Student Name: _____

Teacher: _____

Sydney Technical High School

2026

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 10 minutes
- Working Time – 2 hours
- Write using black pen.
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless work or illegible writing.
- **Begin each question on a new page.**

Total marks: 70

Section I — 10 marks (pages 2-5)

- Attempt Questions 1 - 10
- Allow about 15 minutes for this section.

Section II — 60 marks (pages 6-12)

- Attempt Questions 11 - 14
- Allow about 1 hour and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

1 What is the solution to $|2x - 5| > 7$?

A. $-1 < x < 6$

B. $x < -1$ or $x > 6$

C. $-6 < x < 1$

D. $x < -6$ or $x > 1$

2 The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ is given by $\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|^2}\right)\underset{\sim}{v}$.

What is the projection of $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ onto $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j}$

A. $\frac{4}{5}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

B. $-\frac{4}{5}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

C. $\frac{4}{5}\left(-\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

D. $\frac{4}{5}\left(\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

3 Garrett, Dean, John L. and John T. sit at the kitchen bench in a row.
In how many ways can the two Johns sit together?

A. 12

B. 9

C. 6

D. 3

4 Consider the integral:

$$I = \int_0^3 \frac{1}{9 - x^2}\, dx$$

The substitution $x = 3\sin\theta$ is applied.

Which of the following is obtained?

A.

$$\frac{1}{3}\int_0^{\pi} \text{cosec}\theta d\theta$$

B.

$$\frac{1}{3}\int_0^{\pi} \sec\theta d\theta$$

C.

$$\frac{1}{3}\int_0^{\frac{\pi}{2}} \text{cosec}\theta d\theta$$

D.

$$\frac{1}{3}\int_0^{\frac{\pi}{2}} \sec\theta d\theta$$

5 How many distinct solutions are there to the equation $\sin 3x + \cos x = 0$ for $0 \leq x \leq 2\pi$?

A. 3

B. 4

C. 5

D. 6

6 Given that A is a non-zero constant, which of the following integrals is equal to zero?

A. $\int_{-A}^{A} \cos^{-1}(x) \times \sin(x)\, dx$

B. $\int_{-A}^{A} \sin^{-1}(x) \times \tan(x)\, dx$

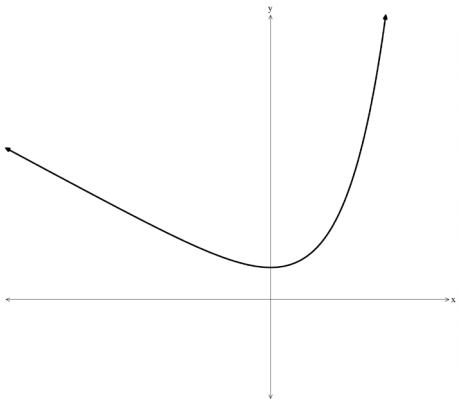
C. $\int_{-A}^{A} \tan^{-1}(x) \times \cos(x)\, dx$

D. $\int_{-A}^{A} \cos^{-1}(x) \times \tan(x)\, dx$

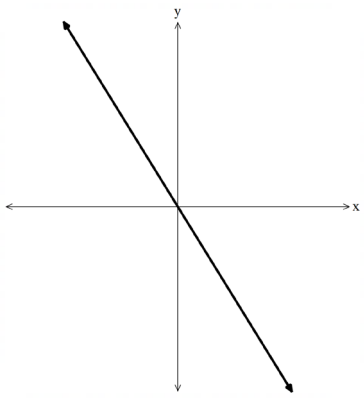
7 Which of the following graphs could be a solution to the differential equation:

$$\frac{dy}{dx} = x + 2y$$

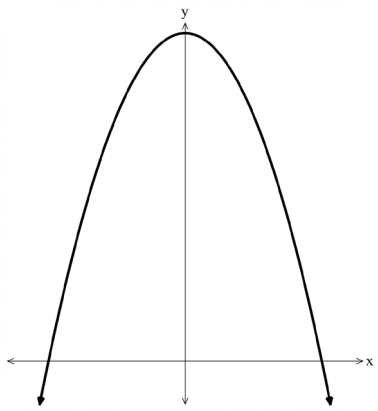
A.



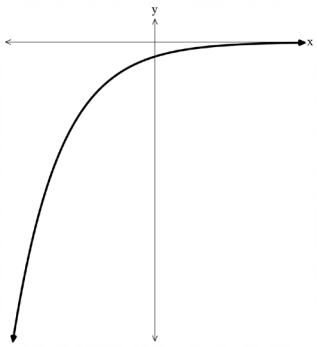
C.



B.



D.



8 Given that $x^3 - 4x^2 + c \equiv (x-1)Q(x) + 3$, where $Q(x)$ is a polynomial, what is the value of c?

A. 3

B. 4

C. 5

D. 6

9 The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have magnitudes of 4 and 7 respectively. Given that the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\frac{2\pi}{3}$ and there exists a vector $\underset{\sim}{c}$ such that $\underset{\sim}{a} + \underset{\sim}{b} = -\underset{\sim}{c}$, what is the magnitude of $\underset{\sim}{c}$ to the nearest whole number?

A. 10

B. 6

C. 7

D. 8

10 Let $g(x)$ be the inverse function of $f(x)$. If we know that $f(2) = 4$, $f(3) = 2$, $f'(3) = -1$ and $f'(2) = -2$, what is the derivative of $g(x)$ when $x = 2$?

A. -1

B. $-\frac{1}{2}$

C. 1

D. $\frac{1}{2}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section.

Answer each question on a new page in the answer booklet.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW page.

(a) Find the inverse function of $f(x) = \frac{x+1}{x-1}$. **2**

(b) Solve $\cos 2\theta + \cos \theta = 1$ for $\theta \epsilon [0,2\pi]$. **3**

(c) Evaluate the following: **3**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2 \sin x \cos 3x \, dx$$

(d) Sketch $y = 3 \sin^{-1} \frac{x}{2}$. **2**

(e) If $\underset{\sim}{u} = \begin{pmatrix} m \\ 4 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ are perpendicular, find the value of m. **1**

Question 11 continues on page 7

(f) The roots of $P(x) = 2x^3 - 3x^2 + x - 6$ are α, β and γ.

Find the value of:

i. $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **2**

ii. $\alpha^2 + \beta^2 + \gamma^2$. **2**

End of Question 11

Question 12 (15 marks) Start a NEW page.

