x)$ is divided by $(x + 2)(x - 5)$ is $3x + 1$.
What is the remainder when $P(x)$ is divided by $x + 2$?

A. 1

B. -5

C. -9

D. 16

$P(x) = (x + 2)(x - 5)\,Q(x) + (3x + 1)$
$P(-2) = (-2 + 2)(-2 - 5)\,Q(-2) + (3(-2) + 1)$
$= -5$
Answer: B

6 Let $t = \tan\frac{x}{2}$. What values of t satisfy the equation $2\cos x + \sin x = -2$?

A. $t = 2$

B. $t = -2$ and $t = \pi$

C. $t = 2$ and $t = \pi$

D. $t = -2$

$2\cos x + \sin x = -2$

$\frac{2(1-t^2)}{1+t^2} + \frac{2t}{1+t^2} = -2$

$2(1-t^2) + 2t = -2(1+t^2)$

$2 - 2t^2 + 2t = -2 - 2t^2$

$2t = -4$

$t = -2$

Answer: D

7 Let $P(x) = 1 - x^2 + x^4 - x^6 + x^8 - x^{10}$.
What is the coefficient of x^7 in the expansion of $P(x-1)$?

A. 0

B. 128

C. 112

D. -128

$P(x-1) = 1 - (x-1)^2 + (x-1)^4 - (x-1)^6 + (x-1)^8 - (x-1)^{10}$

$(x-1)^8 = \sum_{r=0}^{8} {}^8\mathrm{C}_r x^{8-r}(-1)^r$

$(x-1)^{10} = \sum_{r=0}^{10} {}^{10}\mathrm{C}_r x^{10-r}(-1)^r$

x^7 term $= {}^8\mathrm{C}_1 x^{8-1}(-1)^1 - {}^{10}\mathrm{C}_3 x^{10-3}(-1)^3$

Coefficient of $x^7 = -{}^8\mathrm{C}_1 + {}^{10}\mathrm{C}_3$

$= 112$

Answer: C

8 What is the unit vector in the direction of $\overrightarrow{AB}$, where A and B are the points $(-2, 1)$ and $(6, 3)$, respectively?

A. $\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$

B. $8\mathbf{i} + 2\mathbf{j}$

C. $\frac{2}{17}\mathbf{i} + \frac{1}{34}\mathbf{j}$

D. $\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$\overrightarrow{AB} = 6\mathbf{i} + 3\mathbf{j} - (-2\mathbf{i} + \mathbf{j})$
$= 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{i} - \mathbf{j}$
$= 8\mathbf{i} + 2\mathbf{j}$
$\left|\overrightarrow{AB}\right| = \sqrt{8^2 + 2^2}$
$= 2\sqrt{17}$
Unit vector $= \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$
$= \frac{8\mathbf{i} + 2\mathbf{j}}{2\sqrt{17}}$
$= \frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$

Answer: A

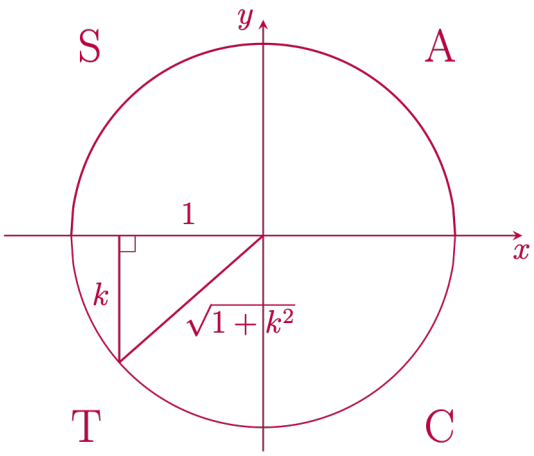
9 Given that $\tan x = k$, where $\pi < x < \frac{3\pi}{2}$, determine the value of $\sin\frac{x}{2}$.

A. $-\sqrt{\frac{1}{2}\left(1+\frac{k}{\sqrt{1+k^2}}\right)}$

B. $\sqrt{\frac{1}{2}\left(1+\frac{k}{\sqrt{1+k^2}}\right)}$

C. $-\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{1+k^2}}\right)}$

D. $\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{1+k^2}}\right)}$



$\cos x = 1 - 2\sin^2\frac{x}{2}$

$-\frac{1}{\sqrt{1+k^2}} = 1 - 2\sin^2\frac{x}{2}$

$\sin^2\frac{x}{2} = \frac{1}{2}\left(1+\frac{1}{\sqrt{1+k^2}}\right)$

$\sin\frac{x}{2} = \sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{1+k^2}}\right)}$ $\left(\text{since } \sin\frac{x}{2} > 0, \text{where } \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}\right)$

Answer: D

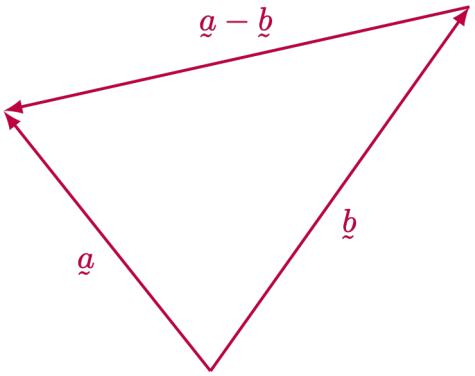
10 If $|\underset{\sim}{a} - \underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}|$, where $\underset{\sim}{a} \neq \underset{\sim}{0}$ and $\underset{\sim}{b} \neq \underset{\sim}{0}$, which one of the following must be true?

