BAULKHAM HILLS HIGH SCHOOL

2022

HSC Assessment Task 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using non-erasable black or blue pen.
- Calculators approved by NESA may be used.
- Marks may be deducted for careless or badly arranged work.
- Relevant formulae from the NESA Reference Sheet are provided on page 2.
- Answer all questions on the appropriate page of your answer booklet.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.
- If extra paper is used, it must be clearly labelled with your NESA number and the question being answered, and stapled inside your answer booklet.

Total marks – 47
Exam consists of 7 pages.

Section I: Pages 2 - 4
5 marks
Attempt Questions 1 – 5
Allow about 8 minutes for this section

Section II: Pages 5 - 7
42 marks
Attempt Questions 6 – 8
Allow about 52 minutes for this section

Formulae from NESA Reference Sheet:

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^n e^{in\theta}$$

Section I: **Marks**

5 marks

Attempt Questions 1 - 5

Use the multiple choice answer sheet in your booklet.

1. If $w = 5 - 2i$, which of the following is $\dfrac{1}{2-w}$? **1**

(A) $\dfrac{3}{13} - \dfrac{2}{13}i$

(B) $-\dfrac{3}{13} + \dfrac{2}{13}i$

(C) $-\dfrac{3}{13} - \dfrac{2}{13}i$

(D) $\dfrac{3}{13} + \dfrac{2}{13}i$

2. For the complex numbers z and w, where $\arg(z) = \dfrac{\pi}{2}$ and $\arg(w) = \dfrac{\pi}{4}$, **1**

the value of $\arg\left(\dfrac{z^5}{w^4}\right)$ is:

(A) $-\dfrac{\pi}{2}$

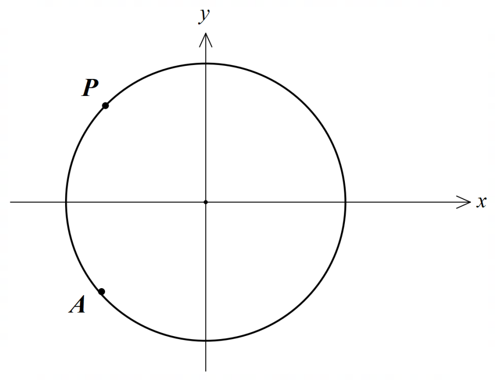
(B) $\dfrac{\pi}{2}$

(C) π

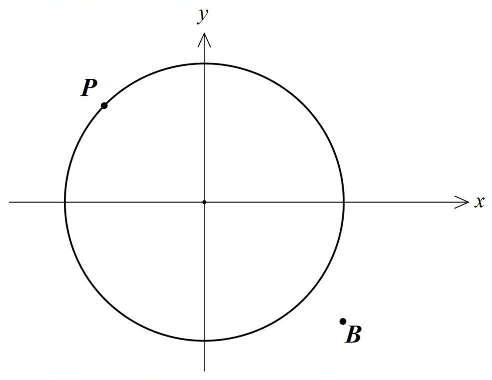
(D) $\dfrac{\pi}{4}$

3. If $\overrightarrow{OP} = z$, $\frac{\pi}{2} < \arg z < \pi$ and $|z| < 1$ then which of the following is a possible location for the complex number $\sqrt{z}$? **1**

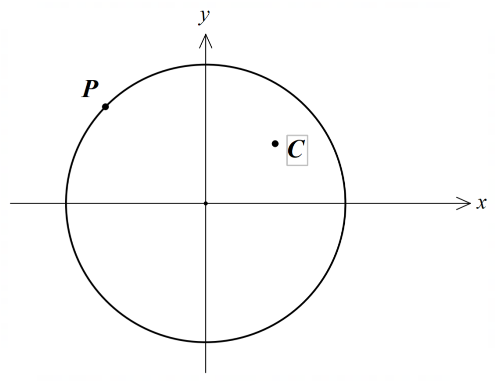
(A) $\overrightarrow{OA}$



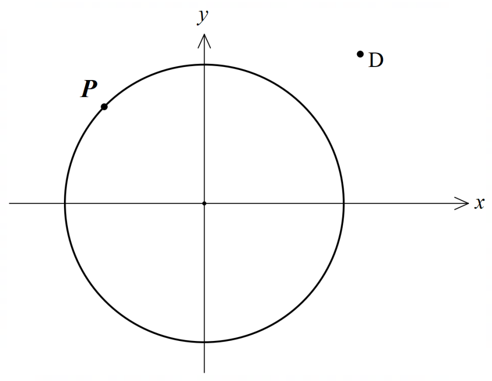
(B) $\overrightarrow{OB}$



(C) $\overrightarrow{OC}$



(D) $\overrightarrow{OD}$



4. The polynomial equation $P(z) = 0$ has real coefficients, and has roots which include $z = -2 + i$ and $z = 2$. The minimum degree of $P(z)$ would be: **1**