(a) If $\sin x = \frac{3}{5}$, where x is acute, find the exact value of $\cos 2x$. **1**

(b) Find $\int \sin^2 2x \, dx$. **2**

(c) By expressing $2\cos\theta - \sqrt{5}\sin\theta$ in terms of $R\cos(\theta + \alpha)$ where, $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$, solve $2\cos\theta = \sqrt{5}\sin\theta - 3$ for $0 \leq \theta \leq 2\pi$. **3**

(d) Solve the differential equation: **3**

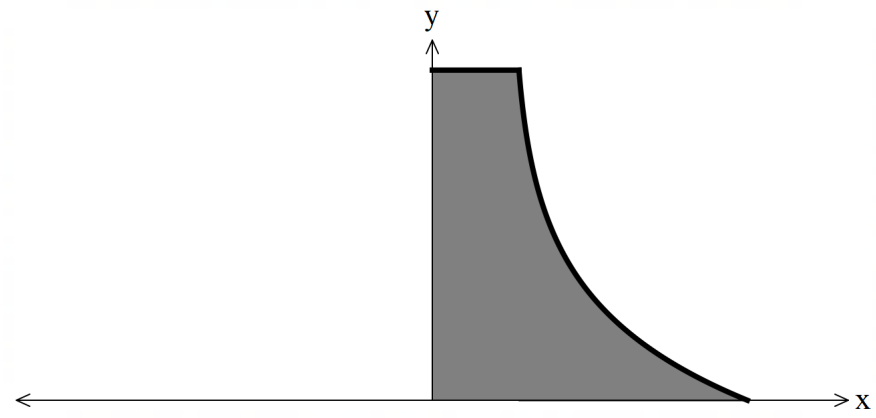
$$\frac{dy}{dx} = xy - x$$

given $\frac{dy}{dx} = -3$ when $x = -3$.

Question 12 continues on page 9

(e) Solve $\frac{x}{3x+1} \geq 1$ **3**

(f) Consider the region between the $x-$ axis, $y-$ axis, the line $y = 10$ and the function $y = -\frac{1}{0.7}\ln\frac{x-1}{3}$, as shown in the diagram below. **3**



When the region is rotated about the $y-$ axis, it forms the volume of the bell of Hannah's clarinet.

Find the volume of the bell.

End of Question 12

Question 13 (15 marks) Start a NEW page.

(a) Find: **3**

$$\int_0^2 \frac{x}{x^4 + 16} dx$$

using the substitution of $u = x^2$.

(b) Given that $\frac{dy}{dx} = \frac{3}{2y-2}$ and $y = 1$ when $x = 0$. **3**

Find y as a function of x.

(c) Briar Hawks is the men's varsity ice hockey team at Briar University.
The team has 12 players that make up **four** forward lines (3 players each) and 6 players that make up the **three** defensive lines (2 players each).
The forward and defensive lines rotate time on the ice during the game.
(Note defensive players and offensive players never swap roles.)

i. In how many ways can the three defensive lines be made? **1**

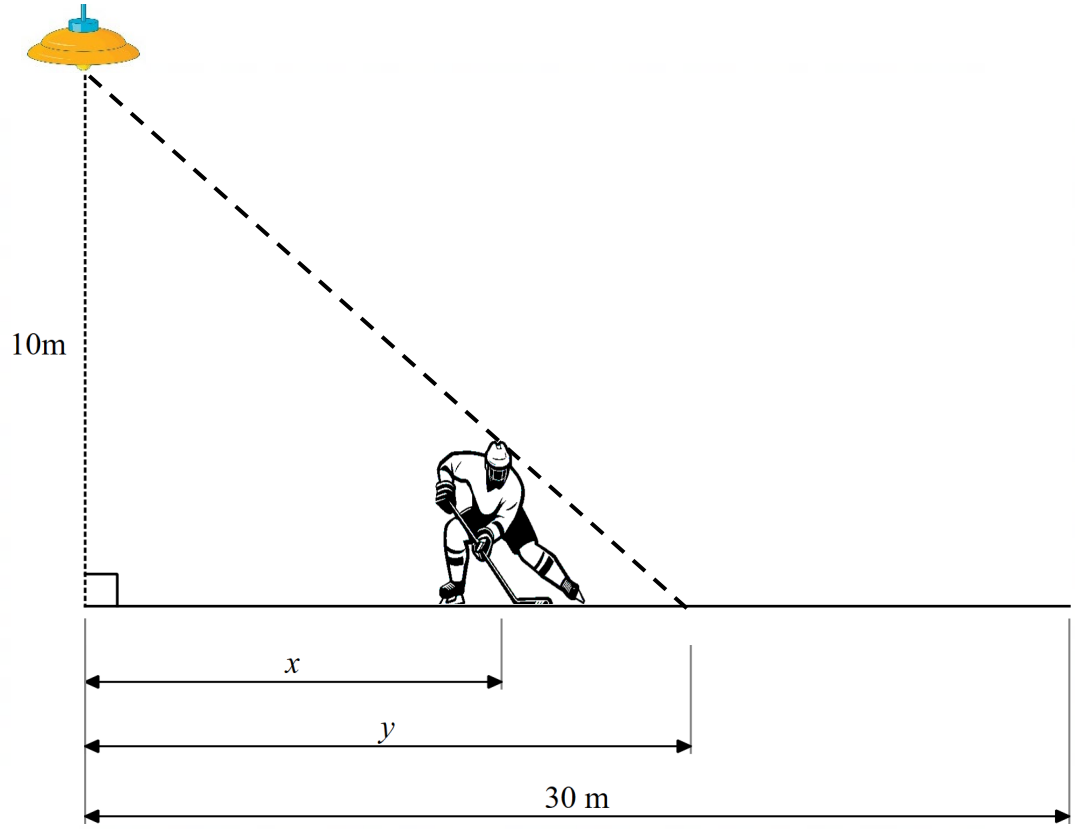
ii. How many player combinations are possible on the ice? **2**

Question 13 continues on page 11

(d) Prove by mathematical induction that the sum of the squares of the first $2n$ positive integers can be given by: **4**

$$1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$$

(e) An ice hockey player, who is 200 cm tall with ice skates on, skates from directly under a single overhead light mounted at a height of 10 m from the ice. He skates at a constant rate of $3ms^{-1}$ away from the light towards the edge of the rink which is 30 m from his starting point. **2**



