STUDENT'S NAME: ______________________________

TEACHER'S NAME: ______________________________

2022

HURLSTONE
AGRICULTURAL
HIGH SCHOOL

Mathematics Extension 2

HSC ASSESSMENT TASK 2

Examiners

- Mr G Huxley
- Mr G Rawson

General Instructions

- Reading time – 3 minutes
- Working time – 40 minutes
- Write using black or blue pen
- NESA-approved calculators may be used
- A Reference Sheet is provided for your use at the back of this question booklet
- In Questions 7 – 8, show relevant mathematical reasoning and/or calculations

Total marks: 30

Section I – 6 marks (pages 2 - 3)

- Attempt Questions 1– 6
- Allow about 8 minutes for this section

Section II – 24 marks (pages 4 – 6)

- Attempt Questions 7 – 8
- Allow about 32 minutes for this section

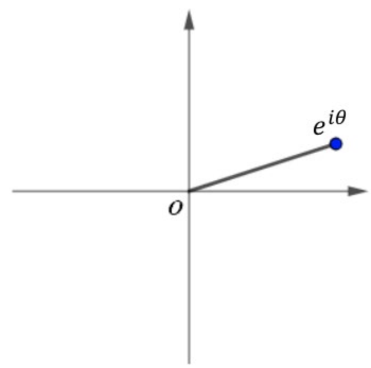
Section I : 6 marks

Attempt Questions 1 – 6.

Allow about 8 minutes for this section.

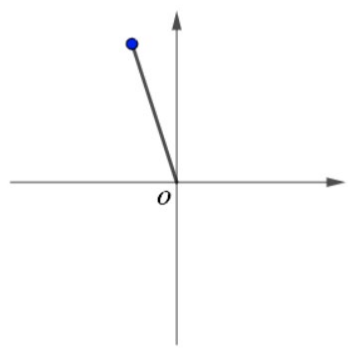
Use the multiple choice answer sheet provided for Questions 1 – 6.

1. The Argand diagram shows the complex number $e^{i\theta}$.

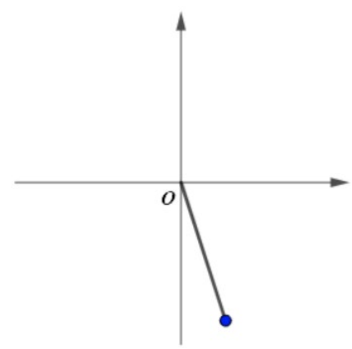


Which of the following diagrams best shows the complex number $-ie^{2i\theta}$?

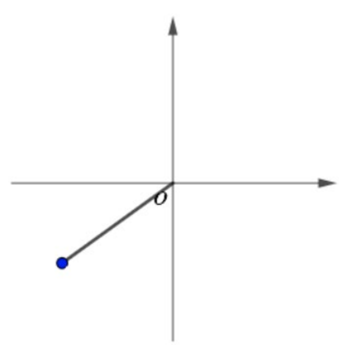
A



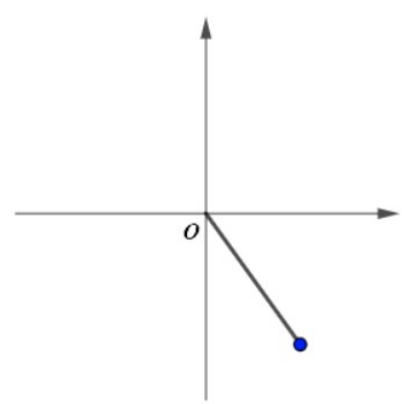
B



C



D



2. If $z = 2e^{-\frac{\pi}{3}i}$, then which of the following is purely real?

A $\bar{z}$

B z^3

C z^i

D $3z$

3. If w is a non-real cube root of unity, then what is the simplest expression equal to $\left(1 - w + w^2\right)^6$?

A $64w^2$

B $64w$

C 64

D $-64w$

4. Which of the following is a solution to:

$|z| \geq 5$ and $\frac{\pi}{2} < Arg\, z < \pi$?

