James Ruse Agricultural High School

2022 **TASK 2 EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/ or calculations

Total marks: 80

Section I – 5 marks (pages 2–3)

- Attempt Questions 1–5
- Allow about 8 minutes for this section

Section II – 75 marks (pages 4–9)

- Attempt Questions 6–10
- Allow about 1 hour and 52 minutes for this section

Section I

5 marks
Attempt Questions 1 – 5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 5.

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1 The modulus of the complex number $1 + \cos(2\theta) + i\sin(2\theta)$, where $0 < \theta < \dfrac{\pi}{2}$ is

A. $4\cos(2\theta)$

B. $4\sin(2\theta)$

C. $2\cos\theta$

D. $2\sin\theta$

2 Which expression is equal to $\displaystyle\int \frac{2}{e^x + e^{-x}}\,dx$?

A. $\ln(e^{2x} + 1) + C$

B. $\ln(e^{-2x} + 1) + C$

C. $2\tan^{-1}(e^x) + C$

D. $2\tan^{-1}(e^{-x}) + C$

3 If P and Q are logical statements, and $P \iff Q$, which of the following is true?

A. $P \iff \neg Q$

B. $\neg P \iff Q$

C. $\neg P \iff \neg Q$

D. None of the above.

4 Consider the following expressions below:

I. $$\int_0^a \Big(f(x) + f(2a - x)\Big)\, dx$$

II. $$\int_a^{2a} \Big(f(x) + f(2a - x)\Big)\, dx$$

Which of these expressions is equivalent to $\displaystyle\int_0^{2a} f(x)\, dx$?

A. Neither I nor II are equivalent

B. Only I is equivalent

C. Only II is equivalent

D. Both I and II are equivalent

5 Which of the following is the negation of the statement

For all odd prime $p < q$, there exists positive non-prime integers $r < s$ such that $p^2 + q^2 = r^2 + s^2$.

A. For all odd primes $p < q$ there exists positive non-primes $r < s$ such that $p^2 + q^2 \neq r^2 + s^2$.

B. There exists odd primes $p < q$ such that for all positive non-primes $r < s$ such that $p^2 + q^2 \neq r^2 + s^2$.

C. For all odd primes $p < q$ and for all positive non-primes $r < s$, $p^2 + q^2 \neq r^2 + s^2$.

D. There exists odd primes $p < q$ and there exists positive non-primes $r < s$ such that $p^2 + q^2 \neq r^2 + s^2$.

Section II

75 marks
Attempt Questions 6 – 10
Allow about 1 hour and 52 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 marks) Use the Question 6 Writing Booklet

(a) Find the principal argument of $z = \dfrac{i}{-2-2i}$. **2**

(b) (i) Express $z^4 + 3z^2 - 4$ in the form $(z^2 + a)(z^2 + b)$ where a and b are real constants to be found. **1**

(ii) Hence, plot the complex numbers representing the roots of $z^4 + 3z^2 - 4 = 0$ on an Argand Diagram. **2**

(c) Write, as an English sentence, the converse of the statement: **1**

"If I am rich, then I have a lot of money."

(d) Prove that the sum of the squares of two odd numbers is even. **2**

(e) Find the following integrals:

(i) $\displaystyle\int \frac{1-\tan x}{1+\tan x}\,dx$ **2**

(ii) $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$ **2**

(iii) $\displaystyle\int \sqrt{3-\sqrt{x}}\,dx$ **3**

End of Question 6

Question 7 (15 marks) Use the Question 7 Writing Booklet

(a) Let $A = \sqrt{a^2 + 1}$ and $\alpha = \operatorname{Arg}(a + i)$.

(i) Write down the two square roots of $a + i$ in Euler form in terms of A and α. **1**

(ii) With the aid of trigonometric identities, show that the square roots obtained in part (i) can be written as $\pm\frac{1}{\sqrt{2}}\left(\sqrt{A + a} + i\sqrt{A - a}\right)$. **3**

(b) (i) Find the real constants A, B, C and D such that **2**

$$\frac{1}{x(x-1)(x^2+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2+2}.$$

(ii) Hence, or otherwise, find **2**

$$\int \frac{dx}{x(x-1)(x^2+2)}.$$

(c) Let n be a natural number. For each of the following statements, decide whether it is true or false. If true, provide a proof for the statement; if false, give a counterexample.

