Name: ______________________________

Class: ______________

Mathematics Extension 2

Task 2, Term 1, 2020

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 6 –10, show relevant mathematical reasoning and/or calculations

Total marks: 80

Section I – 5 marks (pages 2-3)

- Attempt Questions 1–5
- Allow about 7 minutes for this section

Section II – 75 marks (pages 4-8)

- Attempt Questions 6-10
- Allow about 1 hours and 53 minutes for this section

Section I (5 marks)

Write your answers on the multiple-choice answer sheet provided.

1. Given that $(1-2i)^2 = -3-4i$, the roots of the equation: $z^2 - 5z + (7+i) = 0$ are:

   A. $3-i,\ 2-i$

   B. $3+i,\ 2-i$

   C. $3-i,\ 2+i$

   D. $3+i,\ 2+i$

2. $x = 3 + 4\,i$, $y = -1 + 2\,i$, find the value of $|\,x\,|x - \bar{y}y$.

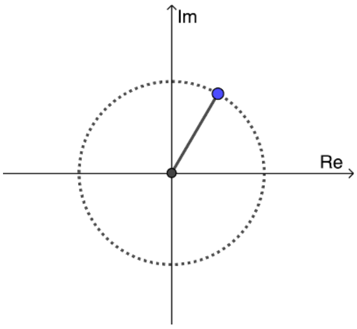
   A. $15 + \sqrt{5} + 20\,i$

   B. $20 + 20\,i$

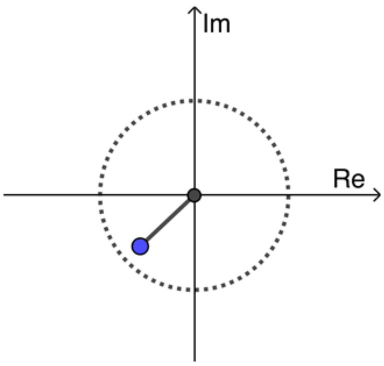
   C. $10 + 20\,i$

   D. $15 - \sqrt{5} + 20\,i$

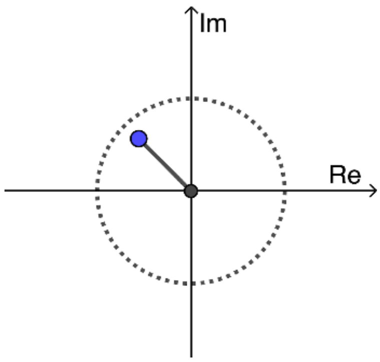
3. The Argand Diagram below shows the complex number $z = x + iy$. Which diagram best represents the complex number $iz + \bar{z}$?



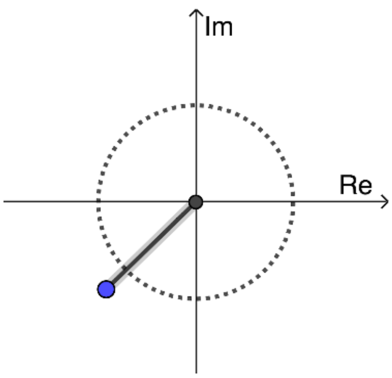
A.



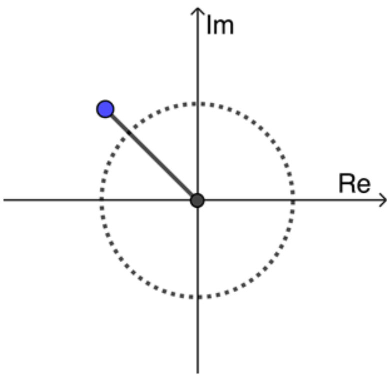
B.



C.



D.



4. If $P \Rightarrow Q$, which of the following is logically equivalent?

A. $Q \Rightarrow P$

B. $\neg P \Rightarrow \neg Q$

C. $P \Leftrightarrow Q$

D. $\neg Q \Rightarrow \neg P$

5. Which of the following is the converse of the conditional statement "When Raj stays up late, it is necessary that Raj sleeps until noon."

A. "If Raj stays up late, then Raj sleeps until noon."
B. "If Raj does not stay up late, then Raj does not sleep until noon."
C. "If Raj does not sleep until noon, then Raj does not stay up late."
D. "If Raj sleeps until noon, then Raj stays up late."

Section II begins on the next page.

Section II (75 marks)

Attempt Questions 6-10

Allow about 1 hour and 53 minutes for this section

Start each question on a new page. Extra paper is available.

In Questions 6-10, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 marks)

Start a new page.

a) Find all the zeros of $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$ over **C**, given that $(1 - i)$ is one of the zeros. **3**

b) (i) Express $3 + \sqrt{3}i$ and $3 - \sqrt{3}i$ in the Modulus Argument Form. **2**

(ii) Find the values of n that satisfy $(\frac{1}{3+\sqrt{3}i})^n + (\frac{1}{3-\sqrt{3}i})^n = 0$. **3**

c) If ω is a non-real cube root of unity, show that $(1 + \omega)^3(1 + 2\omega + 2\omega^2) = 1$. **4**

d) (i) Sketch, without calculating stationary points, the graph of the function $f(x)$ given by $f(x) = (x - p)(x - q)(x - r)$, where $p < q < r$ and $p, q, r \in \mathbb{R}$. **1**

(ii) By considering the equation $f'(x) = 0$, show that $(p + q + r)^2 > 3(qr + rp + pq)$. **2**

Question 7 begins on the next page.

Question 7 (15 marks)

Start a new page.

a) Sketch the following loci on separate Argand diagrams.

