BAULKHAM HILLS HIGH SCHOOL

2021

HSC Assessment Task 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using non-erasable black or blue pen.
- Calculators approved by NESA may be used.
- Marks may be deducted for careless or badly arranged work.
- Relevant formulae from the NESA Reference Sheet are provided on page 2.
- Answer all questions on the appropriate page of your answer booklet.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.
- If extra paper is used, it must be clearly labelled with your name and the question being answered, and stapled inside your answer booklet.

Total marks – 43
Exam consists of 5 pages.

Section I: Pages 2-4
5 marks
Attempt Questions 1-5

Section II: Pages 5-7
38 marks
Attempt Questions 6-9

Formulae from NESA Reference Sheet:

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^n e^{in\theta}$$

Section I: **Marks**

5 marks

Attempt Questions 1-5

Use the multiple choice answer sheet in your booklet.

1. If $z = \sqrt{3} + i$, what is the value of $\overline{\left(\frac{i}{z}\right)}$? **1**

(A) $\frac{1+i\sqrt{3}}{4}$

(B) $\frac{-\sqrt{3}+i}{4}$

(C) $\frac{1-i\sqrt{3}}{4}$

(D) $\frac{\sqrt{3}-i}{4}$

2. ω is a non-real root of the equation $z^5 + 1 = 0$. **1**

Which of the following is not a root of this equation?

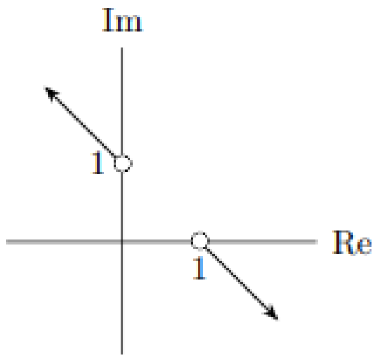
(A) $\overline{\omega}$

(B) ω^2

(C) $\frac{1}{\omega}$

(D) ω^3

3. The locus of a complex number z is shown on the Argand diagram below. **1**



Which of the following is the equation of this locus?

(A) $\arg(z - i) - \arg(z - 1) = 0$

(B) $\arg(z - i) - \arg(z - 1) = \pi$

(C) $\arg(z + i) - \arg(z + 1) = 0$

(D) $\arg(z + i) - \arg(z + 1) = \pi$

4. What is the smallest positive value of θ such that $e^{i\theta} \times e^{2i\theta} = i$? **1**

(A) $\dfrac{\pi}{12}$

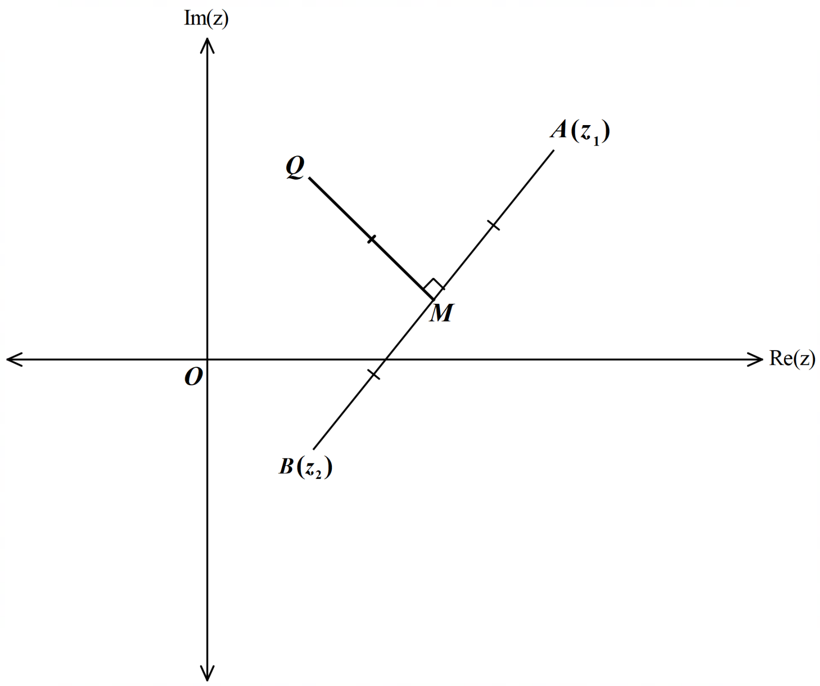
(B) $\dfrac{\pi}{6}$

(C) $\dfrac{\pi}{3}$

(D) $\dfrac{5\pi}{6}$

5. The complex numbers z_1 and z_2 are represented by points A and B respectively. **1**

QM is perpendicular to AB, and $QM = AM = BM$.