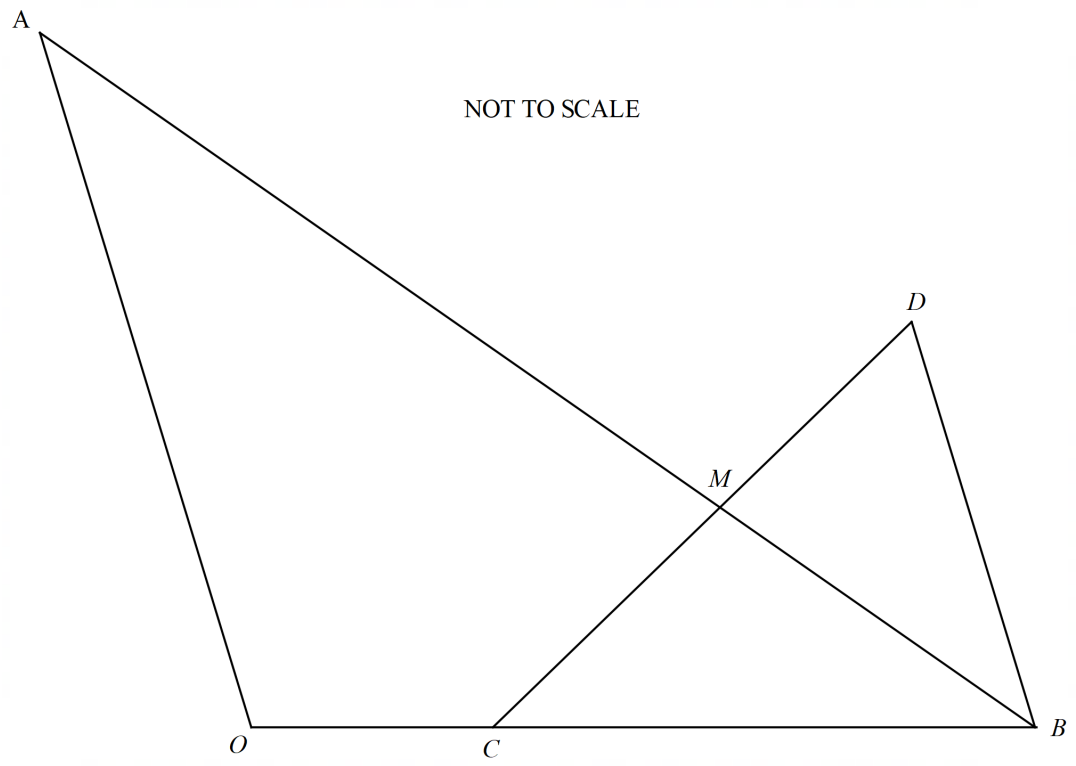
When the ice hockey player is x m from his starting point, is the ice hockey player or the tip of his shadow moving more rapidly?
Justify your solution with calculations.

End of Question 13

Question 14 (15 marks) Start a NEW page.

(a) 25 boys are sitting around a circular table. What is the minimum number of girls required so that when they join the boys around the table, at least 2 girls must be sitting together? Explain your solution. **2**

(b) Consider the diagram below that shows two triangles OAB and BCD. **3**



The point M lies on the point of intersection of $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

The point C lies on $\overrightarrow{OB}$ such that $\overrightarrow{OB} = 4\overrightarrow{OC}$.

$\overrightarrow{OA}$ is parallel to $\overrightarrow{BD}$ and $|\overrightarrow{OA}| = 2|\overrightarrow{BD}|$.

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

Find $\dfrac{\left|\overrightarrow{CM}\right|}{\left|\overrightarrow{CD}\right|}$.

Question 14 continues on page 13

(c) By considering $(1+x)^{1000} + (1+x)^{999} + (1+x)^{998} + \cdots + (1+x)^{950}$ or otherwise, show that: **3**

$$\binom{1000}{50} + \binom{999}{49} + \binom{998}{48} + \cdots + \binom{950}{0} = \binom{1001}{50}$$

(d) The equation $ax^3 + cx + d = 0$ has a double root.

i. Show that $27ad^2 + 4c^3 = 0$ **2**

ii. Hence, and without the use of calculus methods, find the exact values of the $y-$ coordinates of the two stationary points on the curve $y = 3x^3 - 4x + 1$. **2**

(e) Let $\underset{\sim}{x}, \underset{\sim}{y}$ and $\underset{\sim}{z}$ be non-zero vectors. Let α be the angle between $\underset{\sim}{x}$ and $\underset{\sim}{y}$ and β be the angle between $\underset{\sim}{y}$ and $\underset{\sim}{z}$, where $\alpha + \beta \neq \frac{\pi}{2}$ and $\alpha, \beta \in (0, \frac{\pi}{2})$. **3**

If $proj_{\underset{\sim}{y}} \underset{\sim}{x} = \underset{\sim}{u}$, show that:

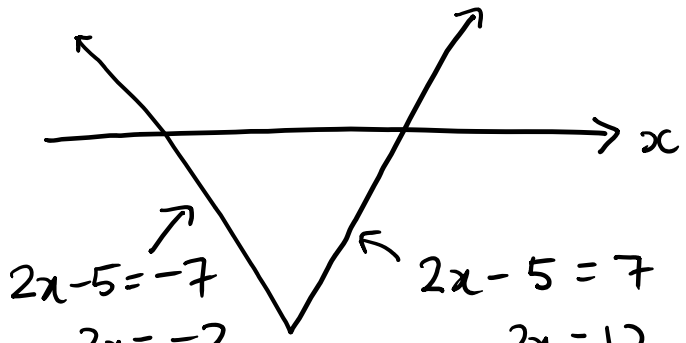
$$proj_{\underset{\sim}{z}} \underset{\sim}{u} = \frac{\cos\alpha \cos\beta}{\cos(\alpha+\beta)} \times proj_{\underset{\sim}{z}} \underset{\sim}{x}$$

End of paper

MC Solutions Ext 1 2026

Sunday, 23 August 2026 4:32 PM

Q1.



$|2x-5| > 7$

$x < -1$ or $x > 6$

Q2. $\underset{\sim}{u} \cdot \underset{\sim}{v} = -2 + 6$

$= 4$

$|\underset{\sim}{v}|^2 = 1^2 + 2^2$

$= 5$

$$\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \underset{\sim}{v} = \frac{4}{5}(-\underset{\sim}{i} + 2\underset{\sim}{j})$$

Q3. Treat the Johns as a group.