(i) $\text{Arg}(z+i) = \text{Arg}(z-1)$ **2**

(ii) $z - \bar{z} \leq |z|^2$ **3**

b) If $z_1 = 3e^{\frac{i\pi}{6}}, z_2 = 2e^{\frac{i\pi}{6}}$, find: **2**

(i) Find $\overline{z_2}$ in the exponential form.

(ii) Find $\frac{z_2}{z_1} - z_1\overline{z_2} + \frac{4}{z_2}$ in the form of $x + iy$. **3**

c) Find the area of a triangle whose vertices are $z, iz, and\ z + iz$. **2**

d) Let a and b be rationals and x irrational. Show that if $\frac{x+a}{x+b}$ is rational then $a = b$. **3**

Question 8 begins on the next page.

Question 8 (15 marks)

Start a new page.

a) Given that $\frac{1+7i}{1+2i} = re^{i\theta}$, $r > 0$ and $-\pi < \theta \le \pi$, find the value of r and θ (Provide the value of θ rounded to two decimal places). **2**

b) $OPQR$ is a parallelogram on an Argand Diagram. The point O is the origin, and the point P and Q represent the complex number $z_1$and $z_2$respectively.

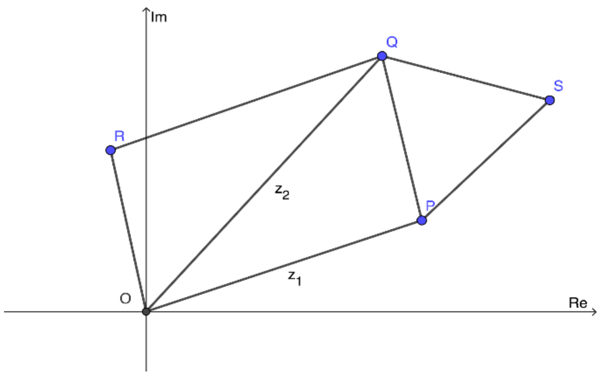
Given $|z_2| = \frac{2\sqrt{3}}{3}|z_1|$, $\arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{6}$, $0 < \arg(z_1) < \arg(z_2) < \frac{\pi}{2}$,

(i) Prove $z_2 = \left(1 + \frac{\sqrt{3}}{3}i\right)z_1$. **2**

(ii) Prove $|\, z_2 \ - z_1| = \frac{1}{2}|z_2|$. **1**

(iii) Find $arg\left(\frac{z_2 - z_1}{z_1}\right)$. **3**

(iv) Given that ΔQPS is an equilateral triangle and the point S represents complex number z_3. Find the expression of z_3 in terms of z_1. **3**



(c) (i) Prove the following for real and positive numbers a, b. **1**

$$\frac{a+b}{2} \ge \sqrt{ab}$$

(ii) Hence, show that $\sqrt{a}\ \sqrt{b+c}\ \le \frac{a+b+c}{2}$. **1**

(iii) Show that $\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2$ for $a, b, c > 0$. **2**

Question 9 begins on the next page.

Question 9 (15 marks)

Start a new page.

a) (i) Express $cos5\theta$ in the form of $acos^5\theta + bcos^3\theta + ccos\theta,$ giving the numerical values of a, b, c. **3**

(ii) Find all the solutions of the equation $z^5 + 1 = 0$. **2**

(iii) Sketch all the solutions of the equation $z^5 + 1 = 0$ on a unit circle. **1**

(iv) Express $z^5 + 1$ as a product of linear and quadratic factors with real coefficients. **2**

(v) Find the value of $cos\frac{\pi}{5} + cos\frac{3\pi}{5} + cos\frac{7\pi}{5} + cos\frac{9\pi}{5},$ giving reasons. **2**

b) Let $x, y \in \mathbb{Z}$. Prove that if $x^2(y^2 - 2y)$ is odd, then x and y are both odd. **2**

c) Prove that a number is divisible by 8 if and only if the last three digits of a number are divisible by 8. **3**

Question 10 begins on the next page.

Question 10 (15 marks)

Start a new page.

a) (i) Factorise $z^4 + 1 = 0$ into quadratic factors of real coefficients. **1**

(ii) Divide by z^2 and hence show $cos2\theta = 2\left(cos\theta - cos\frac{\pi}{4}\right)\left(cos\theta - cos\frac{3}{4}\pi\right)$. **4**

b) If the roots of the equation $z^4 + az^3 + (12 + 9i)z^2 + bz = 0$ are the vertices of a square, and $2b - 13a = x$, find the values of x, where $x \in \mathbb{Z}$ and a and b are complex numbers. **3**

c) Given a, b, c are positive real numbers with $a > b$ and $c^2 > ab$, prove by contradiction that $\frac{a+c}{\sqrt{a^2+c^2}} - \frac{b+c}{\sqrt{b^2+c^2}} > 0$. **4**

d) Given $\frac{\text{x+y+z}}{3} \geq \sqrt[3]{xyz}$, show that $\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \geq \frac{9}{\pi}$ if *A*, *B*, *C* are three interior angles (in radians), of ΔABC. **3**