(A) 1

(B) 2

(C) 3

(D) 4

5. The points corresponding to the four complex numbers given by **1**

$$z_1 = 2\text{cis}\left(\frac{\pi}{3}\right), \quad z_2 = \text{cis}\left(\frac{3\pi}{4}\right), \quad z_3 = 2\text{cis}\left(\frac{-2\pi}{3}\right), \quad z_4 = \text{cis}\left(\frac{-\pi}{4}\right)$$

are the vertices of a parallelogram in the complex plane. Which one of the following statements is **not** true?

(A) The acute angle between the diagonals of the parallelogram is $\frac{5\pi}{12}$

(B) The diagonals of the parallelogram have lengths 2 and 4

(C) $z_1 z_2 z_3 z_4 = 0$

(D) $z_1 + z_2 + z_3 + z_4 = 0$

End of Section I

Section II: 42 marks

Attempt Questions 6 - 8

Answer each question on the appropriate page of the answer booklet, showing all necessary working

Question 6 (14 marks)

a) Let $z = a + ib$, where a and b are real, find:

(i) $\text{Im}(4i - z)$ 1

(ii) $\overline{(3iz)}$ in the form of $x + iy$, where x and y are real numbers 1

b) If $wz = 3 - i$, where $w = 1 - i$ find:

(i) z in the form of $x + iy$, where x and y are real numbers 2

(ii) $\arg w$ 1

c) (i) Express $3 - \sqrt{3}i$ in polar form. 2

(ii) Express $\left(3 - \sqrt{3}i\right)^3$ in the form of $x + iy$, where x and y are real numbers. 1

(iii) Find the integer values of n for which $\left(3 - \sqrt{3}i\right)^n$ is real. 2

d) Sketch the locus of z on separate complex planes:

(i) $|z| = \left|z + 1 - \sqrt{3}i\right|$ 2

(ii) $\dfrac{-2\pi}{3} < \arg(z + 1) \leq 0$ 2

Question 7 (14 marks)

a) Given that $(2+3i)$ is one of the roots of the polynomial $P(x)=4x^3-13x^2+40x+39$, solve $4x^3-13x^2+40x+39=0$ over the complex field. 3

b) (i) Find the square roots of $-8-6i$, giving the answers in the form of $x+iy$, where x and y are real numbers. 2

(ii) Hence, or otherwise, solve the equation $2z^2-8z+12+3i=0$, giving your solutions in the form $a+ib$ where a and b are real numbers. 2

c) (i) If ω is a non-real cube root of unity, show that $1+\omega+\omega^2=0$. 1

(ii) Hence prove that $\dfrac{1}{1+\omega}-\dfrac{1}{1+\omega^2}=-(1+2\omega)$. 2

d) (i) Express $z=-1-\sqrt{3}i$ in the form of $re^{i\theta}$. 2

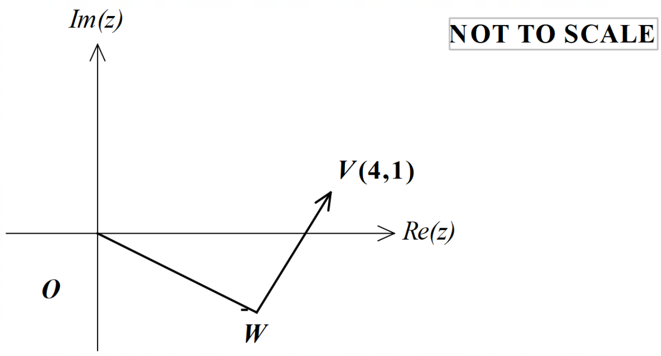
(ii) Hence, find the real values of a and b such that $-1-\sqrt{3}i=e^{a+ib}$. 2

Question 8 (14 marks)

a) (i) Sketch on an Argand diagram, the locus of the point P which satisfies the equation $|z-3|=3$. **2**

(ii) Use the diagram to show that P satisfies the condition $\arg(z-3)=\arg z^2$. **2**

b) The point V represents the complex number $4+i$, $\angle OWV=\dfrac{\pi}{2}$ and $|VW|=|OW|$. Find the complex number represented by the point W in the form of $a+ib$ where a and b are real numbers. **2**



c) (i) By using De Moivre's theorem and the binomial expansion of $(\cos\theta+i\sin\theta)^4$, show that $\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1$. **3**

(ii) Hence solve the equation $8x^4-8x^2+1=0$. **2**

(iii) Hence find the exact value of $\cos\dfrac{\pi}{8}$. **3**

END OF EXAMINATION

Extension-2 Task-1, 2022 Solutions.