So 3! arrangements.

Then arrange Johns : 2! arrangements.

$3! \times 2! = 12$

Q4. $x = 3\sin\theta$

$dx = 3\cos\theta \, d\theta$

when $x = 0 \quad \theta = 0$

$x = 3 \quad \theta = \frac{\pi}{2}$

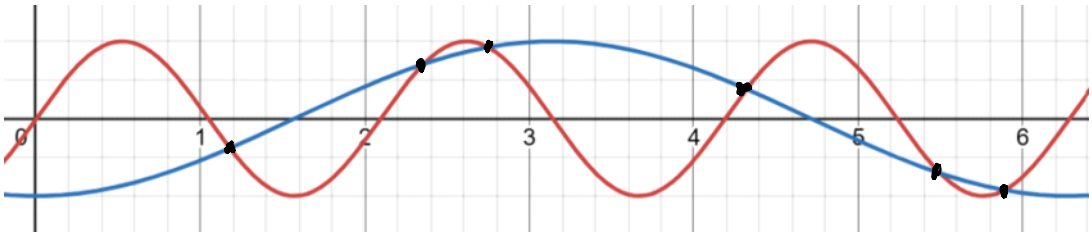
$$I = \int_0^{\frac{\pi}{2}} \frac{1}{9-(3\sin\theta)^2} \times 3\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{9-9\sin^2\theta} \times 3\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{9\cos^2\theta} 3\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}\sec\theta \, d\theta$$

Q5. $\sin 3x = -\cos x$



Q6. Want an odd function: comes from odd function × even function (or vice versa)

$\cos^{-1}x$: neither

$\sin x$: odd

$\sin^{-1}x$: odd

$\tan x$: odd

$\tan^{-1}x$: odd

$\cos x$: even

Q7. $\frac{dy}{dx} = x + 2y$ sub points to test which gives A.

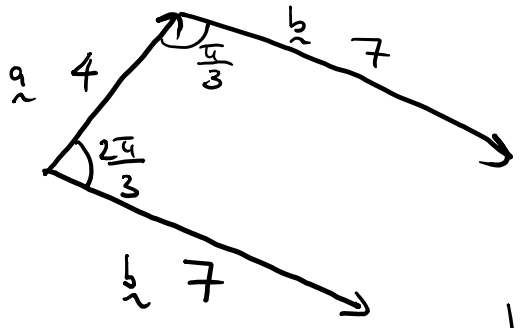
Q8. Sub $x=1$

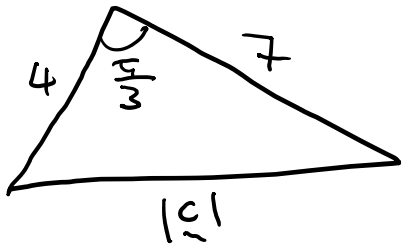
$1-4+c=0+3$

$-3+c=3$

$c=6$

Q9.





$|c|^2 = 4^2 + 7^2 - 2\times4\times7\cos\frac{\pi}{3}$

$= 16 + 49 - 56\times\frac{1}{2}$

$= 37$

$|c| = \sqrt{37}$

Q10. $f(2)=4 \quad f'(2)=-2$

$f(3)=2 \quad f'(3)=-1$

$g(4)=2$

$g(2)=3$

$g'(2) = \frac{1}{f'(3)}$

$= -1$

Solutions Ext 1 2026

Thursday, 23 July 2026 5:34 PM

Section 1

1. B 2. C 3. A 4. D 5. D
6. C 7. A 8. D 9. B 10. A

Section 2

Q11.

a) $x = \frac{y+1}{y-1}$

$= \frac{y-1}{y-1} + \frac{2}{y-1}$

$= 1 + \frac{2}{y-1}$

$x-1 = \frac{2}{y-1}$

$y-1 = \frac{2}{x-1}$

$y = 1 + \frac{2}{x-1}$ OR $y = \frac{x+1}{x-1}$

b) $\cos 2\theta + \cos\theta = 1 \quad , \theta \in [0, 2\pi]$

$2\cos^2\theta - 1 + \cos\theta = 1$

$2\cos^2\theta + \cos\theta - 2 = 0$

$\cos\theta = \frac{-1 \pm \sqrt{1 - 4(2)(-2)}}{2 \times 2}$

$= \frac{-1 \pm \sqrt{17}}{4}$ but $-1 \leq \cos\theta \leq 1$

$\therefore \cos\theta = \frac{-1 + \sqrt{17}}{4}$

$\theta \doteq 0.675, 5.608$ (3 d.p.)

c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sin x \cos 3x \, dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2 \times \frac{1}{2}\left[\sin(x+3x) + \sin(x-3x)\right] dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 4x + \sin(-2x) \, dx$

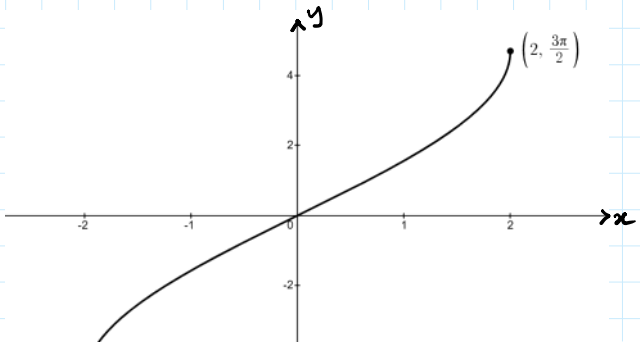
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 4x - \sin 2x \, dx$

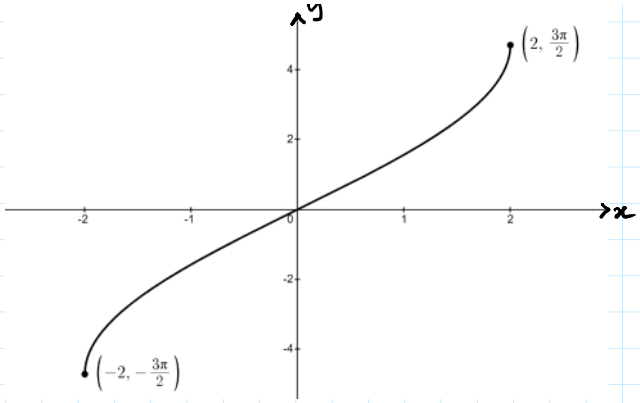
$= \left[-\frac{\cos 4x}{4} + \frac{\cos 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \left(-\frac{\cos 2\pi}{4} + \frac{\cos\pi}{2}\right) - \left(-\frac{\cos\frac{2\pi}{3}}{4} + \frac{\cos\frac{\pi}{3}}{2}\right)$