END OF EXAMINATION

MC Answers:
1C 2C 3C 4D 5D

HSC-Ext 2- T1-2020

MATHEMATICS Extension 2 : Question……6		1/4
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
a) Find all zeros $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$, $(1-i)$ is zero } given $\therefore$ the conjugate root is $(1+i)$	1	zeros ≠ factored form But! No mark was penalised.
$\therefore (1-i)(1+i) = x^2 - 2x + 2$ $x^2 - 2x + 2 \overline{) x^4 - x^3 - 2x^2 + 6x - 4}$, quotient $x^2 + x - 2$ $-(x^4 - 2x^3 + 2x^2)$ $x^3 - 4x^2 + 6x$ $-(x^3 - 2x^2 + 2x)$ $-2x^2 + 4x - 4$ $-(-2x^2 + 4x - 4)$ $-$	1	correct division
$P(x) = (x+2)(x-1)(x-(1+i))(x-(1-i))$ zeros $= (1-i), (1+i), -2, 1$	1	
b) i) $3+\sqrt{3}i$ and $3-\sqrt{3}i$ in mod arg form. mod $= 3^2 + (\sqrt{3})^2$ $= \sqrt{12} = 2\sqrt{3}$ arg $= \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$ $2\sqrt{3}\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$	1	Unfortunately only 2 marks allocated. So the marking was based on correct arg for both and correct mod for both. If not marked this way student would have lost both marks.
$2\sqrt{3}\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$ or $2\sqrt{3}\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)$	1	- pls define cis

MATHEMATICS: Question 6   2/4

Suggested Solutions	Marks Awarded	Marker's Comments

b) ii) $\left(\frac{1}{3+\sqrt{3}i}\right)^n + \left(\frac{1}{3-\sqrt{3}i}\right)^n = 0.$

$= \left(2\sqrt{3}\,\text{cis}\,\frac{\pi}{6}\right)^{-n} + \left[2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)\right]^{-n}$ by DMT (1)

$= \left(2\sqrt{3}\right)^{n}\left(\text{cis}\,-\frac{n\pi}{6} + \text{cis}\,\frac{n\pi}{6}\right)$

$= \left(2\sqrt{3}\right)^{n}\left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6} + \cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$

$= \left(2\sqrt{3}\right)^{n} \times 2\cos\left(\frac{n\pi}{6}\right) = 0.$ (1)

$\cos\left(\frac{n\pi}{6}\right) = \cos(0)$

$\frac{n\pi}{6} = \pm\frac{\pi}{2} + 2k\pi$

$n = \pm 3$ (1)

$n = 6k+3,\ k \in \mathbb{Z}$
(or $12k \pm 3$)

c) ω is non-real cube root of unity } given & show question
show $(1+\omega)^3(1+2\omega+2\omega^2) = 1$

METHOD 1

$\omega^3 = 1$ but $\omega \neq 1$ (1)

$(\omega-1)(1+\omega+\omega^2) = 0$ $(1+\omega+\omega^2) = 0$

$1+\omega = -\omega^2$
$\omega+\omega^2 = -1$

LHS:
$(1+\omega)^3(1+2\omega+2\omega^2)$
$(1+\omega)^3(1+\omega+\omega+\omega^2+\omega^2)$
$(1+\omega)^3(1+\omega+\omega^2+\omega+\omega^2)$ (2)
$(-\omega^2)^3(1+\omega+\omega^2+\omega+\omega^2)$

$= (-1)(0-1)$ (1)

$= 1$
$= \text{RHS}$

Marker's comments:

Lucky !!
– didn't penalise for not mentioning cis &/or DMT

note*.
→ have to consider even and odd func. at this step.
$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

one could have written as +ve or −ve of the function and was marked accordingly. marks was not awarded if ± was not mentioned but cfe was awarded

Too many marks allocated for this question.
(Could have given a mark for showing that ω^2 is the other non real root – it is otherwise assumed but why should it be?)
(But you only used it in method 2!)
(Or is it method 3!).

MATHEMATICS: Question…….. 3/4

Suggested Solutions	Marks Awarded	Marker's Comments

METHOD 2

LHS $(1+\omega)^3(1+2\omega+2\omega^2) = 1$

$(1+3\omega+3\omega^2+\omega^3)(1+2\omega+2\omega^2)$ — 1

$= 1+3\omega+3\omega^2+\omega^3+2\omega+6\omega^2+6\omega^3+2\omega^4$
$\quad +2\omega^2+6\omega^3+6\omega^4+2\omega^5$

$= 1+3\omega+2\omega+3\omega^2+6\omega^2+2\omega^2+\omega^3+6\omega^3$
$\quad +6\omega^3+2\omega^4+6\omega^4+2\omega^5$