(i) If n^2 is a multiple of 9 then so is n. **2**

(ii) If n is a multiple of 9 then so is n^2. **2**

(d) Prove that there is no complex number z such that $|z| - z = i$. **3**

End of Question 7

Question 8 (15 marks) Use the Question 8 Writing Booklet

(a) (i) Show that $\dfrac{\cot\theta + i}{\cot\theta - i} = e^{2i\theta}$. **2**

(ii) Write $\dfrac{\sqrt{3}+i}{\sqrt{3}-i}$ in the form of $e^{2i\theta}$. **1**

(iii) Hence, find the roots of the equation $z^4 = \dfrac{\sqrt{3}+i}{\sqrt{3}-i}$. **2**

(b) (i) Show that **3**

$$\int \sec^3 x \, dx = \frac{1}{2}\Big(\sec x \tan x + \ln|\sec x + \tan x|\Big) + C.$$

(ii) Hence, by first rationalising the denominator, find **3**

$$\int \frac{dx}{x + \sqrt{1+x^2}}.$$

(c) Suppose $a, b \in \mathbb{Z}$. Consider the statement:

If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

(i) Write the contrapositive of the statement. **1**

(ii) Use the contrapositive to prove the statement is true. **3**

End of Question 8

Question 9 (14 marks) Use the Question 9 Writing Booklet

(a) (i) Show that, for $\theta > 0$, **1**

$$\tan^{-1}(\theta) + \tan^{-1}\left(\frac{1}{\theta}\right) = \frac{\pi}{2}.$$

(ii) Define the function **4**

$$T(x) := \int_0^x \frac{\tan^{-1}(y)}{y}\, dy.$$

Using part (i), or otherwise, show that for $x > 0$

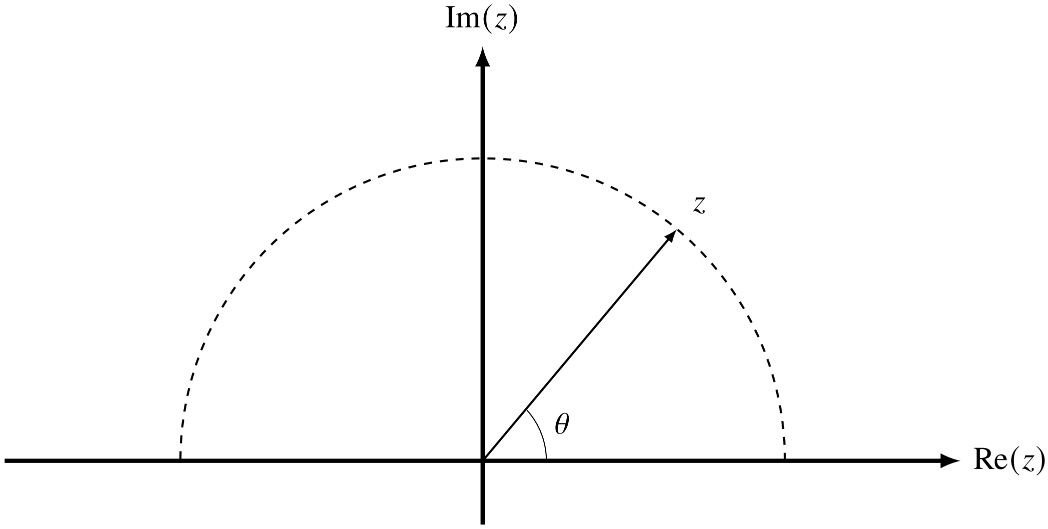
$$T(x) - T\left(\frac{1}{x}\right) = \frac{\pi}{2}\ln x.$$

(b) (i) Prove that the product of three consecutive numbers is always divisible by 6. **3**

(ii) Hence or otherwise, prove that for all natural numbers n, $\frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6}$ is a natural number. **2**

Question 9 continues on page 8

(c) A complex number z lies on the upper half of a unit semi-circle.



Copy the diagram neatly onto your page.

(i) Sketch the vectors $z + 1$ and $z - 1$ on the same diagram. **2**

(ii) Using your diagram, or otherwise, prove that $\operatorname{Arg}\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$. **2**

(iii) Hence, prove that $\operatorname{Arg}(z) = 2\tan^{-1}\left|\dfrac{z-1}{z+1}\right|$. **1**

End of Question 9

Question 10 (15 marks) Use the Question 10 Writing Booklet

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to evaluate $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{2 - \cos x}\, dx$. **3**

(b) Suppose $u = u(x)$ and $v = v(x)$ are differentiable functions of x, with $v \neq 0$.