$= -\frac{9}{8}$

d) $y = 3\sin^{-1}\frac{x}{2}$





e) If $\underset{\sim}{u} \perp \underset{\sim}{v}$ then $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

$2m + 12 = 0$

$m = -6$

f) i. $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

$= \alpha\beta\gamma(\alpha + \beta + \gamma)$

$\alpha + \beta + \gamma = \frac{3}{2}$

$\alpha\beta\gamma = 3$

$\therefore \alpha\beta\gamma(\alpha+\beta+\gamma) = \frac{3}{2} \times 3$

$= \frac{9}{2}$

ii. $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{1}{2}$

$\therefore (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma) = (\frac{3}{2})^2 - 2(\frac{1}{2})$

$= \frac{5}{4}$

Q12.

a) $\sin x = \frac{3}{5}$

$\cos 2x = 1 - 2\sin^2 x$

$= 1 - 2(\frac{3}{5})^2$

$= \frac{7}{25}$

b) $\int \sin^2 2x \, dx$

Now $\cos 4x = 1 - 2\sin^2 2x$

$\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$

$\therefore \int \sin^2 2x \, dx = \int \frac{1}{2}(1 - \cos 4x) \, dx$

$= \frac{1}{2}\int 1 - \cos 4x \, dx$

$= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right] + C$

c) $2\cos\theta - \sqrt{5}\sin\theta = R\cos(\theta + \alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\therefore R\cos\alpha = 2$

$R\sin\alpha = \sqrt{5}$

$R^2 = 4 + 5$

$= 9$

$R = 3$, $R > 0$.

$\tan\alpha = \frac{\sqrt{5}}{2}$

$\alpha = \tan^{-1}\frac{\sqrt{5}}{2}$

So $2\cos\theta = \sqrt{5}\sin\theta - 3$ gives:

$2\cos\theta - \sqrt{5}\sin\theta = -3$

So $2\cos\theta = \sqrt{5}\sin\theta - 3$ gives:

$2\cos\theta - \sqrt{5}\sin\theta = -3$

$3\cos(\theta + \tan^{-1}\frac{\sqrt{5}}{2}) = -3$

$\cos(\theta + \tan^{-1}\frac{\sqrt{5}}{2}) = -1$

$\theta + \tan^{-1}\frac{\sqrt{5}}{2} = \pi$

$\theta = \pi - \tan^{-1}\frac{\sqrt{5}}{2}$

d) $\frac{dy}{dx} = -3$ when $x = -3$

$\frac{dy}{dx} = xy - x \rightarrow -3 = -3y + 3$

$-6 = -3y$

$\frac{dy}{dx} = x(y-1)$ $\quad y = 2$

$\int \frac{dy}{y-1} = \int x\,dx$

$\ln|y-1| = \frac{x^2}{2} + C$

Sub $y = 2$, $x = -3$

$\ln|2-1| = \frac{9}{2} + C$

$0 = \frac{9}{2} + C$

$C = -\frac{9}{2}$

$\therefore \ln|y-1| = \frac{x^2}{2} - \frac{9}{2}$

e) $\frac{x}{3x+1} \geq 1 \qquad x \neq -\frac{1}{3}$

$x(3x+1) \geqslant (3x+1)^2$

$0 \geqslant (3x+1)^2 - x(3x+1)$

$0 \geqslant (3x+1)[(3x+1) - x]$

$0 \geqslant (3x+1)(2x+1)$



$\therefore -\frac{1}{2} \leq x < -\frac{1}{3}$

f) $V = \pi\int_0^{10} x^2\,dy$

$y = -\frac{1}{0.7}\ln\frac{x-1}{3}$

$-0.7y = \ln\frac{x-1}{3}$

$e^{-0.7y} = \frac{x-1}{3}$

$x - 1 = 3e^{-0.7y}$

$x = 1 + 3e^{-0.7y}$

$x^2 = (1 + 3e^{-0.7y})^2$

$= 1 + 6e^{-0.7y} + 9e^{-1.4y}$

$\therefore V = \pi\int_0^{10} 1 + 6e^{-0.7y} + 9e^{-1.4y}\,dy$

$= \pi\left[y + \frac{6e^{-0.7y}}{-0.7} + \frac{9e^{-1.4y}}{-1.4}\right]_0^{10}$

$= \pi\left[10 + \frac{6e^{-7}}{-0.7} + \frac{9e^{-14}}{-1.4} - 0 - \frac{6}{-0.7} - \frac{9}{-1.4}\right]$

$= \pi\left[25 - \frac{60}{7}e^{-7} - \frac{45}{7}e^{-14}\right]$ units^3

Q13.

a) $\int_0^2 \frac{x}{x^4+16}\,dx$

let $u = x^2$

a) $\int_0 \frac{x}{x^4+16}\,dx$

let $u = x^2$

x	0	2
u	0	4

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\therefore \int_0^2 \frac{x}{x^4+16}\,dx = \int_0^4 \frac{\frac{1}{2}\,du}{u^2+16}$

$= \frac{1}{2}\int_0^4 \frac{1}{u^2+16}\,du$

$= \frac{1}{2}\left[\frac{1}{4}\tan^{-1}\frac{u}{4}\right]_0^4$

$= \frac{1}{8}\left[\tan^{-1}\frac{4}{4} - \tan^{-1}\frac{0}{4}\right]$

$= \frac{1}{8}\left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{32}$

b) $\frac{dy}{dx} = \frac{3}{2y-2}$

$\int (2y-2)\,dy = \int 3\,dx$

$y^2 - 2y = 3x + C$

Sub $y=1, x=0$

$1 - 2 = 0 + C$

$C = -1$

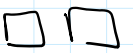
$\therefore y^2 - 2y = 3x - 1$

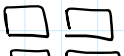
$y^2 - 2y + 1 = 3x$

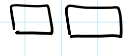
$(y-1)^2 = 3x$

$y - 1 = \pm\sqrt{3x}$

$y = 1 \pm \sqrt{3x}$

c) i. Line 1  ${}^6C_2$

Line 2  ${}^4C_2$

Line 3  ${}^2C_2$

3 equal groups $\rightarrow \div 3!$

$\therefore {}^6C_2 \times {}^4C_2 \times {}^2C_2 \div 3! = 15$ ways.

ii. Choose 3 people from 12 for forward.