A. $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

B. $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$

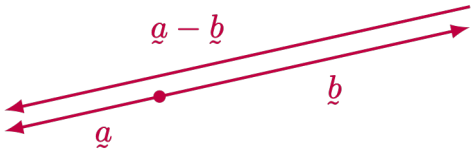
C. $\underset{\sim}{a} = \underset{\sim}{b}$

D. $\underset{\sim}{a} \perp \underset{\sim}{b}$



$\Rightarrow$

When $|\underset{\sim}{a} - \underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}|$



$\therefore \underset{\sim}{a} \parallel \underset{\sim}{b}$

Answer: B

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section.

- Answer the questions in the spaces provided.
- Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate Writing Booklet

(a) Solve $|4x - 3| > 11$. **2**

$4x - 3 > 11$ or $4x - 3 < -11$

$4x > 14$ or $4x < -8$

$x > \frac{7}{2}$ or $x < -2$

(1) (1)

1 mark for each correct solution.

Question was done well.

(b) A cup of coffee is heated to 90 °C. The room temperature is maintained at a constant 20 °C, and the coffee cools to 50 °C after 8 minutes. The temperature of the coffee can be modelled by the equation $T = 20 + 70e^{-kt}$, where T is the temperature of the coffee in degrees Celsius, t is the time in minutes since the coffee was placed in the room, and k is a constant. **2**

Determine the time taken for the temperature of the coffee to reach 25 °C.

Give your answer correct to 1 decimal place.

(1) calculate formula for T

$T = 20 + 70e^{-kt}$

Sub. $t = 8$ and $T = 50$,

$50 = 20 + 70e^{-8k}$

$70e^{-8k} = 30$

$e^{-8k} = \frac{3}{7}$

$-8k = \ln\frac{3}{7}$

$\therefore k = -\frac{1}{8}\ln\frac{3}{7}$

$\therefore T = 20 + 70e^{\frac{1}{8}t\ln\frac{3}{7}}$

(1) correct sub and calculate t.

When $T = 25$,

$25 = 20 + 70e^{\frac{1}{8}t\ln\frac{3}{7}}$

$e^{\frac{1}{8}t\ln\frac{3}{7}} = \frac{1}{14}$ ←

$\frac{1}{8}t\ln\frac{3}{7} = \ln\frac{1}{14}$

$t = 8\ln\frac{1}{14} \div \ln\frac{3}{7}$

$t = 24.9$ minutes (1 d.p.)

Question was done well.

many students wrote '14' then wrote '4' in the next line.

(c) A solid is formed by rotating the region bounded by the curves $y = 4 - x^2$ and $y = 2 - x$ about the x-axis. Find the exact value of the volume of the solid. **4**

(1) calculate intersection points

For the points of intersection,
$4 - x^2 = 2 - x$
$x^2 - x - 2 = 0$
$(x - 2)(x + 1) = 0$
$x = 2, -1$

Common mistake: students calculated intersection points for x correctly and then used '0 to 2' as limits instead.

(1) sub correct form and simplify

$$V = \pi \int_{-1}^{2} (4 - x^2)^2 \, dx - \pi \int_{-1}^{2} (2 - x)^2 \, dx$$
$$= \pi \int_{-1}^{2} ((4 - x^2)^2 - (2 - x)^2) \, dx$$
$$= \pi \int_{-1}^{2} (16 - 8x^2 + x^4 - (4 - 4x + x^2)) \, dx$$
$$= \pi \int_{-1}^{2} (x^4 - 9x^2 + 4x + 12) \, dx$$

common mistake: students didn't collect terms properly.

(1) Integration step

$$= \pi \left[\frac{x^5}{5} - 3x^3 + 2x^2 + 12x \right]_{-1}^{2}$$
$$= \pi \left[\left(\frac{32}{5} - 24 + 8 + 24 \right) - \left(-\frac{1}{5} + 3 + 2 - 12 \right) \right]$$

(1) final answer

$$= \frac{108\pi}{5} \text{ unit}^3$$

(d) (i) Complete the square for $x^2 - 6x + 13$, expressing the answer in the form $[f(x)]^2 + a$, where a is a constant and $f(x)$ is a linear function of x. **1**

$$x^2 - 6x + 13 = x^2 - 6x + 9 + 4$$
$$= (x - 3)^2 + 4$$

(1) completing the square.

(ii) Hence, find $\int \frac{1}{x^2 - 6x + 13} dx$. **2**

$$\int \frac{1}{x^2 - 6x + 13} dx = \int \frac{1}{4 + (x - 3)^2} dx \quad \text{(from (a))}$$
$$= \frac{1}{2} \tan^{-1} \frac{x - 3}{2} + C$$

(1) correct sub
(1) using correct identity.

Question was done well

(e) A standard pack of 52 playing cards consists of 13 hearts, 13 diamonds, 13 clubs and 13 spades. Fifteen cards are drawn with replacement from the pack.

(i) Let X be the number of diamonds drawn. Find the mean and standard deviation of X, giving your answers correct to 2 decimal places, where applicable. 2

$n = 15, \quad p = \frac{13}{52}, \quad q = 1 - \frac{1}{4}$
$= 0.25, \quad = 0.75$

$\mu = np,$
$= 15 \times 0.25$
$= 3.75$ (1)

$\sigma^2 = npq$
$= 15 \times 0.25 \times 0.75$
$\sigma = \sqrt{2.8125}$
$= 1.68$ (1)

Question was done well.
Common mistake: students did not square root variance.
1 mark each, mean and standard deviation

(ii) Determine the probability that X is within one standard deviation of the mean. 2
Give your answer correct to 3 decimal places.

$P(3.75 - 1.68 \leq X \leq 3.75 + 1.68)$ (1)
$= P(2.07 \leq X \leq 5.43)$
$= P(X = 3) + P(X = 4) + P(X = 5)$
$= {}^{15}C_3(0.25)^3(0.75)^{12} + {}^{15}C_4(0.25)^4(0.75)^{11} + {}^{15}C_5(0.25)^5(0.75)^{10}$ (1)
$= 0.616$

± standard deviation from mean.
correctly calculate probability.