expand &/or collect like terms — 1

$= 1+5\omega+11\omega^2+13\omega^3+8\omega^4+2\omega^5$ → coz $\omega^3 = 1$

$= 1+5\omega+11\omega^2+13+8\omega+2\omega^2$

$= 1+5\omega+8\omega+11\omega^2+2\omega^2$ — 1

$= 14+5\omega+8\omega+11\omega^2$

$1+\omega+\omega^2 = 0$
$\therefore \omega+\omega^2 = -1$

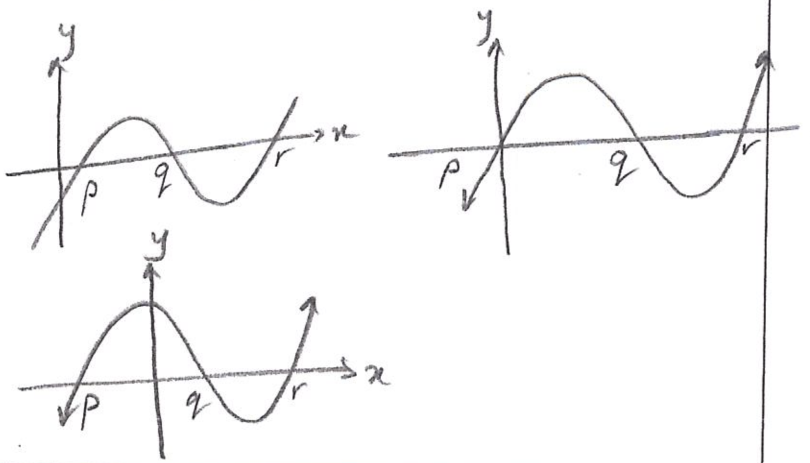
$= 14+13(\omega+\omega^2)$

$= 14+13(-1)$ — 1

$= 1$

$= RHS$

d) i) sketch $f(x) = (x-p)(x-q)(x-r)$
$p<q<r$, $p,q,r \in \mathbb{R}$



1

any was accepted with roots in correct order and +ve cubic function.

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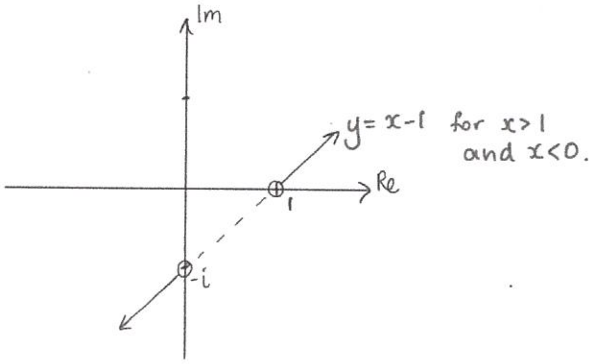
MATHEMATICS: Question…….. 4/4

Suggested Solutions	Marks Awarded	Marker's Comments
dii) by considering $f'(x)=0$		
show $(p+q+r)^2>3(qr+rp+pq)$		
$f(x)=(x-p)(x-q)(x-r)$ — expand out		
$=x^3-x^2(p+q+r)+x(pq+qr+pr)-pqr$		
$f'(x)=3x^2-2x(p+q+r)+(pq+qr+pr)$		1
for stationary pts $f'(x)=0$		
there are 2 distinct stnry pts.		
ie $\Delta>0$		
$\Delta=b^2-4ac$		
$=4(p+q+r)^2-4(3)(pq+qr+pr)>0$		1
$4(p+q+r)^2>12(pq+qr+pr)$		
$(p+q+r)^2>3(pq+qr+pr)$	shown	

Question 7 solutions

Sunday, 19 April 2020 8:39 pm

(i)



① intercepts and arrows in correct direction

① showing line is $y = x - 1$
by - labelling the line
or - showing dotted line between intercepts
- showing angles between line and axes.

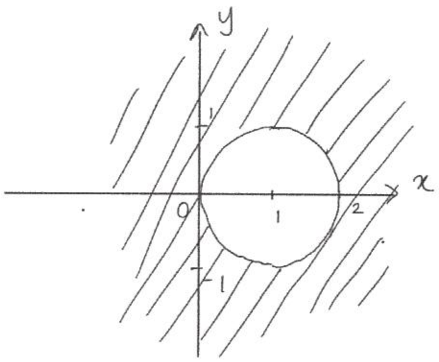
(ii) $z + \bar{z} \leq |z|^2$

$x + iy + (x - iy) \leq (x^2 + y^2)$

$2x \leq (x^2 + y^2)$

$x^2 + y^2 - 2x \geq 0$

$(x-1)^2 + y^2 \geq 1$



[Some students were not aware of the correction from $z - \bar{z}$ to $z + \bar{z}$
-no penalty for no diagram when $z - \bar{z}$ used]

① Inequality with $2x$

① $x^2 + y^2$

① correct diagram

b) $z_1 = 3e^{i\pi/6}$, $z_2 = 2e^{i\pi/6}$

(i) $z_2 = 2e^{i\pi/6}$

$= 2(\cos \pi/6 + i \sin \pi/6)$

$\bar{z}_2 = 2(\cos \pi/6 - i \sin \pi/6)$

$= 2(\cos(-\pi/6) + i \sin(-\pi/6))$

$= 2e^{-i\pi/6}$

Bald answer ②

①

①

(ii) $\frac{z_2}{z_1} - z_1 \bar{z}_2 + \frac{4}{z_2}$

$= \frac{2e^{i\pi/6}}{3e^{i\pi/6}} - 3e^{i\pi/6} \times 2e^{-i\pi/6} + \frac{4}{2e^{i\pi/6}}$

$= \frac{2}{3} - 6 + 2e^{-i\pi/6}$

$= -5\frac{1}{3} + 2(\cos(-\pi/6) + i \sin(-\pi/6))$

$= -5\frac{1}{3} + 2(\cos \pi/6 - i \sin \pi/6)$

$= (-5\frac{1}{3} + \sqrt{3}) - i$

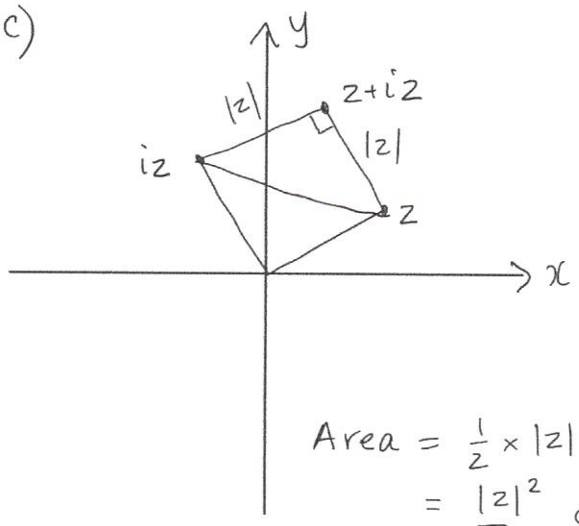
$= \frac{3\sqrt{3} - 16}{3} - i$

① for $\frac{z_2}{z_1} = \frac{2}{3}$

① for $z_1 \bar{z}_2 = 6$

① for $\frac{4}{z_2} = \sqrt{3} - i$

c)