What is the complex number that corresponds to Q?

(A) $i\left(\dfrac{z_1 - z_2}{2}\right)$

(B) $i\left(\dfrac{z_1 + z_2}{2}\right)$

(C) $\left(\dfrac{z_1 + z_2}{2}\right) + i\left(\dfrac{z_1 + z_2}{2}\right)$

(D) $\left(\dfrac{z_1 + z_2}{2}\right) + i\left(\dfrac{z_1 - z_2}{2}\right)$

End of Section I

Section II: 38 marks

Attempt Questions 6-9

Question 6 (11 marks)

a) Simplify

(i) $|2+3i|$ **1**

(ii) $A\text{rg}\left(-1-i\sqrt{3}\right)$ **2**

(iii) $\dfrac{4+3i}{1-2i}$ **2**

b) Find the square roots of $9e^{i\frac{\pi}{5}}$ expressing your answers in simplest exponential form. **2**

c) Given $|z-2-2i|=1$:

(i) Draw the locus of z on the Argand diagram. **2**

(ii) Find the maximum value of $|z|$ **1**

(iii) Find the maximum value of $Arg(z)$, giving your answer in radians correct to 3 decimal places. **1**

Question 7 (9 marks)

a) (i) Find the two square roots of $15 - 8i$, giving the answers in the form $x + iy$, where x and y are real numbers. **2**

(ii) Hence, or otherwise, solve the equation $z^2 - (2 + 3i)z - 5 + 5i = 0$ giving your solutions in the form $a + ib$ where a and b are real numbers. **3**

b) The polynomial $P(z) = 3z^4 - z^3 + 3z^2 + 29z - 10$ has a root $z_1 = 1 - 2i$.

(i) Explain why $1 + 2i$ is also a root. **1**

(ii) Hence express $P(z)$ as a product of real linear and real quadratic factors. **3**

Question 8 (10 marks)

a) z is a complex number which satisfies $\arg(z + 2) - \arg(z - 2) = \frac{\pi}{4}$.

(i) Sketch the locus of z. **2**

(ii) Find the Cartesian equation of this locus, noting any restrictions. **2**

b) (i) By using De Moivre's theorem and the binomial expansion of $(\cos\theta + i\sin\theta)^5$, show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **3**

(ii) Hence solve the equation $\cos 5\theta = \cos\theta$ for $0 \le \theta \le 2\pi$. **3**

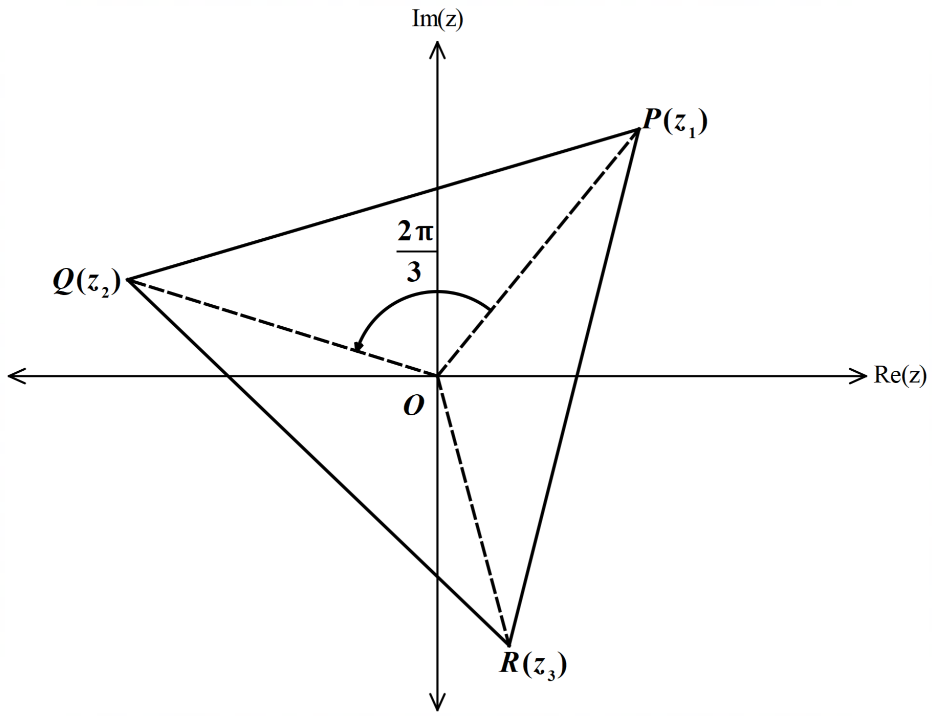
Question 9 (8 marks)