${}^{12}C_3 = 220$

Choose 2 people from 6 for defense.

${}^6C_2 = 15$

Player combinations: Forward × Defense

$= 220 \times 15$

$= 3300$ ways.

d) $1^2 + 2^2 + 3^2 + \dots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$

Show $n=1$ is true.

LHS $= 1^2 + (2\times1)^2$

$= 1 + 4$

$= 5$

RHS $= \frac{1(2\times1+1)(4\times1+1)}{3}$

$= 5$

$RHS = \frac{1(2\times1+1)(4\times1+1)}{3}$

$= \frac{1(3)(5)}{3}$

$= 5$

$LHS = RHS$

$\therefore$ True for $n=1$

Assume true for some $n=k$

i.e. $1^2+2^2+3^2+\dots+(2k)^2 = \frac{k(2k+1)(4k+1)}{3}$

Prove true for some $n=k+1$

i.e. Prove $1^2+2^2+3^2+\dots+(2(k+1))^2 = \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}$

$= \frac{(k+1)(2k+2+1)(4k+4+1)}{3}$

$= \frac{(k+1)(2k+3)(4k+5)}{3}$

$= \frac{(k+1)(8k^2+22k+15)}{3}$

$= \frac{8k^3+22k^2+15k+8k^2+22k+15}{3}$

$= \frac{8k^3+30k^2+37k+15}{3}$

$LHS = 1^2+2^2+3^2+\dots+(2k+2)^2$

$= 1^2+2^2+3^2+\dots+(2k)^2+(2k+1)^2+(2k+2)^2$

$= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2+(2k+2)^2$ By assumption.

$= \frac{k(8k^2+6k+1)}{3} + \frac{3(2k+1)^2}{3} + \frac{3(2k+2)^2}{3}$

$= \frac{8k^3+6k^2+k+3(4k^2+4k+1)+3(4k^2+8k+4)}{3}$

$= \frac{8k^3+6k^2+k+12k^2+12k+3+12k^2+24k+12}{3}$

$= \frac{8k^3+30k^2+37k+15}{3}$

$= RHS$

OR:

Assume true for $n=k$

$1^2+2^2+3^2+\dots+(2k)^2 = \frac{k(2k+1)(4k+1)}{3}$

Need to show true for $n=k+1$

$1^2+2^2+3^2+\dots+(2k)^2+(2k+1)^2+(2k+2)^2$

From assumption gives:

$\frac{k(2k+1)(4k+1)}{3} + (2k+1)^2+(2k+2)^2$

$= \frac{k(8k^2+6k+1)+3(4k^2+4k+1)+3(4k^2+8k+4)}{3}$

$\frac{8k^3+6k^2+k+12k^2+12k+3+12k^2+24k+12}{3}$

$$= \frac{8k^3+6k^2+k+12k^2+12k+3+12k^2+24k+12}{3}$$

$$= \frac{8k^3+30k^2+37k+15}{3}$$

$$= \frac{(k+1)(8k^2+22k+15)}{3} \quad \text{(Factor theorem)}$$

$$= \frac{(k+1)(2k+3)(4k+5)}{3}$$

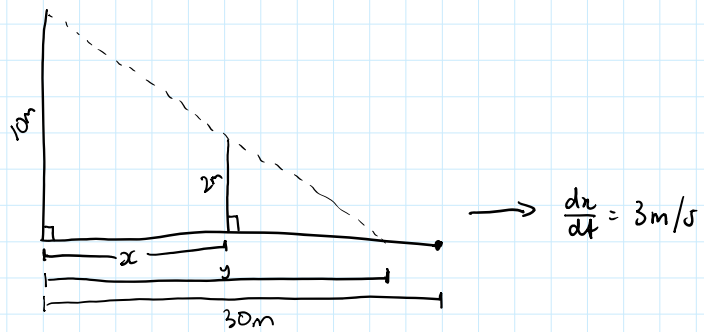
$$= \frac{(k+1)(2k+2+1)(4k+4+1)}{3}$$

$$= \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}$$

$\therefore$ $n=k+1$ is true if $n=k$ is true.

$\therefore$ Since $n=1$ is true, and from above $n=k+1$ is true if $n=k$ is true it must follow that $n=2$ is true given $n=1$ is true and $n=3$ is true given $n=2$ is true and so on for n positive integers.

e)



By similar triangles: $\frac{y}{y-x} = \frac{10}{2}$

$y = 5(y-x)$

$y = 5y-5x$

$5x = 4y$

$y = \frac{5}{4}x$

$\frac{dy}{dx} = \frac{5}{4}$

We want $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= \frac{5}{4} \times 3$

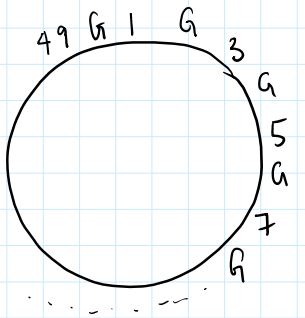
$= \frac{15}{4}$

$= 3.75$ m/s

$\therefore$ Shadow is moving faster.

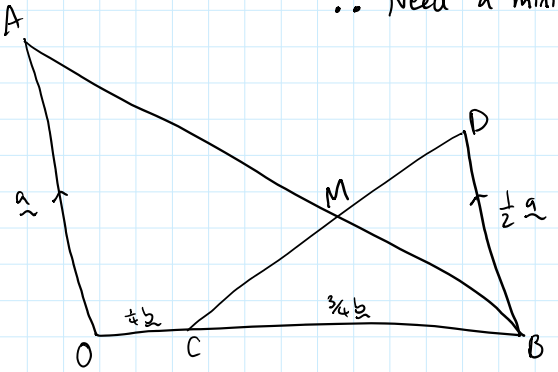
Q14.

a)



Worst case scenario is that boys sit in all the odd seats. Leaving 25 empty gaps in the circle.
25 girls would fill all the in between spots, (n holes)
The 26th girl then must sit next to a girl (n+1 pigeon)
∴ Need a minimum of 26 girls.

b)