Overall done well

- Some students only calculated top limit. (no marks)

- Many students calculated $P(2.07 \leq X \leq 5.43)$ and didn't continue
or calculated $P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$

- Rounding should have been better.

Question 12 (15 marks) Use a separate Writing Booklet

(a) There are 5 men and 6 women to choose from. In how many ways can 3 men and 5 women be selected and arranged in a row? **2**

Done very well ✓

A group of 3 men and 5 wemen can be choosen in ${}^5C_3 \times {}^6C_5$ ways.
Since each group can be arranged in a row in 8! ways,
Number of arrangements $= {}^5C_3 \times {}^6C_5 \times 8!$ (1)
$= 2419200$ (1)

(b) Consider the function $f(x) = \sqrt{3}\cos x + 3\sin x$ for $0 \le x \le 2\pi$.

(i) By first expanding $\cos(x-\alpha)$, express $f(x)$ in the form $R\cos(x-\alpha)$. **3**

$\sqrt{3}\cos x + 3\sin x = R\cos(x-\alpha)$
$= R\cos x\cos\alpha + R\sin x\sin\alpha$
Equate coefficients,
$R\cos\alpha = \sqrt{3}$ [1]
$R\sin\alpha = 3$ [2]
$[1]^2 + [2]^2$, $R^2\cos^2\alpha + R^2\sin^2\alpha = 12$
$R^2(\cos^2\alpha + \sin^2\alpha) = 12$
$R^2 = 12$
$\therefore R = 2\sqrt{3}$ (since $R > 0$) (1)

Many students did not justify why angle was in the first quadrant (*)
→ Did not penalise.

From [1], $\cos\alpha = \dfrac{\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{2} > 0$

From [2], $\sin\alpha = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2} > 0$

* Since $\cos\alpha > 0$ and $\sin\alpha > 0$, α is in the 1st quadrant with related angle $\dfrac{\pi}{3}$.
$\therefore \alpha = \dfrac{\pi}{3}$ (1)
$\therefore \sqrt{3}\cos x + 3\sin x = 2\sqrt{3}\cos\left(x - \dfrac{\pi}{3}\right)$ (1)

(ii) Hence, solve $\sqrt{3}\cos x + 3\sin x = 1$, correct to 2 decimal places. **2**

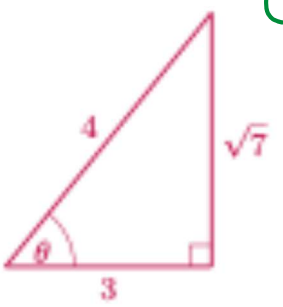
Done very well ✓

$2\sqrt{3}\cos\left(x - \dfrac{\pi}{3}\right) = 1$, where $-\dfrac{\pi}{3} \le x - \dfrac{\pi}{3} \le \dfrac{5\pi}{3}$
$\cos\left(x - \dfrac{\pi}{3}\right) = \dfrac{1}{2\sqrt{3}}$
$x - \dfrac{\pi}{3} = \cos^{-1}\dfrac{1}{2\sqrt{3}}$ (1)
$= 1.278$
Since $\cos\left(x - \dfrac{\pi}{3}\right) > 0$, $x - \dfrac{\pi}{3}$ lies in the first and fourth quadrants.
$x - \dfrac{\pi}{3} = 1.278$ or $2\pi - 1.278$
$\therefore x = 2.33$ or 6.05 (1)

(c) Express $\sin\left(2\cos^{-1}\frac{3}{4}\right)$ in the form $a\sqrt{b}$, where a and b are rational numbers. **2**

Let $\theta = \cos^{-1}\frac{3}{4}$

Then $\cos\theta = \frac{3}{4}$



(1)

$\text{Opposite} = \sqrt{4^2 - 3^2}$

$= \sqrt{7}$

$\sin\theta = \frac{\sqrt{7}}{4}$ (since $\cos\theta > 0$ and $0 \le \cos^{-1}x \le \pi$)

$\sin\left(2\cos^{-1}\frac{3}{4}\right) = \sin(2\theta)$

$= 2\sin\theta\cos\theta$

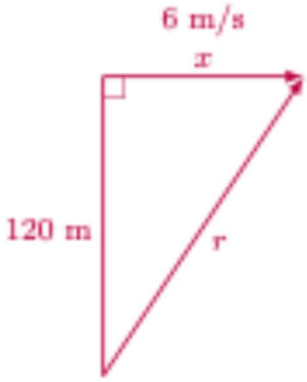
$= 2\left(\frac{\sqrt{7}}{4}\right)\left(\frac{3}{4}\right)$

$= \frac{3}{8}\sqrt{7}$ (1)

Done mostly well.
→ Some didn't draw the correct triangle which messed up the whole solution.

(d) A drone is flying horizontally at a constant height of 120 m above the ground. It moves horizontally away from a person at a constant speed of 6 m/s. **3**

Determine the rate at which the distance between the drone and the person is increasing when the drone is 150 m from the person.



$\frac{dx}{dt} = 6$, $\frac{dr}{dt} = ?$, $r^2 = x^2 + 120^2$

$r = \sqrt{x^2 + 120^2}$ (1)

$\frac{dr}{dx} = \frac{1}{2}(x^2 + 14400)^{-\frac{1}{2}}(2x)$

$= \frac{x}{\sqrt{x^2 + 14400}}$

$\frac{dr}{dt} = \frac{dr}{dx} \times \frac{dx}{dt}$

$= \frac{x}{\sqrt{x^2 + 14400}} \times 6$ (1)

$= \frac{6x}{\sqrt{x^2 + 14400}}$

When $r = 150$, $150^2 = x^2 + 120^2$

$x^2 = 8100$

$x = 90$

$\frac{dr}{dt} = \frac{6 \times 90}{\sqrt{90^2 + 14400}}$

$= \frac{360}{\sqrt{22500}}$

$= \frac{18}{5}$ m/s (1)

- Most students recieved 2/3 marks.
- Common error : - $x = 150$.
  - Incorrect related rate (reciprocal).