a) Given $1 - i$ is a root of $P(z) = z^3 - (1 + 2i)z^2 + (1 + 3i)z + 2 - 2i$, find the other roots. **3**

b) The points P, Q and R on the Argand diagram represent the complex numbers z_1, z_2 and z_3

so that PQR is an equilateral triangle whose centre is the origin (ie. $OP = OQ = OR$) and

$\angle POQ = \dfrac{2\pi}{3}$.



Let $\omega = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$.

(i) Show that $1 + \omega + \omega^2 = 0$. **1**

(ii) Find an expression for z_2 in terms of z_1 and ω. **1**

(iii) The complex number z_4 is represented by point S such that $PQRS$ is a parallelogram. **3**

Using vector methods, express z_4 in terms of z_1 and ω giving your answer in its **simplest** form.

END OF EXAMINATION

X2 - Dec. Task 2021.

Section I.

1. $\overline{\left(\frac{i}{z}\right)} = \frac{-i}{\sqrt{3}-i} \cdot \frac{\sqrt{3}+i}{\sqrt{3}+i}$

$= \frac{1-i\sqrt{3}}{4}$ (C)

2. $\omega^5 = -1$ since ω is a root.

If $z = \omega^2$, $z^5 + 1 = (\omega^2)^5 + 1$

$= (\omega^5)^2 + 1$

$= (-1)^2 + 1$

$\neq 0$ (B)

3. (A)

4. $e^{i\theta} \times e^{2i\theta} = i$

$e^{3i\theta} = e^{i\pi/2}$

$3\theta = \frac{\pi}{2} + 2k\pi \quad (k \in \mathbb{Z})$

$\theta = \frac{\pi}{6}$. (B).

5. Q and $\vec{OQ}$ represent the same complex number.

$\vec{OQ} = \vec{OM} + \vec{MQ}$

$= \frac{z_1 + z_2}{2} + i\left(\frac{z_1 - z_2}{2}\right)$ (D)

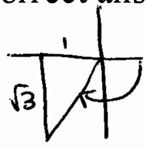
SECTION II

Question 6.

a) (i) $\sqrt{2^2+3^2} = \sqrt{13}$ — (1) correct answer

(ii) $Arg(-i+i\sqrt{3}) = Arg(-1-i\sqrt{3})$

$= -\frac{2\pi}{3}$



(2) correct soln
(1) obtains pi/3

(iii) $\frac{4+3i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{4-6+11i}{5}$

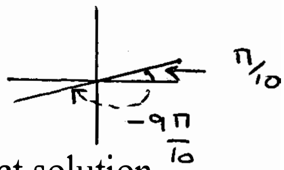
$= -\frac{2}{5} + \frac{11}{5}i$

$\therefore Re\left(\frac{4+3i}{1-2i}\right) = -\frac{2}{5}$

(2) correct answer
(1) multiply by conjugate

b) $\sqrt{9e^{i\pi/5}} = \pm 3e^{i\pi/10}$

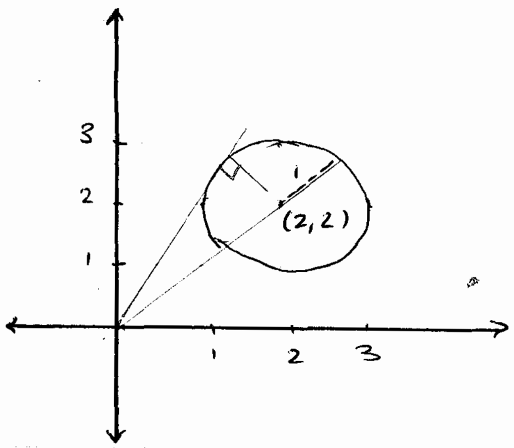
$= 3e^{i\pi/10}, \quad 3e^{i(-9\pi/10)}$