$\frac{|\vec{CM}|}{|\vec{CD}|} = \lambda$

So $\vec{CD} = \vec{CB} + \vec{BD}$
$= \frac{3}{4}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$

$\vec{CM} = \lambda\vec{CD}$
$= \lambda\left(\frac{1}{2}\underset{\sim}{a} + \frac{3}{4}\underset{\sim}{b}\right) \longrightarrow (1)$

But $\vec{CM} = \vec{CO} + \vec{OA} + \vec{AM}$
$= -\frac{1}{4}\underset{\sim}{b} + \underset{\sim}{a} + \vec{AM}$

Let $\frac{\vec{AM}}{\vec{AB}} = \mu$

$\vec{AM} = \mu(\vec{AB})$
$= \mu(\underset{\sim}{b} - \underset{\sim}{a})$

$\therefore \vec{CM} = -\frac{1}{4}\underset{\sim}{b} + \underset{\sim}{a} + \mu\underset{\sim}{b} - \mu\underset{\sim}{a}$
$= \underset{\sim}{a}(1-\mu) + \underset{\sim}{b}(\mu - \frac{1}{4}) \longrightarrow (2)$

Equate (1) and (2)

$\frac{1}{2}\lambda = 1-\mu \rightarrow \mu = 1 - \frac{1}{2}\lambda$

$\frac{3}{4}\lambda = \mu - \frac{1}{4}$

$\frac{3}{4}\lambda = 1 - \frac{1}{2}\lambda - \frac{1}{4}$

$3\lambda = 4 - 2\lambda - 1$

$5\lambda = 3$

$\lambda = \frac{3}{5}$

$\therefore \frac{|\vec{CM}|}{|\vec{CB}|} = \frac{3}{5}$

c) $(1+x)^{1000} + (1+x)^{999} + (1+x)^{998} + \cdots + (1+x)^{950}$

$\downarrow$ $\quad\downarrow$ $\quad\downarrow$ $\quad\downarrow$

$^{1000}C_{50}$ $\quad$ $^{999}C_{49}$ $\quad$ $^{998}C_{48}$ $\quad$ $^{950}C_{0}$ } Pascal's symmetry of $^nC_r = {}^nC_{n-r}$

$\downarrow$ $\quad\downarrow$ $\quad\downarrow$ $\quad\downarrow$

$^{1000}C_{950}$ $\quad$ $^{999}C_{950}$ $\quad$ $^{998}C_{950}$ $\quad$ $^{950}C_{950}$

The coefficient of x^{950}

So:

$(1+x)^{1000} + (1+x)^{999} + (1+x)^{998} + \cdots + (1+x)^{950}$

So:

$$(1+x)^{1000} + (1+x)^{999} + (1+x)^{998} + \cdots + (1+x)^{950}$$
$$= (1+x)^{950}\underbrace{\left[(1+x)^{50} + (1+x)^{49} + (1+x)^{48} + \cdots 1\right]}$$

GP: $a = 1$ → go backwards. $r = (1+x)$, $n = 51$

$$= (1+x)^{950}\left[\frac{1\left((1+x)^{51}-1\right)}{(1+x)-1}\right]$$
$$= (1+x)^{950}\left[\frac{(1+x)^{51}-1}{x}\right]$$
$$= \frac{(1+x)^{950}(1+x)^{51}}{x} - \frac{(1+x)^{950}}{x}$$
$$= \frac{(1+x)^{1001}}{x} - \frac{(1+x)^{950}}{x}$$

Now we must consider the x^{950} coefficient.

$\frac{(1+x)^{950}}{x}$ will NOT have a x^{950} term (only goes to x^{949})

$\frac{(1+x)^{1001}}{x}$ will have one but it will come from the $\frac{{}^{1001}C_{951}x^{951}}{x} = {}^{1001}C_{951}x^{950}$

So ${}^{1001}C_{951} = {}^{1001}C_{1001-951}$
$= {}^{1001}C_{50}$

By comparing coefficients of x^{950} we get:

$$\binom{1000}{50} + \binom{999}{49} + \binom{998}{48} + \cdots + \binom{950}{0} = \binom{1001}{50}$$

OR: Using ${}^{n+1}C_{k+1} = {}^{n}C_{k+1} + {}^{n}C_{k}$

$$\text{RHS} = {}^{1001}C_{50} = {}^{1000}C_{50} + {}^{1000}C_{49}$$
$$= {}^{1000}C_{50} + {}^{999}C_{49} + {}^{999}C_{48}$$
$$= {}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + {}^{998}C_{47}$$
$$\vdots$$
$$= {}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + \cdots + {}^{951}C_{1} + {}^{951}C_{0}$$
$$= {}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + \cdots + {}^{951}C_{1} + {}^{950}C_{0}$$

since ${}^{951}C_0 = {}^{950}C_0 = 1$

OR using: $(1+x)^{1001} = (1+x)(1+x)^{1000}$
$= (1+x)^{1000} + x(1+x)^{1000}$
$= (1+x)^{1000} + x(1+x)(1+x)^{999}$
$= (1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{999}$

$$= (1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{999}$$

$$\vdots$$

$$= (1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \cdots + x^{50}(1+x)^{950}$$

Then compare term x^{50}

$(1+x)^{1001}$ gives ${}^{1001}C_{50}x^{50}$

The RHS gives ${}^{1000}C_{50}x^{50} + x \times {}^{999}C_{49}x^{49} + x^2 \times {}^{998}C_{48} + \cdots x^{50} \times {}^{950}C_0 x^0$

So ${}^{1001}C_{50} = {}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + \cdots + {}^{950}C_0$

d) i. Let roots of the polynomial $P(x) = ax^3 + cx + d$ be α, α, β (double root).