(e) Using the substitution $u = \sqrt{6x-1}$, show that **3**

$$\int \frac{x}{\sqrt{6x-1}}\,dx = \frac{1}{27}(3x+1)\sqrt{6x-1} + C.$$

$$\int \frac{x}{\sqrt{6x-1}}\,dx \qquad u = \sqrt{6x-1}, \quad u^2 = 6x-1, \quad x = \frac{1}{6}(u^2+1)$$

$$= \frac{1}{18}\int (u^2+1)\,du \qquad \frac{du}{dx} = \frac{1}{2}(6x-1)^{-\frac{1}{2}} \times 6$$

$$= \frac{1}{18}\left(\frac{u^3}{3} + u\right) + C \quad (1) \qquad \frac{du}{dx} = \frac{3}{\sqrt{6x-1}}$$

$$= \frac{1}{18}\left(\frac{\left(\sqrt{6x-1}\right)^3}{3} + \sqrt{6x-1}\right) + C \qquad \frac{1}{3}du = \frac{1}{\sqrt{6x-1}}dx$$

$$= \frac{1}{54}\left((6x-1)\sqrt{6x-1} + 3\sqrt{6x-1}\right) + C$$

$$= \frac{1}{54}\sqrt{6x-1}(6x-1+3) + C$$

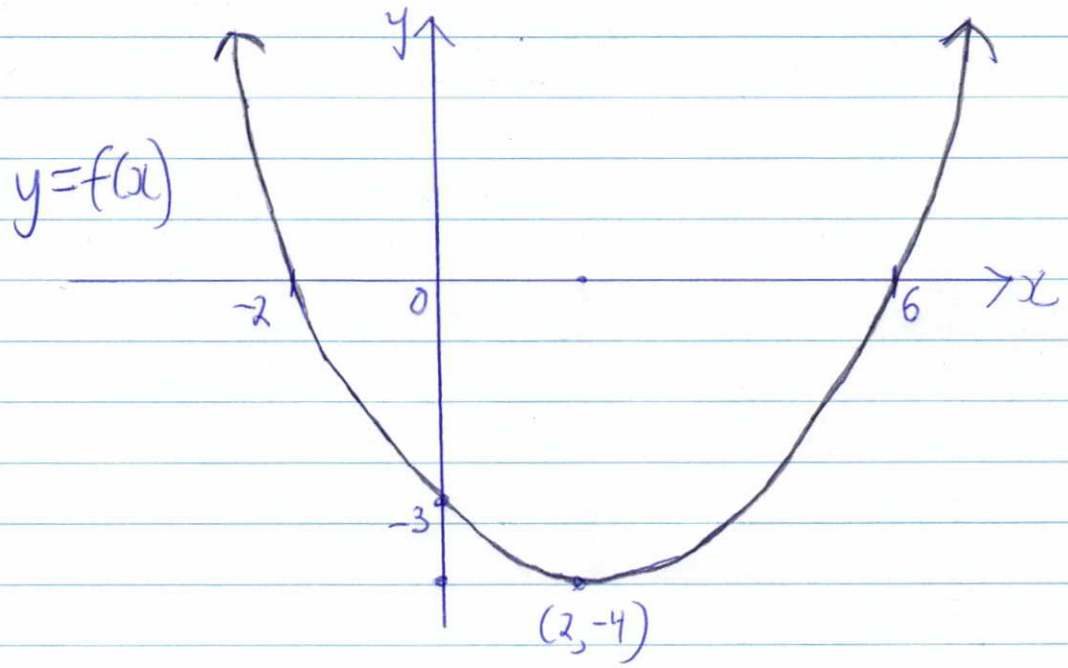
$$= \frac{1}{54}\sqrt{6x-1}(6x+2) + C$$

$$= \frac{1}{27}(3x+1)\sqrt{6x-1} + C \quad (1)$$

(1)

- Done well ✓
- Do not skip too many steps!