(2) correct solution
(1) one correct root

c) $|z-2-2i| = 1$

(i)



← Locus is a circle centre (2,2), radius 1 unit.

(2) correct locus
(1) circle with correct centre or radius

(ii) $Max\ |z| = \sqrt{8} + 1$

$= 2\sqrt{2} + 1$

(1) correct answer

(iii) $Max\ arg\ z = \frac{\pi}{4} + \sin^{-1}\left(\frac{1}{\sqrt{8}}\right)$

$= 1.147$

(1) correct answer

Question 7.

a) i) Let $z = x + iy$: $(x+iy)^2 = 15 - 8i$ $\quad (x, y \in R)$

$$x^2 - y^2 + 2ixy = 15 - 8i$$

$$x^2 - y^2 = 15 \quad \text{— ①}$$

$$2xy = -8$$

$$xy = -4$$

$$y = \frac{-4}{x} \quad \text{— ②}$$

$$x^2 - \frac{16}{x^2} = 15$$

$$x^4 - 15x^2 - 16 = 0$$

$$(x^2 - 16)(x^2 + 1) = 0$$

$x^2 = 16 \qquad \cancel{x^2 = -1}$ $(x \in R)$

$\begin{cases} x = \pm 4 \\ y = \mp 1 \end{cases}$

$\therefore z = 4 - i, \; -4 + i$

ie $\underline{z = \pm(4 - i)}$

(2) correct soln
(1) obtains simultaneous eqns and attempts to solve

(ii) $z^2 - (2+3i)z - (5-5i) = 0$

$$\Delta = [-(2+3i)]^2 - 4 \cdot 1 \cdot -(5-5i)$$

$$= 4 - 9 + 12i + 20 - 20i$$

$$= 15 - 8i$$

$$\therefore z = \frac{2+3i \pm (4-i)}{2}$$

$$= \frac{6+2i}{2}, \; \frac{-2+4i}{2}$$

$$= \underline{3+i, \; -1+2i}$$

(3) correct soln
(2) attempts to link result from (i) to quadratic formula
(1) finds discriminant

Question 7 (cont)

b) (i) Since $1-2i$ is a root, and all the coefficients are real, its conjugate must also be a root ($1+2i$)

(1) correct explanation

(ii) There is a quadratic factor : ($S=2$, $P=5$)

z^2-2z+5

$\therefore 3z^4-z^3+3z^2+29z-10$

$= (z^2-2z+5)(3z^2+5z-2)$

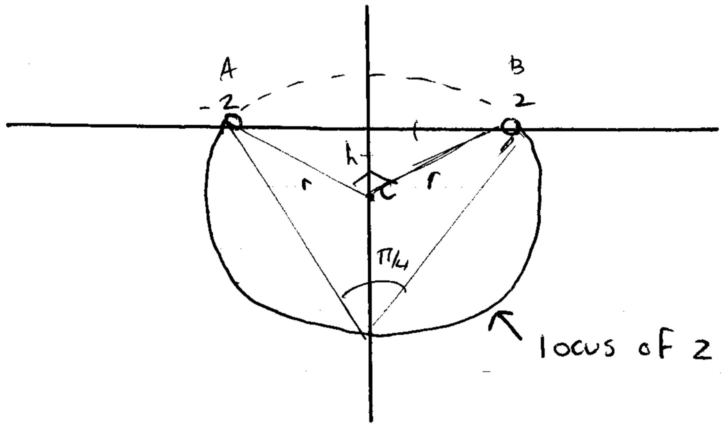
$= \underline{(z^2-2z+5)(3z-1)(z+2)}$

15
-2

(3) correct soln
(2) finds a second quadratic factor
(1) finds one quadratic factor

Question 8

a) (i)



(2) correct arc, endpoints excluded
(1) a major arc with correct endpoints

(ii) $2r^2 = 4^2$
$r^2 = 8$
$r = 2\sqrt{2}$
$h = 2$
$\therefore$ Centre $(0, -2)$

$\therefore$ ~~Cent~~

$\therefore$ Eqn: $x^2 + (y+2)^2 = 8$ with $y < 0$.