$\alpha + \alpha + \beta = 0$ (sum)

$\beta = -2\alpha$

$\alpha^2 + \alpha\beta + \alpha\beta = \frac{c}{a}$ (sum of pairs)

$\alpha^2 + 2\alpha\beta = \frac{c}{a}$

$\alpha^2 + 2\alpha(-2\alpha) = \frac{c}{a}$

$-3\alpha^2 = \frac{c}{a}$

$\alpha^2 = -\frac{c}{3a}$

$\alpha^2\beta = -\frac{d}{a}$ (roots multiplied)

$\alpha^2(-2\alpha) = -\frac{d}{a}$

$-2\alpha^3 = -\frac{d}{a}$

$\alpha^3 = \frac{d}{2a}$

So if $\alpha^2 = -\frac{c}{3a}$

$\alpha^6 = \left(-\frac{c}{3a}\right)^3$

and if $\alpha^3 = -\frac{d}{2a}$

$\alpha^6 = \left(-\frac{d}{2a}\right)^2$

$\therefore \left(-\frac{c}{3a}\right)^3 = \left(-\frac{d}{2a}\right)^2$

$-\frac{c^3}{27a^3} = \frac{d^2}{4a^2}$

$-\frac{c^3}{27a} = \frac{d^2}{4}$

$-4c^3 = 27ad^2$

$\therefore 27ad^2 + 4c^3 = 0$

OR: Double root: $P'(x) = 0$

$P'(x) = 3ax^2 + c = 0$

$x^2 = -\frac{c}{3a}$

$ax^3 + cx + d = 0$

$x(ax^2 + c) = -d$

$x^2(ax^2 + c)^2 = d^2$

$\left(-\frac{c}{3a}\right)\left(a\left(-\frac{c}{3a}\right) + c\right)^2 = d^2$

$-\frac{c}{3a}\left(-\frac{c}{3} + c\right)^2 = d^2$

$-\frac{c}{3a}\left(\frac{2c}{3}\right)^2 = d^2$

$\frac{c}{}(4c^2) = d^2$

$$-\frac{c}{3a}\left(\frac{2c}{3}\right)^2 = d^2$$

$$-\frac{c}{3a}\left(\frac{4c^2}{9}\right) = d^2$$

$$\frac{-4c^3}{27a} = d^2$$

$$-4c^3 = 27ad^2$$

$$0 = 27ad^2 + 4c^3$$

as the alternative

ii. Let the 2 stationary points be at $y = m$ for $y = 3x^3 - 4x + 1$

$\therefore\ m = 3x^3 - 4x + 1$

$0 = 3x^3 - 4x + 1 - m$

And this has a double root

$\therefore\ a = 3$

$c = -4$

$d = 1 - m$

From part i): $27ad^2 + 4c^3 = 0$

$27 \times 3 \times (1-m)^2 + 4 \times (-4)^3 = 0$

$81(1-m)^2 - 256 = 0$

$(1-m)^2 = \frac{256}{81}$

$1 - m = \pm\frac{16}{9}$

$m = 1 \pm \frac{16}{9}$

$= \frac{25}{9}$ or $-\frac{7}{9}$

$\therefore$ y-coordinate of stationary points are at $y = \frac{25}{9}$ and $y = -\frac{7}{9}$

e) $\text{proj}_{\underset{\sim}{z}}\underset{\sim}{u} = \frac{\cos\alpha\cos\beta}{\cos(\alpha+\beta)}\text{proj}_{\underset{\sim}{z}}\underset{\sim}{x}$

So: $\text{proj}_{\underset{\sim}{z}}\underset{\sim}{x} = \frac{\underset{\sim}{x}\cdot\underset{\sim}{z}}{|\underset{\sim}{z}|^2}\underset{\sim}{z}$

$\underset{\sim}{u} = \text{proj}_{\underset{\sim}{y}}\underset{\sim}{x}$

$= \frac{\underset{\sim}{x}\cdot\underset{\sim}{y}}{|\underset{\sim}{y}|^2}\underset{\sim}{y}$

$\underset{\sim}{x}\cdot\underset{\sim}{y} = |\underset{\sim}{x}||\underset{\sim}{y}|\cos\alpha$

$\cos\alpha = \frac{\underset{\sim}{x}\cdot\underset{\sim}{y}}{|\underset{\sim}{x}||\underset{\sim}{y}|}$

Similarly

$\cos\beta = \frac{\underset{\sim}{y}\cdot\underset{\sim}{z}}{|\underset{\sim}{y}||\underset{\sim}{z}|}$

and

$\cos(\alpha+\beta) = \frac{\underset{\sim}{x}\cdot\underset{\sim}{z}}{|\underset{\sim}{x}||\underset{\sim}{z}|}$

Now: RHS $= \frac{\cos\alpha\cos\beta}{\cos(\alpha+\beta)} \times \text{proj}_{\underset{\sim}{z}}\underset{\sim}{x}$

$= \frac{\underset{\sim}{x}\cdot\underset{\sim}{y}}{|\underset{\sim}{x}||\underset{\sim}{y}|} \times \frac{\underset{\sim}{y}\cdot\underset{\sim}{z}}{|\underset{\sim}{y}||\underset{\sim}{z}|}$ ~~$\frac{\underset{\sim}{x}\cdot\underset{\sim}{z}}{|\underset{\sim}{x}||\underset{\sim}{z}|}$~~

$\cos(\alpha+\beta)$

$$= \frac{\frac{\underset{\sim}{x}\cdot\underset{\sim}{y}}{|\underset{\sim}{x}||\underset{\sim}{y}|} \times \frac{\underset{\sim}{y}\cdot\underset{\sim}{z}}{|\underset{\sim}{y}||\underset{\sim}{z}|}}{\frac{\underset{\sim}{x}\cdot\underset{\sim}{z}}{|\underset{\sim}{x}||\underset{\sim}{z}|}} \times \frac{\underset{\sim}{w}\cdot\underset{\sim}{z}}{|\underset{\sim}{z}|^2}\underset{\sim}{z}$$

$$= \frac{\underset{\sim}{x}\cdot\underset{\sim}{y}}{|\underset{\sim}{y}|} \times \frac{\underset{\sim}{y}\cdot\underset{\sim}{z}}{|\underset{\sim}{y}|} \times \frac{\underset{\sim}{z}}{|\underset{\sim}{z}|^2}$$

$$= \boxed{\frac{\underset{\sim}{x}\cdot\underset{\sim}{y}}{|\underset{\sim}{y}|^2}\underset{\sim}{y}} \cdot \underset{\sim}{z} \times \frac{\underset{\sim}{z}}{|\underset{\sim}{z}|^2}$$

$$\Downarrow$$

$$= \underset{\sim}{u}\cdot\underset{\sim}{z} \times \frac{\underset{\sim}{z}}{|\underset{\sim}{z}|^2}$$

$$= \frac{\underset{\sim}{u}\cdot\underset{\sim}{z}}{|\underset{\sim}{z}|^2} \times \underset{\sim}{z}$$

$$= \text{proj}_{\underset{\sim}{z}}\underset{\sim}{u}$$

$$= \text{LHS}.$$