① lengths of sides

① area expression

$$\text{Area} = \frac{1}{2} \times |z| \times |z|$$
$$= \frac{|z|^2}{2} \text{ square units}$$

d) Method 1: contradiction

Suppose $\frac{x+a}{x+b}$ is rational

ie. $\frac{x+a}{x+b} = r$ where r is rational

Assume $r \neq 1$

$\frac{x+a}{x+b} = r$

$x + a = rx + rb$

$x(1-r) = rb - a$

$x = \frac{rb-a}{1-r}$

But a, b, r are rationals

$\therefore$ x is rational

But x is irrational

$\therefore$ $r = 1$ by contradiction, which is rational

$\therefore$ $\frac{x+a}{x+b} = 1$ which is rational

$x + a = x + b$

$\therefore a = b$ if $\frac{x+a}{x+b}$ is rational.

Method 2: Contradiction

If $\frac{x+a}{x+b}$ is rational and a,b are rational

then $\frac{x+a}{x+b} = \frac{p}{q}$ where $p,q \in \mathbb{Z}$. Assume $p \neq q$

$qx + qa = px + pb$

$qx - px = pb - qa$

$x(q-p) = pb - qa$

but x is irrational

$x = \frac{pb-qa}{q-p}$

if $a \neq b$ then x is rational as $\frac{pb-qa}{q-p}$ is rational

if $a = b$ then $x = \frac{a(p-q)}{q-p}$

$= -a$ which is rational

but x is irrational

$\therefore p=q$ by contradiction of assumption

Now if $p=q$ then $\frac{p}{q} = 1$

$\frac{x+a}{x+b} = 1$

So $x+a = x+b$

$\therefore a=b$

or $x = \frac{b(p-q)}{q-p}$

$= -b$

$\therefore -a = -b$

$a = b$

(1) expression for x following assumption $p \neq q$

(1) show by contradiction that $p=q$

(1) show $a=b$

Note: Only 1 mark if $p=q$ not considered.

Method 3: Using contrapositive and contradiction

Prove by contrapositive if $a \neq b$ then $\frac{x+a}{x+b}$ is irrational

$a \neq b$, a,b are rational

$\frac{x+a}{x+b} = \frac{x+b}{x+b} - \frac{b-a}{x+b}$

$= 1 - \frac{b-a}{x+b}$

(1) stating contrapositive and finding x

Consider $\frac{b-a}{x+b}$. For this to be rational $b-a=0$,

Since $x+b$ is irrational.

But $a \neq b$ in assumption

(1) contradiction

$\therefore \frac{b-a}{x+b}$ is irrational

$\therefore \frac{x+a}{x+b}$ is irrational if $a \neq b$

(1) proof of contrapositive

$\therefore$ If $\frac{x+a}{x+b}$ is rational then $a=b$ by contrapositive

Method 4: Equating rational & irrational parts

If $\frac{x+a}{x+b}$ is rational let $\frac{x+a}{x+b} = \frac{p}{q}$ $p,q \in \mathbb{Z}$

$q(x+a) = p(x+b)$

$qx + qa = px + pb$ ①

Since a,b,p,q are rational qa and pb are rational

Equating irrational parts of same form — ① Recognising irrational parts

$qx = px$

$\therefore p = q$ — ① Showing $p=q$

$\therefore \frac{x+a}{x+b} = 1$

$x+a = x+b$

$\therefore a = b$ — ① Showing $a=b$

Suggested Solutions	Marks	Marker's Comments
a) Method 1		
$\frac{1+7i}{1+2i} = \frac{(1+7i)(1-2i)}{(1+2i)(1-2i)}$		Too many people left argument in degrees where radians are clearly required in question.
$= \frac{1+14+5i}{5}$		
$= \frac{15+5i}{5}$		
$= 3+i$		
$\therefore r = \sqrt{3^2+1^2} = \underline{\underline{\sqrt{10}}}$	1	
$\theta = \tan^{-1}(1/3) = \underline{\underline{0.32}}$ (to 2DP)	1	Usually 1 mark for $\sqrt{10}$ and 1 for 0.32. Where a slip was made in the initial calculation, 1 mark if rest of answer followed.
Method 2		
$\frac{1+7i}{1+2i} = \frac{\sqrt{50}\, e^{i(\tan^{-1}7)}}{\sqrt{5}\, e^{i(\tan^{-1}2)}}$		
$= \sqrt{10}\, e^{i(\tan^{-1}7 - \tan^{-1}2)}$		
$= \sqrt{10}\, e^{i(0.32)}$		
$\therefore \underline{\underline{r = \sqrt{10}}} \quad \underline{\underline{\theta = 0.32}}$		
b) i)		
$\arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{6} \qquad 0 \le \arg z_1 < \arg z_2 < \frac{\pi}{2}$		1 for modulus and argument shifts. 1 for final answer.
$\therefore \arg z_2 = \arg z_1 + \frac{\pi}{6}$		
$z_2 = \lvert z_2 \rvert e^{i(\arg z_1 + \pi/6)}$	1	
$= \frac{2\sqrt{3}}{3} \lvert z_1 \rvert e^{i \arg z_1} \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$		Several people lost second mark by omitting last line of working *. Essential when result is given.
$= z_1 \frac{2\sqrt{3}}{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$		
$= z_1\left(\frac{2\sqrt{3}}{3}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\right)$ *		
$= \underline{\underline{z_1\left(1 + \frac{\sqrt{3}}{3} i\right)}}$	1	