(2) correct eqn and restriction
(1) correct eqn

b) (i) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ (by De Moivre's theorem)

$(\cos\theta + i\sin\theta)^5 = c^5 + 5c^4 is + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$

$= (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5)$

where $c = \cos\theta$, $s = \sin\theta$.

Equating the real parts:

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(3) correct soln
(2) 2 correct expansions and attempt to equate real parts
(1) 2 correct expansions

Question 8 (cont).

b) (ii) $\cos 5\theta - \cos\theta = 0 \quad [0 \le \theta \le 2\pi]$

$16\cos^5\theta - 20\cos^3\theta + 4\cos\theta = 0$

$4\cos\theta\,(4\cos^4\theta - 5\cos^2\theta + 1) = 0$

$4\cos\theta\,(4\cos^2\theta - 1)(\cos^2\theta - 1) = 0$

$\cos\theta = 0 \qquad \cos\theta = \pm\frac{1}{2} \qquad \cos\theta = \pm 1$

$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, 0, \pi, 2\pi$

(3) All correct solutions
(2) Solves 2 of the three cases correctly
(1) Factorises LHS of the eqn

Question 9.

a) Let roots be α, β, $1-i$

$$\alpha+\beta+\cancel{1}-i = \cancel{1}+2i$$

$$\alpha+\beta = 3i \quad \text{——} \ (1)$$

$$\alpha\beta(1-i) = -2+2i$$

$$\alpha\beta = \frac{-2+2i}{1-i} = \frac{-2(1-i)}{1-i} = -2 \quad \text{——} \ (2)$$

$$\alpha(3i-\alpha) = -2$$

$$3i\alpha - \alpha^2 = -2$$

$$\alpha^2 - 3i\alpha - 2 = 0$$

$$(\alpha-2i)(\alpha-i) = 0$$

(3) correct soln.
(2) Solves simultaneous eqns arising from sum and product
(1) Forms eqns using sum and product of roots.

$\therefore$ The other two roots are $\underline{2i, i}$.

(note: if $\alpha=2i$, $\beta=i$ and if $\alpha=i$, $\beta=2i$).

b) $$1+\omega+\omega^2 = 1+\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}+\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

(by De Moivre's)

$$= 1+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)+\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$

$$= 1-\frac{1}{2}+\frac{i\sqrt{3}}{2}-\frac{1}{2}-\frac{i\sqrt{3}}{2}$$

$$= 0.$$

OR

$$1+\omega+\omega^2 = 1+\operatorname{cis}\frac{2\pi}{3}+\operatorname{cis}\frac{-2\pi}{3}$$

$$= 1+2\cos\frac{2\pi}{3}$$

$$= 1+2\left(-\frac{1}{2}\right)$$

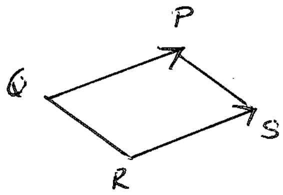
$$= 0.$$

(1) correctly shown

(ii) $z_2 = z_1\omega$

(1) correct answer

(iii) $\vec{QP} = \vec{RS}$ since



opposite sides of parallelogram are parallel & equal.

$$z_1 - z_2 = z_4 - z_3$$

$$z_4 = z_1 - z_2 + z_3$$

$$= z_1 - z_1\omega + z_1\omega^2$$

$$= z_1(1 - \omega + \omega^2)$$

From (i), $1 + \omega + \omega^2 = 0$

$$\omega^2 = -1 - \omega$$

$$\therefore \quad z_4 = z_1(1 - \omega - 1 - \omega) = \underline{-2\omega z_1}$$

(3) Correct soln
(2) Uses vectors to obtain z4 in terms of z1 and w with reason
(1) Uses vectors to obtain a relationship between z1, z2, z3, z4