Section-1

Multiple Choice:

1) $\frac{1}{2-\omega} = \frac{1}{2-5+2i} = \frac{1}{-3+2i} \times \frac{-3-2i}{-3-2i} = \frac{-3-2i}{(-3)^2-(2i)^2} = \frac{-3}{13} - \frac{2}{13}i$ $\boxed{C}$

2) $\arg\left(\frac{z^5}{w^4}\right) = \arg z^5 - \arg w^4$

$= 5\arg z - 4\arg w$

$= 5\times\frac{\pi}{2} - 4\times\frac{\pi}{4} = \frac{3\pi}{2} = -\frac{\pi}{2}$ $\boxed{A}$

3) Since $|z| < 1$, therefore $|\sqrt{z}| > 1$

so $\sqrt{z}$ lies outside the circle.

Also if $\frac{\pi}{2} < \arg z < \pi$ then $0 < \arg\sqrt{z} < \frac{\pi}{2}$ $\boxed{D}$

as $\arg(\sqrt{z}) = \frac{1}{2}\arg z$

4) Since $P(z)$ has real coeff. $\therefore$ $-2+i$, $-2-i$ and z are roots.

Therefore minimum degree of $P(z)$ is 3. $\boxed{C}$

5) $|z_1| = 2$, $\arg z_1 = \frac{\pi}{3}$ $|z_3| = 2$, $\arg z_3 = -\frac{2\pi}{3}$

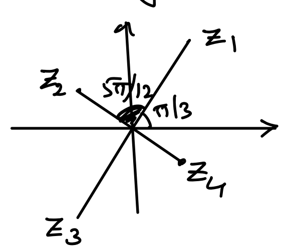
$|z_2| = 1$, $\arg z_2 = \frac{3\pi}{4}$ $|z_4| = 1$, $\arg z_4 = -\frac{\pi}{4}$

A Correct

B Correct

C Incorrect as $z_1 z_2 z_3 z_4 \neq 0$

D Correct



$\boxed{C}$

Section II

Question 6

a) i) $z = a + ib$ (1 mark Correct answer)

$4i - z = 4i - a - ib$

$= -a + (4-b)i$

$\text{Im}(4i - z) = 4 - b$

ii) $\overline{3iz} = \overline{3i(a+ib)}$ (1 mark Correct answer.)

$= \overline{i3a - 3b}$

$= -3b - 3ia$

b) i) $(1-i)z = 3 - i$

$z = \dfrac{3-i}{1-i} \times \dfrac{1+i}{1+i}$

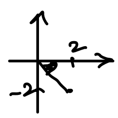
$= \dfrac{4+2i}{2}$

$= 2 + i$

2 marks correct answer.

1 mark to make z as subject and multiply & divide by $(1-i)$

ii) $\arg z = -\dfrac{\pi}{4}$

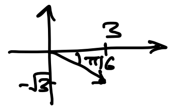


(1 mark correct answer)

c) i) Let $z = 3 - \sqrt{3}i$

$|z| = \sqrt{(3)^2 + (-\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

$\arg z = -\dfrac{\pi}{6}$



$\therefore z = 2\sqrt{3}\,\text{Cis}\left(-\dfrac{\pi}{6}\right)$

2 marks correct solution

1 mark either correct

$|z|$ or $\arg z$

ii) $(3 - \sqrt{3}i)^3 = \left[2\sqrt{3}\,\text{Cis}\left(-\dfrac{\pi}{6}\right)\right]^3$

$= 24\sqrt{3}\,\text{Cis}\left(-\dfrac{\pi}{2}\right)$

$= -24\sqrt{3}i$

1 mark correct solution.

Question 6 (continues)

c) iii) $(3-\sqrt{3}i)^n = (2\sqrt{3})^n \operatorname{Cis}(-\frac{n\pi}{6})$

$$= (2\sqrt{3})^n \left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$$

2 marks correct solution

1 mark attempt to equate imaginary part equal to zero.