Suggested Solutions	Marks	Marker's Comments
b) ii) $z_2 - z_1 = \frac{\sqrt{3}}{3} i z_1$ (Using (i)) $\therefore \lvert z_2 - z_1 \rvert = \left\lvert \frac{\sqrt{3}}{3} \right\rvert \lvert z_1 \rvert$ $= \frac{\sqrt{3}}{3} \cdot \frac{3}{2\sqrt{3}} \lvert z_2 \rvert$ (Since $\lvert z_2 \rvert = \frac{2\sqrt{3}}{3} \lvert z_1 \rvert$) $= \frac{\lvert z_2 \rvert}{2}$	1	Again several people did not satisfactorily explain the last line. Essential when result given
iii) $\arg\left(\frac{z_2 - z_1}{z_1}\right) = \arg(z_2 - z_1) - \arg(z_1)$	1	Not really worth 3 marks but take what you can!
$= \arg\left(z_1 + \frac{\sqrt{3}}{3} i z_1 - z_1\right) - \arg(z_1)$ (Using (i)) $= \arg\left(\frac{\sqrt{3}\, i}{3}\right) + \arg(z_1) - \arg(z_1)$ $= \arg\left(\frac{\sqrt{3}\, i}{3}\right)$ $= \pi/2$ as this is purely imag^y	2	A lot of students used a longer geometrical approach ending with an answer of $\pi/2$ but with no reasoning why the argument was not $-\pi/2$. They generally got 2 marks of the 3.
iv) $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS}$	1	1 for the (implied) vector addition
$z_3 = z_1 + (z_2 - z_1) e^{-i\pi/3}$	1	1 for a correct rotation in the equilateral triangle
$= z_1 + \frac{\sqrt{3}\, i}{3}\left(\cos\frac{\pi}{3} - i \sin\frac{\pi}{3}\right) z_1$ $= z_1 + \frac{\sqrt{3}\, i}{3}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_1$ $= \left(1 + \frac{\sqrt{3}\, i}{6} + \frac{3}{6}\right) z_1$ $= \left(\frac{3}{2} + \frac{\sqrt{3}\, i}{6}\right) z_1$	1	1 for getting to the final answer.
*See note on next page.		

Suggested Solutions	Marks	Marker's Comments
* There was an awful mixture of notations in (b) and especially part (iv). Vectors and complex numbers should not be mixed e.g $\overrightarrow{OS} = z_1 + \overrightarrow{PS}$ (NO) Others used S when they meant $\overrightarrow{OS}$. And all of these things were mixed up at random. Generally I marked what they would have intended but it went a bit against the grain.		
c) i) $(\sqrt{a} - \sqrt{b})^2 \geq 0$		
$a + b - 2\sqrt{a}\sqrt{b} \geq 0$		
$\therefore (a+b)/2 \geq \sqrt{ab}$ (since $a>0, b>0$)	1	
ii) So $(x+y)/2 \geq \sqrt{x}\sqrt{y}$ $(x, y > 0)$		
Substitute $x = a$ $y = b+c$ $(a, b, c > 0)$		
$\therefore (a+(b+c))/2 \geq \sqrt{a}\sqrt{(b+c)}$	1	
or $\sqrt{a}\sqrt{b+c} \leq \dfrac{a+b+c}{2}$		
iii) $\sqrt{a}\sqrt{b+c} \leq \dfrac{a+b+c}{2}$		Only a handful of students took the reciprocal. Most got bogged down in algebra. The first mark was for taking a useful reciprocal.
Take reciprocals, $\therefore \dfrac{1}{\sqrt{a}\sqrt{b+c}} \geq \dfrac{2}{a+b+c}$ (all >0)	1	
$\therefore \dfrac{\sqrt{a}}{\sqrt{b+c}} \geq \dfrac{2a}{a+b+c}$ (1)		
Similarly $\dfrac{\sqrt{b}}{\sqrt{c+a}} \geq \dfrac{2b}{a+b+c}$ (2)		
$\dfrac{\sqrt{c}}{\sqrt{a+b}} \geq \dfrac{2c}{a+b+c}$ (3)		

Suggested Solutions	Marks	Marker's Comments
c) iii) (Continued)		
Add lines ①, ② & ③		
$\frac{\sqrt{a}}{\sqrt{b+c}}+\frac{\sqrt{b}}{\sqrt{c+a}}+\frac{\sqrt{c}}{\sqrt{a+b}}\geq\frac{2a+2b+2c}{(a+b+c)}$		
$\frac{\sqrt{a}}{\sqrt{b+c}}+\frac{\sqrt{b}}{\sqrt{c+a}}+\frac{\sqrt{c}}{\sqrt{a+b}}\geq 2.$	1	Getting this far got the second mark. (Could have done with a 3rd mark)
But equality holds in original statement only when $a=b+c$. The other parts require $b=c+a$, $c=a+b$		
These are only simultaneously possible when $a=0, b=0, c=0$ which is not the case since $a,b,c>0$. Therefore the final sign is strictly $>$.		Two people managed to explain the $>$, others didn't try. Unfortunately there was no mark left.
$\frac{\sqrt{a}}{\sqrt{b+c}}+\frac{\sqrt{b}}{\sqrt{c+a}}+\frac{\sqrt{c}}{\sqrt{a+b}}>2$		

Q 9a JRAHS T1 Ext 2 1/4

a i) Let $c = \cos\theta$ $s = \sin\theta$

1 both lines

$(\text{cis}\,\theta)^5 = c^5 + 5c^4 is + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$

$= (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5)$

$(\text{cis}\,\theta)^5 = \cos 5\theta + i\sin 5\theta$ (De Moivre's Th)

1

Equating real coefficient.