A $z = 4 + 3i$

B $z = 5 - 3i$

C $z = -5 + 3i$

D $z = -3 + 3i$

5. Which of the following can be proven using the Arithmetic Mean/Geometric Mean inequality? (Consider x and y as positive real numbers)

A $\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{2}{xy}$

B $\frac{1}{x^2} + \frac{1}{y^2} \leq \frac{2}{xy}$

C $\frac{1}{x^2 y^2} \geq 2\left(\frac{1}{x^2} + \frac{1}{y^2}\right)$

D $\frac{2}{xy} \leq \frac{1}{x} + \frac{1}{y}$

6. Assume that a and b are negative real numbers with $a > b$. Which of the following is false?

A $\frac{1}{a - b} < 0$

B $\frac{a}{b} - \frac{b}{a} < 0$

C $a + b > 2b$

D $2a > 3b$

Section II

24 marks

Attempt Questions 7 – 8

Allow about 32 minutes for this section

Answer each question in a separate writing booklet.

All necessary working should be shown in every question.

Question 7 (12 marks) Start a new booklet **Marks**

(a) Express $-2+2\sqrt{3}\ i$ in the form $ke^{i\theta}$, where k is real **2**

(b) $z_1 = 2+3i$, $z_2 = 3+2i$ and $z_3 = a+bi$, where a and b are real numbers.

(i) Find the exact value of $\left|z_1 + z_2\right|$. **1**

(ii) Find the exact value of $\mathbf{Re}\left(z_1 \overline{z_2}\right)$. **1**

(iii) Given that $w = \dfrac{z_1 z_3}{z_2}$, find w in terms of a and b,
giving your answer in the form $x + iy$, where x and y are real numbers. **2**

(c) Consider the statement: $\forall x \in \square, \exists y \in \square$ such that $x^2 + y^2 = 2$.

Is the statement true or false? Justify your answer. **1**

Question 7 continues on the next page

(d) If x and y are positive real numbers, then $x+y \geq 2\sqrt{xy}$. (Do NOT prove this)

If a and b are positive real numbers, show that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$. **2**

(e) Three lines have equations:

$$y = px + b_1$$
$$y = qx + b_2$$
$$y = rx + b_3$$

Where $p, q, r, b_1, b_2,$ and b_3 are real constants and p, q, and r are distinct.

Use proof by contradiction to show algebraically that these three lines cannot be perpendicular to each other. **3**

Question 8 (12 marks) Start a new booklet **Marks**

(a) Let the complex number z represent the point $P\ (x, y)$ in the Argand Diagram.

(i) Sketch the set of points P that satisfy the equation:

$$|z-1-i|=4$$

Include in your sketch the co-ordinates that show the domain and range for $P\ (x, y)$. **2**

(ii) Find the maximum value of $|z|$ for points that lie on the solution for part (i). **1**

(b) Find all the solutions to the equation $z^4=-16$.
You may provide your solutions in any suitable form. **2**

(c) Use the method of mathematical induction to prove DeMoivre's Theorem.

That is, prove that $\left[r(\cos\theta+i\sin\theta)\right]^n=r^n(\cos n\theta+i\sin n\theta)$ for all positive integers n. **3**

(d) Consider the following statement:

"Every positive integer power of 5 can be written as the sum of the squares of two distinct positive integers."

i.e. Statement $P\ (k)$ can be written $5^k=x^2+y^2$ where x and y are distinct positive integers.
Different values of k would result in different values of x and y.

(i) Find values of x and y that will satisfy $P\ (1)$ and $P\ (2)$. **1**

(ii) Show that $P(k)\Rightarrow P(k+2)$ **2**

(iii) Explain how your answers to parts (i) and (ii) prove the original statement. **1**

End of Task

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \; r \neq 1$$

$$S = \frac{a}{1-r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

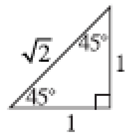
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

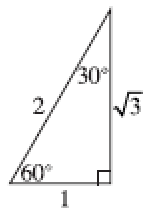
$A = \dfrac{1}{2}ab\sin C$



$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

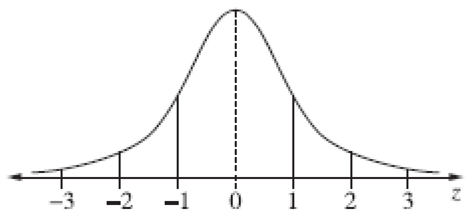
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq r) = \displaystyle\int_a^r f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^{2}(x-c)$$