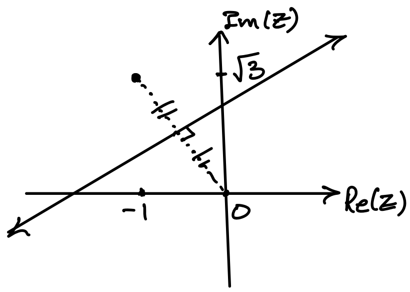
For real; $\sin\frac{n\pi}{6} = 0$

$\frac{n\pi}{6} = k\pi$, where k is integer.

$\therefore n = 6k$

$\therefore$ n is any integral multiple of 6.

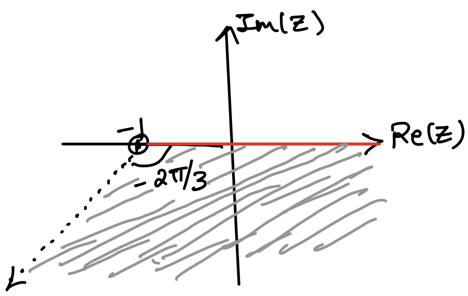
d) i)



2 marks correct locus with all important features.

1 mark to recognise it is a perp. bisector of two incorrect points or finding the algebraic equation for the perp. bisector line or either correct bisector/perp. of two given points.

ii)



2 marks correct locus with all important features

1 mark for correct boundaries or shade the correct region

Question-7

a) Since $2+3i$ is a root and all the coefficients of poly. $P(x)$ are real, therefore its conjugate $(2-3i)$ must also be a root.

Let α be the third root of $P(x) = 4x^3 - 13x^2 + 40x + 39$

$$\text{Sum of roots} = -\frac{b}{a} = \frac{13}{4}$$

$$2+\cancel{3i} + 2 - \cancel{3i} + \alpha = \frac{13}{4}$$

$$\alpha = \frac{13}{4} - 4$$

$$\alpha = -\frac{3}{4}$$

Hence three roots are; $2+3i, 2-3i, -\frac{3}{4}$

3 marks correct solution
2 marks use correct method to attempt to solve poly $P(x)=0$
1 mark to identify $2-3i$ as another root.

b) i) Let $z = x+iy$, where $x, y \in \mathbb{R}$

$$\therefore (x+iy)^2 = -8-6i$$

$$x^2 - y^2 + 2ixy = -8-6i$$

Equate real and imaginary coefficients:

$$x^2 - y^2 = -8 \quad \text{— (1)} \quad \& \quad 2xy = -6$$

$$xy = -3 \quad \text{— (2)}$$

Solve (1) & (2) simultaneously:

$$x^2 - \left(-\frac{3}{x}\right)^2 = -8$$

$$x^2 - \frac{9}{x^2} = -8$$

$$x^4 + 8x^2 - 9 = 0$$

$$(x^2+9)(x^2-1) = 0$$

$$x^2 - 1 = 0 \qquad (\because x^2 \neq -9 \text{ as } x \in \mathbb{R})$$

$$x = \pm 1$$

$$y = \mp 3$$

$$\therefore z = \pm(1-3i)$$

2 marks correct solution
1 mark to obtain two simultaneous equation and attempt to solve or any other equivalent method.

Question 7 (continues)

b) ii) $2z^2 - 8z + 12 + 3i = 0$

$$\Delta = b^2 - 4ac$$
$$= (-8)^2 - 4(2)(12+3i)$$
$$= 64 - 96 - 24i$$
$$= -32 - 24i$$
$$= 4(-8-6i)$$

$$\therefore z = \frac{8 \pm 2(1-3i)}{4}$$
$$= \frac{8+2-6i}{4}, \quad \frac{8-2+6i}{4}$$
$$= \frac{5}{2} - \frac{3i}{2}, \quad \frac{3}{2} + \frac{3}{2}i$$

2 marks correct solution
1 mark to find discriminant or attempt to link results from part i) using quad. formula.

c) i) Let z is cube root of 1

then $z^3 = 1$

$$z^3 - 1 = 0$$
$$(z-1)(z^2+z+1) = 0$$

Since ω is non real root

$$\therefore \omega^2 + \omega + 1 = 0 \quad (\because \omega \neq 1)$$

(1 mark correct solution)

ii) $$\frac{1}{1+\omega} - \frac{1}{1+\omega^2} = \frac{1}{-\omega^2} - \frac{1}{-\omega}$$

($\because$ $1+\omega = -\omega^2$ and $1+\omega^2 = -\omega$)

$$= -\frac{1}{\omega^2} + \frac{1}{\omega}$$
$$= -\frac{1+\omega}{\omega^2} \times \frac{\omega}{\omega}$$
$$= -\omega + \omega^2$$
$$= -\omega - \omega - 1$$
$$= -2\omega - 1$$

Hence proved.