$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$

$= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2$

$= c^5 - 10c^3 + 10c^5 + 5c(1-2c^2+c^4)$

$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$

$= 16c^5 - 20c^3 + 5c$

1 Answer $= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

ii) 1 $z^5 = -1 = \text{cis}(\pi + 2n\pi)$ $n \in J$

Let $z = \text{cis}\,\theta$

$z^5 = \text{cis}\,5\theta$ (De Moivre's Th)

$\therefore$ $5\theta = \pi + 2n\pi$

$\theta = \dfrac{\pi + 2n\pi}{5}$

$\therefore$ $z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$, $\cos -\frac{\pi}{5} + i\sin(-\frac{\pi}{5})$

1 Answers $\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$, $\cos -\frac{3\pi}{5} + i\sin(-\frac{3\pi}{5})$

$\cos\pi + i\sin\pi$ $(=-1)$

9b) Method 1 : using contrapositive 3/4

ie If either x or y is even, then $x^2(y^2-2y)$ is even

If x is even let $x = 2m$, $m \in \mathbb{Z}$

$x^2(y^2-2y) = (2m)^2(y^2-2y)$

$x^2(y^2-2y) = 4(2m^2)(y^2-2y)$

$x^2(y^2-2y) = 2k$ $\quad k \in \mathbb{Z}$

$\therefore$ even /

If y is even, let $y = 2n$ $\quad n \in \mathbb{Z}$

$x^2(y^2-2y) = x^2[4n^2-4n]$

$= 2 \times 2[x^2][n^2-n]$ /

$\therefore x^2(y^2-2y)$ is even

Alternatively students can prove

Case 1	x odd	y even	if only 2 of these 3 cases only 1 m
2	x even	y odd	
3	x, y both even		

Method 2 : Direct Proof

ie If $x^2(y^2-2y)$ is odd, then x and y are both odd

A product of 2 number is only odd only if both are odd

$\therefore x^2$ is odd, y^2-2y is odd / for x odd

But $x^2 \equiv x \times x$, $y^2-2y = y \times (y-2)$

$\therefore$ both x, y need to be odd

If y is odd $y-2$ is also odd / for y odd

$\therefore x, y$ are both odd

* No marks given to 'converse'

$\frac{4}{4}$

9c) To prove $P \iff Q$, need to prove $P \to Q$, and prove $Q \to P$

To prove $P \to Q$:

Let the number $N = 1000a + 100b + 10c + d$, $a, b, c, d \in \mathbb{Z}$

$= 8(125a) + 100b + 10c + d$ ☆

If N is divisible by 8, say $N = 8m$, $m \in \mathbb{Z}$

$8m = 8(125a) + 100b + 10c + d$

$8(m - 125a) = 100b + 10c + d$

since $m - 125a$ is an integer

$\therefore$ $100b + 10c + d$ is divisible by 8

To prove $Q \to P$:

If the last 3 digits is divisible by 8,

Let $100b + 10c + d = 8p$ $p \in \mathbb{Z}$

FROM ☆ $N = 8(125a) + 8p$

$N = 8(125a + p)$

$N = 8q$ $q \in \mathbb{Z}$

$\therefore$ N will be divisible by 8

Note ① must see $1000a = 8 \times 125a$ to get any mark

② Some students have misunderstood each of the last 3 digits had to be divisible by 8 0m.

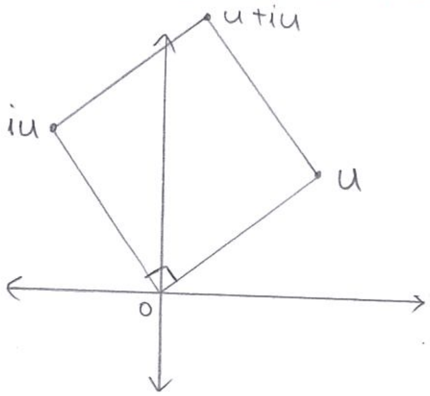
③ some forgot to define new variables, eg $m \in \mathbb{Z}$