Q13a

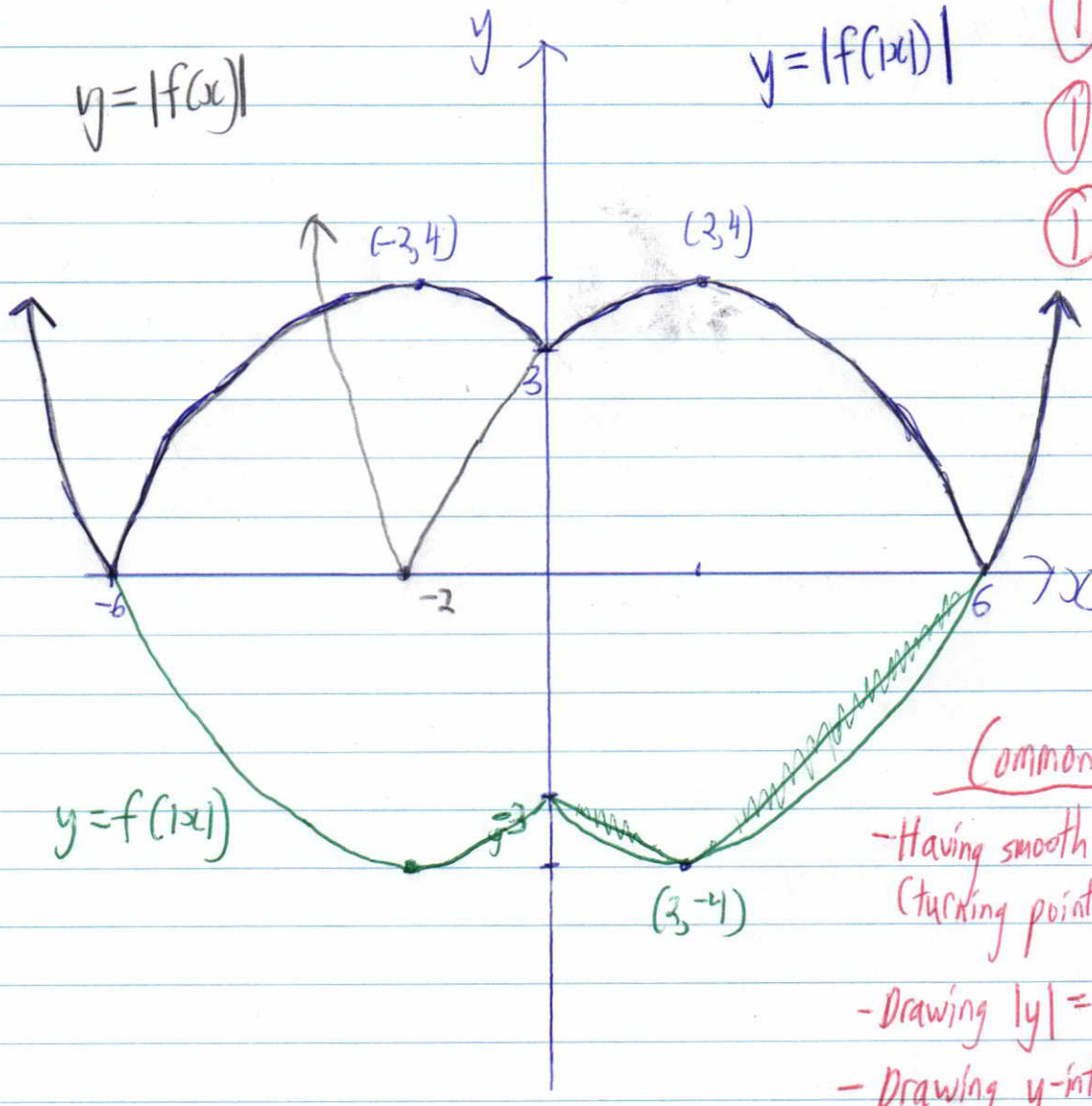


Criteria

① $|f(x)|$

① $f(|x|)$

① Key points/shape



Common Errors:

- Having smooth x-intercepts (turning points), not sharp changes
- Drawing $|y| = f(|x|)$
- Drawing y-intercept at the origin not y=3
- Poor scales (2 is not half of 6)

Q13b

Step 1: Prove true for $n=1$

$4(3^1)+8^1 = 12+8$
$= 20$
$= 5(4)$ (Divisible by 5)

① Base case

$\therefore$ the statement is true for $n=1$

Step 2: Assume true for $n=k$ where k is an integer ≥ 1

i.e. assume $4(3^k)+8^k = 5P$, $P \in \mathbb{Z}$
$8^k = 5P - 4(3^k)$

① Assumption and RTP

Alternate:
$4(3^k) = 5P - 8^k$

Step 3: Given assumption, prove true for $n=k+1$

ie. prove $4(3^{k+1})+8^{k+1} = 5Q$ $(Q \in \mathbb{Z})$

$LHS = 4(3^{k+1})+8^{k+1} = 4(3^k \times 3) + 8(8^k)$
$= 12(3^k) + 8(5P - 4(3^k))$
$= 12(3^k) + 40P - 32(3^k)$
$= 40P - 20(3^k)$
$= 5(8P - 4(3^k))$
$= 5Q$ $(Q = 8P - 4(3^k) \in \mathbb{Z})$

① Using assumption (Using assumption)

(Using assumption)
$= 3(5P - 8^k) + 8(8^k)$
$= 15P - 3(8^k) + 8(8^k)$
$= 15P + 5(8^k)$
$= 5(3P + 8^k)$
$= 5Q$ $(Q = 3P + 8^k \in \mathbb{Z})$

$\therefore$ if the result is true for $n=k$, then it is also true for $n=k+1$.

① Simplifies and conclusion

Step 4: It follows from steps 1, 2, and 3, by the principle of mathematical induction, the statement is true for all integers $n \geq 1$

$4(3^1)+8^1 \neq 15$

$4(3^k \times 3) \neq 12 \times 12^k$

Feedback

- Done well
- [most mistakes were silly substitution, factorisation, or algebra
- too many students still not writing (using assumption)

Q13c

$= 40P - 20(3^k)$

$= 5(8P - 4(3^k))$, which is divisible by 5.

Step 3 By the principle of mathematical induction, the statement is true for all integers $n \geq 0$.

Feedback

(c) Consider the differential equation $\frac{dy}{dx} = \frac{y(1-x)}{x}$.

- Poorly done for such an easy question.