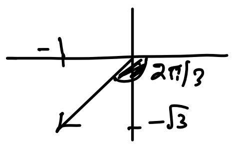
2 marks correct proof
1 mark attempt to prove by using part i)

Question 7 (Continues)

d) i) $z = -1 - \sqrt{3}i$

$|z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$

$\arg z = -\frac{2\pi}{3}$



$\therefore z = 2e^{-i2\pi/3}$

{ 2 marks correct soluti
1 mark either correct
$|z|$ or $\arg z$

ii) $-1 - \sqrt{3}i = e^{a+ib}$

$2e^{-i2\pi/3} = e^{a} \cdot e^{ib}$

Equating both sides;

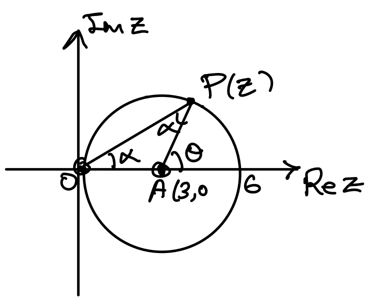
$2 = e^{a}$ and $b = -\frac{2\pi}{3}$

$a = \ln 2$, $b = -\frac{2\pi}{3}$

{ 2 marks correct answer
1 mark either correct
a or b

Question - 8

a) i)



{ 2 marks correct locus
1 mark either correct centre
or radius of circle

ii) $\arg(z-3) = \theta$

$\arg z = \alpha$

{ 2 marks correct proof
1 mark attempt to prove by using
geometry

$\angle OPA = \alpha$ ($OA = AP$, opp. angles to equal sides of isosceles Δ)

$\therefore \theta = 2\alpha$ (Ext angle of ΔOPA is equal to sum of int. opp. $\angle$'s)

$\therefore \arg(z-3) = 2\arg z$ or $\arg(z-3) = \arg z^2$

Question 8 (continues)

b) Let $W = x + iy$

$\overrightarrow{OW} = -i\overrightarrow{WV}$

$x + iy = -i(4 + i - x - iy)$

$= -4i + 1 + ix - y$

$= (1 - y) + i(-4 + x)$

Equating real & imaginary coefficients:

$x = 1 - y$ — ① $\quad y = -4 + x$ — ②

$y = -4 + 1 - y$

$y = -\frac{3}{2}, \quad x = \frac{5}{2}$

$\therefore W = \frac{5}{2} - i\frac{3}{2}$

2 marks correct answer
1 mark attemp to relate $\overrightarrow{OW}$ and $\overrightarrow{WV}$ and solve the equation.

c) i) By De Moivre theorem:

$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

By Binomial expansion;

$(\cos\theta + i\sin\theta)^4 = c^4 + 4c^3(is) + 6c^2(i^2s^2) + 4c(is)^3 + (is)^4$

where $c = \cos\theta$, $s = \sin\theta$

$= (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3)$

Equating real parts;

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

3 marks correct solution
2 marks for correct expansions and equates real part
1 mark for both correct expansions

Question-8 (Continues)

c) ii) To solve $8x^4 - 8x^2 + 1 = 0$

Let $x = \cos\theta$; $\therefore \cos 4\theta = 0$

$$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$

$$\therefore x = \cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}$$

2 marks correct solution
1 mark attempt to solve the equation by letting $x = \cos\theta$ and attempting to solve $\cos 4\theta = 0$

iii) $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$ are solutions of $8\cos^4\theta - 8\cos^2\theta + 1 = 0$

Also, by quad. formula

$$\cos^2\theta = \frac{8 \pm \sqrt{(-8)^2 - 4\times 8\times 1}}{16}$$

$$= \frac{8 \pm \sqrt{32}}{16}$$

$$= \frac{2 \pm \sqrt{2}}{4}$$

$$\cos\theta = \pm\frac{\sqrt{2 \pm \sqrt{2}}}{2}$$

Since $\frac{\pi}{8}$ is in Quadrant I $\therefore \cos\frac{\pi}{8}$ will be positive and either $\frac{\sqrt{2+\sqrt{2}}}{2}$ or $\frac{\sqrt{2-\sqrt{2}}}{2}$

As $\cos\frac{\pi}{8} > \cos\frac{3\pi}{8}$ Hence $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$

3 marks correct solution
2 marks uses the fact $\cos\theta > 0$ OR $\cos\frac{\pi}{8} > \cos\frac{3\pi}{8}$ to identify exact value OR provides exact value with no justification
1 mark uses quad. formula to find the exact value of $\cos^2\theta$.

END OF SOLUTIONS