1

MATHEMATICS Extension 2: Question 10

Suggested Solutions	Marks	Marker's Comments
a) i) $z^4+1=0$		
$z^4+2z^2+1-2z^2=0$		
$(z^2+1)^2-2z^2=0$		
$(z^2-\sqrt{2}z+1)(z^2+\sqrt{2}z+1)=0$	1	
ii) $z^4+1=(z^2-2\cos\frac{\pi}{4}z+1)(z^2-2\cos\frac{3\pi}{4}+1)$	1	changing $\sqrt{2}$ to correct values
$z^2+\frac{1}{z^2}=(z-2\cos\frac{\pi}{4}+\frac{1}{z})(z-2\cos\frac{3\pi}{4}+\frac{1}{z})$	1	dividing correctly
let $z=\cos\theta+i\sin\theta$		
$z^n=\cos n\theta+i\sin n\theta$ (De moivre's Theorem)		
$z^{-n}=\cos(-n\theta)+i\sin(-n\theta)$ (De moivre's Theorem)		
$=\cos n\theta-i\sin n\theta$		
$\therefore z^n+z^{-n}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$		
$=2\cos n\theta$		
$\therefore z^2+z^{-2}=2\cos 2\theta$	1	showing these are true.
$z+z^{-1}=2\cos\theta$		
$\therefore 2\cos 2\theta=(2\cos\theta-2\cos\frac{\pi}{4})(2\cos\theta-2\cos\frac{3\pi}{4})$		
$2\cos 2\theta=4(\cos\theta-\cos\frac{\pi}{4})(\cos\theta-\cos\frac{3\pi}{4})$	1	correct calculation.
$\therefore \cos 2\theta=2(\cos\theta-\cos\frac{\pi}{4})(\cos\theta-\cos\frac{3\pi}{4})$		

MATHEMATICS Extension 2: Question..10.....

Suggested Solutions	Marks	Marker's Comments
b) $z^4 + az^3 + (12+9i)z^2 + bz = 0$ $z(z^3 + az^2 + (12+9i)z + b) = 0$ $\therefore$ 0 is a root. $\therefore$ 0 is a vertice of the square.  $\therefore$ the other vertices are: u, iu and $u+iu$ for the equation $z^3 + az^2 + (12+9i)z + b = 0$	1	correct roots.
sum of roots two at a time: $u(iu) + u(u+iu) + iu(\cancel{u+}iu) = 12+9i$ $iu^2 + u^2 + iu^2 - u^2 = 12+9i$ $3iu^2 = 12+9i$ $u^2 = 3-4i$ $\therefore \quad u = \pm(2-i)$	1	correct 'u'
sum of roots: $u + iu + u + iu = -a$ $(2+2i)u = -a$ $\therefore a = -(2+2i)(2-i)$ $= -6-2i$ or $a = -(2+2i)(-2+i)$ $= 6+2i$ Product of roots: $u(iu)(u+iu) = -b$ $b = u^3(1-i)$		

MATHEMATICS Extension 2: Question 10

Suggested Solutions	Marks	Marker's Comments
$\therefore\ b=(2-i)^3(1-i)$		
$\quad = -9-13i$		
or $b=(-2+i)^3(1-i)$		
$\quad = 9+13i$		
$\therefore\ x = 2b-13a$		
$\quad = 2(-9-13i)-13(-6-2i)$		
$\quad = 60$		
or $x = 2(9+13i)-13(6+2i)$		
$\quad = -60$		
$\therefore\ x = \pm 60$	1	correct x value.
c) Assume		
$\dfrac{a+c}{\sqrt{a^2+c^2}} - \dfrac{b+c}{\sqrt{b^2+c^2}} \leq 0$	1	correct assumption
$\dfrac{a+c}{\sqrt{a^2+c^2}} \leq \dfrac{b+c}{\sqrt{b^2+c^2}}$		
$(a+c)(\sqrt{b^2+c^2}) \leq (b+c)(\sqrt{a^2+c^2})$		
$(a+c)^2(b^2+c^2) \leq (b+c)^2(a^2+c^2)$		
$(a^2+2ac+c^2)(b^2+c^2) \leq (b^2+2bc+c^2)(a^2+c^2)$		
$\cancel{a^2b^2}+2ab^2c+c^2b^2+\cancel{a^2c^2}+2ac^3+\cancel{c^4}$		
$\leq \cancel{a^2b^2}+2a^2bc+\cancel{a^2c^2}+b^2c^2+2bc^3+\cancel{c^4}$		
$\therefore\ 2ab^2c+2ac^3 \leq 2a^2bc+2bc^3$		
$ab^2+ac^2 \leq a^2b+bc^2$	1	working
$ab^2+ac^2-a^2b-bc^2 \leq 0$		

MATHEMATICS Extension 2: Question 10

Suggested Solutions	Marks	Marker's Comments
$ab(b-a) + c^2(a-b) \leq 0$		
$\therefore (a-b)(c^2 - ab) \leq 0$		
as $a > b$ $\therefore (a-b) > 0$		
and $c^2 > ab$ $\therefore (c^2 - ab) > 0$	1	correct explanation.
$\therefore (a-b)(c^2-ab) > 0$		
$\therefore$ This is a contradiction		
hence, $\dfrac{a+c}{\sqrt{a^2+c^2}} - \dfrac{b+c}{\sqrt{b^2+c^2}} > 0$	1	correct conclusion following correct explanation.
d) $A + B + C = \pi$ (interior angle sum of triangle)		
$A > 0,\ B > 0,\ C > 0$		
$\dfrac{A+B+C}{3} \geq \sqrt[3]{ABC}$ (given)		
$\therefore A + B + C \geq 3\sqrt[3]{ABC}$	1	
$\dfrac{\frac{1}{A} + \frac{1}{B} + \frac{1}{C}}{3} \geq \sqrt[3]{\frac{1}{A} \times \frac{1}{B} \times \frac{1}{C}}$ (given)		
$\therefore \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \geq \dfrac{3}{\sqrt[3]{ABC}}$	1	
$\therefore (A+B+C)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \geq 3\sqrt[3]{ABC} \times \dfrac{3}{\sqrt[3]{ABC}}$	1	
$\pi\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \geq 9$ as $A+B+C=\pi$		
$\therefore \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \geq \dfrac{9}{\pi}$		