(i) State any constant solutions. **1**

Substitute $\frac{dy}{dx} = 0$ for the constant solution,

$\frac{y(1-x)}{x} = 0$

$\therefore y = 0$

- Constant solutions are constant functions (horizontal lines)
- No mark if $x=0$ was included

(ii) Find the general solution of the differential equation by separating the variables. **3**

$\frac{dy}{dx} = \frac{y(1-x)}{x}$

$\frac{1}{y}dy = \frac{1-x}{x}dx$ - (1) Seperation of variables

Integrating both sides,

$\int \frac{1}{y}dy = \int \frac{1-x}{x}dx$

$\int \frac{1}{y}dy = \int \left(\frac{1}{x} - 1\right)dx$

$\ln|y| = \ln|x| - x + C$ - (1) Integrates

$\ln|y| - \ln|x| = -x + C$

$\ln\left|\frac{y}{x}\right| = -x + C$

$\left|\frac{y}{x}\right| = e^{-x+C}$

$\frac{y}{x} = \pm e^C e^{-x}$

$\therefore y = Axe^{-x}$, where A is a nonzero constant - (1) Makes y the subject

↓ not correct if you remove either $|y|$ or $|x|$ without explanation

Feedback:

- Poor mathematical understanding overall despite decent attempts
- Integrate both sides at the same time. Otherwise you should have 2 constants (not penalised)
- $\pm Ae^{-x}$ $(A = e^C > 0)$ is not a proper set of general solutions. Each constant should give 1 curve, not 2 (not penalised)
- Many students still don't properly understand how to deal with absolute values and arbitrary constants

Q13c - Alternate

ii) $\frac{dy}{dx} = \frac{y(1-x)}{x}$

$\frac{1}{y}dy = \left(\frac{1}{x} - 1\right)dx$ (variable seperable) (1)

$\int \frac{1}{y}dy = \int \frac{1}{x} - 1\, dx$

$\log_e|y| = \log_e|x| - x + C$ (1)

$|y| = e^{\log_e|x| - x + C}$

$|y| = e^{\log_e|x|} \cdot e^{-x} \cdot e^{C}$

$|y| = A|x|e^{-x}$ $(A = e^C > 0)$

$\therefore\ y = A|x|e^{-x}$ (A is a positive or negative constant) (1)

$|y| = e^C|x|e^{-x}$ to $y = Axe^{-x}$ is not valid working

- If you did not combine absolute values, the $|x|$ cannot disappear without explanation

$|y| = |x|f(x) \rightarrow \left|\frac{y}{x}\right| = f(x)$

$\frac{y}{x} = \pm f(x)$

$y = \pm x f(x)$

Q13d

(d) Find all possible values of k given that the length of the projection of $\mathbf{u}$ onto $\mathbf{v}$ is $\frac{108}{17}$, where $\mathbf{u} = k\mathbf{i} + 9\mathbf{j}$ and $\mathbf{v} = 15\mathbf{i} - 8\mathbf{j}$. **4**

$$\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u} = \frac{(\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{v}}{|v||\underset{\sim}{v}|}$$
$$= \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|}\,\frac{\underset{\sim}{v}}{|\underset{\sim}{v}|}$$
$$= \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|}\,\hat{\underset{\sim}{v}}$$

Length of the projection $= \left|\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|}\right|$ – (1) Uses length of projection

$$\left|\frac{(k\mathbf{i}+9\mathbf{j})\cdot(15\mathbf{i}-8\mathbf{j})}{\sqrt{15^2+(-8)^2}}\right| = \frac{108}{17}$$

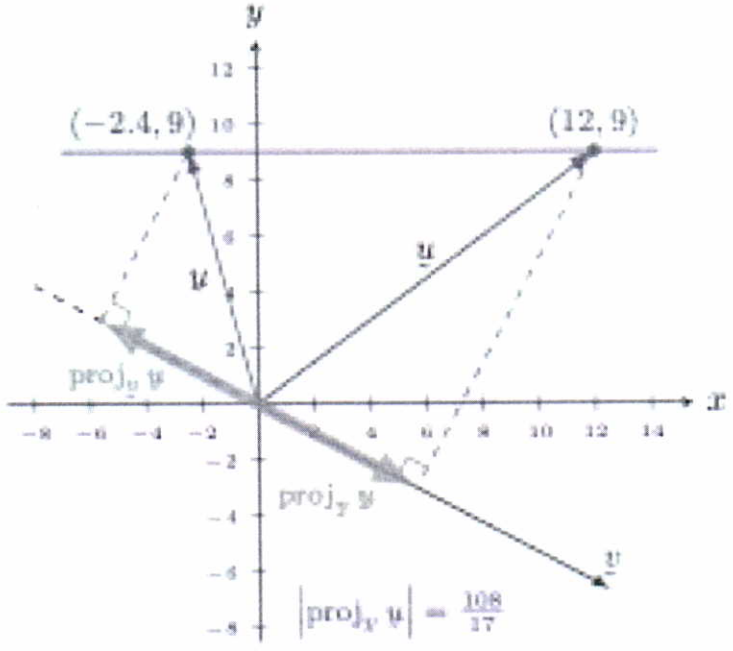
$$\left|\frac{15k-72}{17}\right| = \frac{108}{17}$$ – (1) Simplifies

$$|15k-72| = 108$$

$15k - 72 = 108$ or $15k - 72 = -108$

$15k = 180$ or $15k = -36$

$\therefore k = 12$ – (1) or $k = -\frac{12}{5}$ (or -2.4) – (1)



Feedback:

- Many students forget absolute values.
- Length means absolute value and $|15k-72|$ is not known to be positive.