Year 12 HSC Assessment Task 2

Mathematics Extension 2 – 2022 HSC

Section I – Multiple Choice Answer Sheet

Student Name ____

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ● B ⊗ C ○ D ○

---

1. A ○ B ○ C ○ D ○

2. A ○ B ○ C ○ D ○

3. A ○ B ○ C ○ D ○

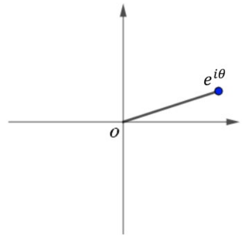
4. A ○ B ○ C ○ D ○

5. A ○ B ○ C ○ D ○

6. A ○ B ○ C ○ D ○

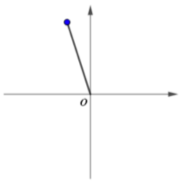
YR12 Ext 2 AT2 2022 Multiple Choice Solutions

1. The Argand diagram shows the complex number $e^{i\theta}$.

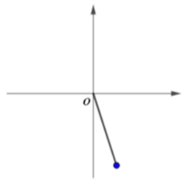


Which of the following diagrams best shows the complex number $-ie^{2i\theta}$?

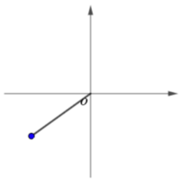
A.



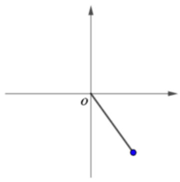
B.



C.



D.

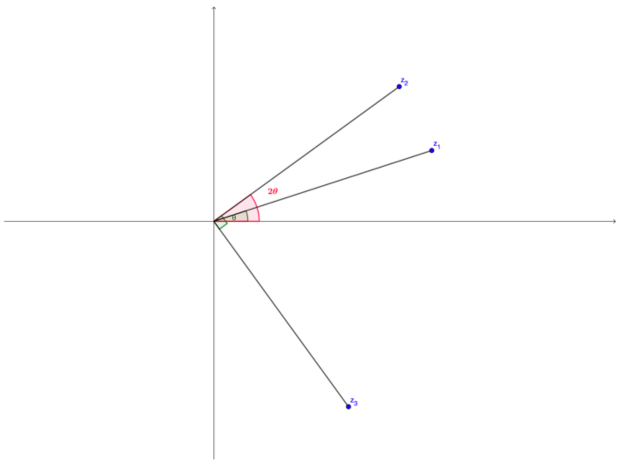


Solution: D

$$e^{i\theta} \rightarrow e^{2i\theta} \rightarrow -ie^{2i\theta}$$

$$z_1 \rightarrow z_2 \rightarrow z_3$$

$$\begin{pmatrix} \text{double} \\ \text{the} \\ \text{angle} \end{pmatrix} \quad \begin{pmatrix} \text{rotate} \\ \text{clockwise} \\ 90° \end{pmatrix}$$



2. If $z = 2e^{-\frac{\pi}{3}i}$, then which of the following is purely real?

A. $\overline{z}$ B. z^3 C. z^i D. $3z$

Solution:

$$\left(2e^{-\frac{\pi}{3}i}\right)^3 = 8e^{i\pi}$$

$\therefore$ Purely real B

3. If w is a non-real cube root of unity, then

$$w^2 + w + 1 = 0$$

$$\therefore \left(1 - w + w^2\right)^6 = \left(-2w\right)^6 = 64\left(w^3\right)^2 = 64$$

Answer = C

4. The interpretation of the symbols is the co-ordinates on the Argand Diagram must be at least 5 units from the origin, and the argument tells us that it is in the 2nd quadrant.
The only response that satisfies both conditions is $z = -5 + 3i$

Answer = C

5. AM/GM inequality: $\dfrac{a+b}{2} \geq \sqrt{ab}$

Rearranging gives $a + b \geq 2\sqrt{ab}$

Which is equivalent to $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{2}{xy}$

Answer = A

6. Assume that *a* and *b* are negative real numbers with *a* > *b*. Which of the following might be false?

A. $\dfrac{1}{a-b} < 0$

B. $\dfrac{a}{b} - \dfrac{b}{a} < 0$

C. $a + b > 2b$

D. $2a > 3b$

Solution:

$$a > b$$

$$a - b > 0$$

$$\frac{1}{a-b} > 0 \quad \text{so A is false}$$