Q13d - Alternate

$\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\underset{\sim}{v}$

$= \frac{(k\underset{\sim}{i}+9\underset{\sim}{j})\cdot(15\underset{\sim}{i}-8\underset{\sim}{j})}{\left(\sqrt{15^2+(-8)^2}\right)^2}(15\underset{\sim}{i}-8\underset{\sim}{j})$

(1) Finds $\underset{\sim}{u}\cdot\underset{\sim}{v}$ and $|\underset{\sim}{v}|$

$= \frac{15k-72}{17^2}(15\underset{\sim}{i}-8\underset{\sim}{j})$

Let $|\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}| = \frac{108}{17}$

$\left|\frac{15k-72}{17^2}(15\underset{\sim}{i}-8\underset{\sim}{j})\right| = \frac{108}{17}$ → If fully expanded: Equation: $65025k^2 - 624240k - 1872720 = 0$

$\left|\frac{1}{17^2}\begin{pmatrix}225k-1080\\-120k+576\end{pmatrix}\right| = \frac{108}{17}$ (1) Writes equation for projection

$\left|\begin{pmatrix}225k-1080\\-120k+576\end{pmatrix}\right| = 17\times 108$

$\sqrt{(225k-1080)^2+(-120k+576)^2} = 17\times 108$

$(225k-1080)^2+(-120k+576)^2 = 17^2\times 108^2$

$15^2(15k-72)^2+8^2(-15k+72)^2 = 17^2\times 108^2$

$17^2(15k-72)^2 = 108^2\times 17^2$

$15k-72 = \pm 108$ (1) Simplifies equation OR

$\therefore 15k-72 = 108$ or $15k-72 = -108$

$15k = 180$ $\quad 15k = -36$

$k = 12$ $\quad k = \frac{-12}{5}$ ← (1) Finds solutions

$\rightarrow 5k^2 - 48k - 144 = 0$

$k = \frac{48 \pm \sqrt{(-48)^2-4(5)(-144)}}{2(5)}$

$= \frac{48 \pm 72}{10}$

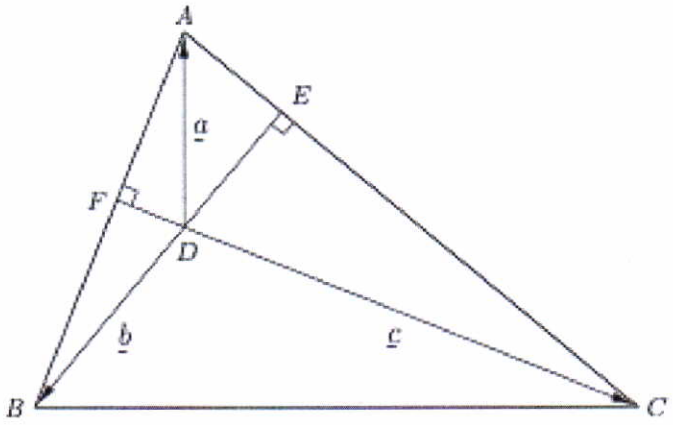
$= 12$ or $\frac{-12}{5}$

The majority of students who used the full formula, particularly the many that did the green method made algebra/calculation errors.

The formula: $\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|}\hat{\underset{\sim}{v}}$ (multiple of a unit vector) should be known

Question 14 (15 marks) Use a separate Writing Booklet

(a) The ΔABC shown below has altitudes BE and CF, where an altitude is a line segment drawn from a vertex perpendicular to the opposite side. The altitudes intersect at D.
Let $\overrightarrow{DA} = \underset{\sim}{a}$, $\overrightarrow{DB} = \underset{\sim}{b}$ and $\overrightarrow{DC} = \underset{\sim}{c}$.



(i) Show that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot \underset{\sim}{c}$. **3**

$\overrightarrow{AC} \cdot \overrightarrow{DB} = 0$ (since $AC \perp BE$)
$(\underset{\sim}{c} - \underset{\sim}{a}) \cdot \underset{\sim}{b} = 0$
$\underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$
$\therefore \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{b}$

$\overrightarrow{AB} \cdot \overrightarrow{DC} = 0$ (since $AB \perp FC$)
$(\underset{\sim}{b} - \underset{\sim}{a}) \cdot \underset{\sim}{c} = 0$
$\underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$
$\therefore \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c}$

$\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot \underset{\sim}{c}$

Many students started with $\vec{AC} \perp \vec{BE}$ then label $\vec{DB} = \lambda \vec{BE}$ and made it complicated.
- overall, it was not done well.

(ii) Hence, or otherwise, prove that the three altitudes of ΔABC are concurrent; that is, they all pass through a single point. **2**

$\overrightarrow{BC} \cdot \overrightarrow{DA} = (\underset{\sim}{c} - \underset{\sim}{b}) \cdot \underset{\sim}{a}$
$= \underset{\sim}{c} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{a}$
$= \underset{\sim}{c} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{a}$ (since $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{c}$)
$= 0$
$\therefore BC \perp AD$
$\therefore$ The extension of AD to meet BC is an altitude of the ΔABC.
$\therefore$ The three altitudes of ΔABC are concurrent.

similarly, many students introduced λ, μ for $\vec{BC} \perp \vec{DA}$, and could not arrive at the final step.

(b) A ball is projected from the base of a cliff with initial speed u m/s at an angle of θ to the horizontal. At the same time, a second ball is projected horizontally with initial speed w m/s from the top of the same cliff, which is h m high. Take the base of the cliff as the origin $(0,0)$. The two balls collide.

(i) Show that the position vector of the first ball is **3**

$$\mathbf{r}_1(t) = (ut\cos\theta)\mathbf{i} + \left(ut\sin\theta - \frac{g}{2}t^2\right)\mathbf{j}.$$

$\mathbf{a}_1(t) = -g\mathbf{j}$

$\mathbf{v}_1(t) = \int(-g\mathbf{j})\,dt$

$= -gt\mathbf{j} + \mathbf{c}_1$

Sub. $t = 0$ and $\mathbf{v}_1(0) = u\cos\theta\,\mathbf{i} + u\sin\theta\,\mathbf{j}$,

$\therefore \mathbf{c}_1 = u\cos\theta\,\mathbf{i} + u\sin\theta\,\mathbf{j}$

$\therefore \mathbf{v}_1 = -gt\mathbf{j} + u\cos\theta\,\mathbf{i} + u\sin\theta\,\mathbf{j}$

$= u\cos\theta\,\mathbf{i} + (u\sin\theta - gt)\mathbf{j}$

$\mathbf{r}_1 = \int(u\cos\theta\,\mathbf{i} + (u\sin\theta - gt)\mathbf{j})\,dt$

$= ut\cos\theta\,\mathbf{i} + \left(ut\sin\theta - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{d}_1$

Sub. $t = 0$ and $\mathbf{r}_1(0) = \mathbf{0}$,

$\therefore \mathbf{d}_1 = \mathbf{0}$

$\therefore \mathbf{r}_1 = ut\cos\theta\,\mathbf{i} + \left(ut\sin\theta - \frac{g}{2}t^2\right)\mathbf{j}$

comments :

This part was done well, except

- few students forgot to find constants for velocity and displacement.

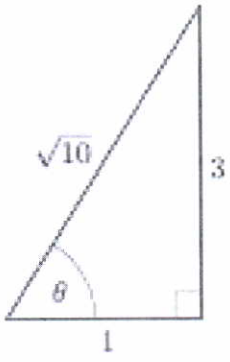
- overall okay.

(ii) It is given that $\theta = \tan^{-1}3$ and the position vector of the second ball is **2**

$$\mathbf{r}_2(t) = wt\mathbf{i} + \left(h - \frac{g}{2}t^2\right)\mathbf{j}.$$

Show that $\cos\theta = \dfrac{1}{\sqrt{10}}$, and hence deduce that $u = \sqrt{10}w$.

$\tan\theta = 3$



hypotenuse $= \sqrt{3^2 + 1^2}$

$= \sqrt{10}$

$\therefore \sin\theta = \dfrac{3}{\sqrt{10}}$

$\therefore \cos\theta = \dfrac{1}{\sqrt{10}}$

- well done !!

At the point of collision, $\mathbf{r}_1(t) = \mathbf{r}_2(t)$.

Equating x components,

$ut\cos\theta = wt$

$u\dfrac{1}{\sqrt{10}} = w$

$\therefore u = \sqrt{10}w$

(iii) Hence, prove that the balls collide at time $t = \frac{h}{3w}$, and that the height of the balls at the point of collision is $h - \frac{gh^2}{18w^2}$. **2**

From $\mathbf{r}_1(t) = \mathbf{r}_2(t)$, equating y components,

$ut\sin\theta - \frac{g}{2}t^2 = h - \frac{g}{2}t^2$

$ut\sin\theta = h$

$\sqrt{10}w\left(\frac{3}{\sqrt{10}}\right)t = h$

$\therefore t = \frac{h}{3w}$

well done !!

Substitute $t = \frac{h}{3w}$ into $y_2 = h - \frac{g}{2}t^2$ for the height,

$y_2 = h - \frac{g}{2}\left(\frac{h}{3w}\right)^2$

$\therefore y_2 = h - \frac{gh^2}{18w^2}$

well done !!

(iv) At the instant of collision, the velocity vectors of the two balls are perpendicular. **3**

Show that $h = \frac{3(3 \pm \sqrt{5})w^2}{2g}$.

At $t = \frac{h}{3w}$, $\quad \mathbf{v}_1(t) = u\cos\theta\,\mathbf{i} + (u\sin\theta - gt)\mathbf{j}$

$= \sqrt{10}w\left(\frac{1}{\sqrt{10}}\right)\mathbf{i} + \left(\sqrt{10}w\left(\frac{3}{\sqrt{10}}\right) - g\left(\frac{h}{3w}\right)\right)\mathbf{j}$

$= w\mathbf{i} + \left(3w - \frac{gh}{3w}\right)\mathbf{j}$

$\mathbf{v}_2(t) = w\mathbf{i} - gt\mathbf{j}$

$= w\mathbf{i} - g\left(\frac{h}{3w}\right)\mathbf{j}$

$= w\mathbf{i} - \frac{gh}{3w}\mathbf{j}$

- some students forgot to write $V_1(t) \cdot V_2(t) = 0$
- some students arrive at the final step with their incorrect working.

$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \quad$ (since $\mathbf{v}_1 \perp \mathbf{v}_2$)

$w^2 - \frac{gh}{3w}\left(3w - \frac{gh}{3w}\right) = 0$

$\left(\frac{gh}{3w}\right)^2 - 3w\left(\frac{gh}{3w}\right) + w^2 = 0$

$\frac{gh}{3w} = \frac{3w \pm \sqrt{9w^2 - 4w^2}}{2}$

$\frac{gh}{3w} = \frac{3w \pm \sqrt{5}w}{2}$

$\frac{gh}{3w} = \frac{(3 \pm \sqrt{5})w}{2}$

$\therefore h = \frac{3(3 \pm \sqrt{5})w^2}{2g}$

- overall, it was not done well.

Alternative:

$$\frac{g^2}{9w^2}h^2 - gh + w^2 = 0$$

$$h = \frac{g \pm \sqrt{g^2 - \frac{4g^2}{9}}}{\frac{2g^2}{9w^2}}$$

$$= \left(g \pm g\sqrt{1 - \frac{4}{9}}\right) \times \frac{9w^2}{2g^2}$$

$$= \left(1 \pm \sqrt{\frac{5}{9}}\right) \times \frac{9w^2}{2g}$$

$$= \frac{9w^2\left(1 \pm \frac{1}{3}\sqrt{5}\right)}{2g}$$

$$= \frac{3w^2(3 \pm \sqrt{5})}{2g}$$

$$\therefore h = \frac{3(3 \pm \sqrt{5})w^2}{2g}$$

- Few students used this method.
- Again, few students show incorrect working to arrive at the final step.
- Overall, it was not done so well.

End of paper