(i) Show that **2**

$$\int \frac{u}{v^2}\frac{dv}{dx}\, dx = -\frac{u}{v} + \int \frac{1}{v}\frac{du}{dx}\, dx.$$

(ii) Define the family of integrals **4**

$$I_n = \int \frac{(1+x^2)^n}{x^2}\, dx$$

for $n \geq 1$. By using part (i), or otherwise, show that

$$I_n = \frac{1}{2n-1}\frac{(1+x^2)^n}{x} + \frac{2n}{2n-1}I_{n-1}.$$

(c) Let n be a fixed positive integer. A complex number w is called a *primitive* nth root of unity if $w^n = 1$ and $w^p \neq 1$ for $p = 1, 2, \ldots n-1$.

(i) Given that any nth root of unity can be written in the from $e^{i\frac{2\pi k}{n}}$ for integer k, use a proof by contradiction to prove that if $e^{i\frac{2\pi k}{n}}$ is a primitive nth root of unity, then k and n have no common factors except 1. **3**

You are given that the converse of the statement in part (i) is true. (Do NOT prove this)

(ii) State the primitive 6th roots of unity. **1**

(iii) Write down the monic polynomial $Q_6(x)$ of smallest degree with real coefficients whose roots are the primitive 6th roots of unity. **2**

End of Question 10

End of Examination.

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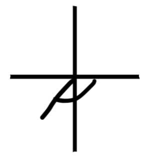
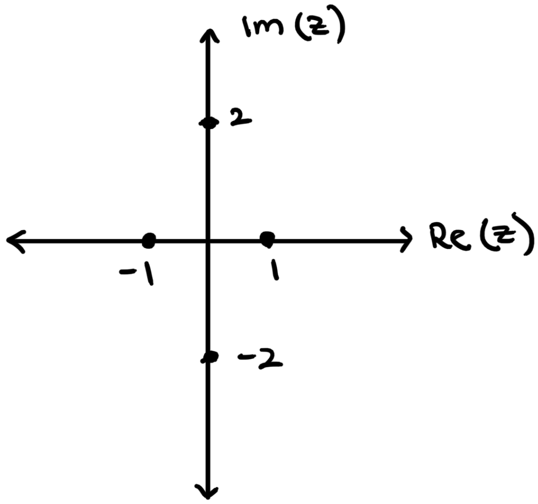
Multiple Choice:

1. C
2. C
3. C
4. D
5. B

MATHEMATICS Extension 2: Question 6

Suggested Solutions	Marks	Marker's Comments
a) $z = \dfrac{i}{-2-2i}$ $= \dfrac{i(-2+2i)}{4+4}$ $= -\dfrac{1}{4} - \dfrac{i}{4}$ 	1	
$\text{Arg}(z) = \dfrac{-3\pi}{4}$	1	
b) i) $z^4 + 3z^2 - 4 = (z^2-1)(z^2+4)$	1	
ii) 		① correct Re(z) ① correct Im(z)
c) If I have a lot of money, then I am rich.		
d) Let the two odd numbers be $x = 2m+1$ and $y = 2n+1$ where x, y, m and $n \in \mathbb{Z}$ $\therefore x^2 + y^2 = (2m+1)^2 + (2n+1)^2$	1	must mention m and n are integers.
$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$		

MATHEMATICS Extension 2: Question…6…

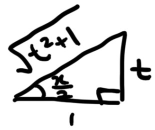
Suggested Solutions	Marks	Marker's Comments
$= 4m^2 + 4m + 4n^2 + 4n + 2$		
$= 2(2m^2 + 2m + 2n^2 + 2n + 1)$		
$= 2k$		-must state k is an integer
where $k = 2m^2 + 2m + 2n^2 + 2n + 1 \in \mathbb{Z}$ as it is closed under multiplication and addition	1	- must also include the conclusion.
$\therefore x^2 + y^2$ is even		
$\therefore$ the sum of the squares of two odd numbers is even.		
e) i) method 1		
$\int \frac{1 - \tan x}{1 + \tan x}\, dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$	1	
$= \ln\lvert \cos x + \sin x \rvert + C$	1	must have absolute values.
method 2		
$\int \frac{1 - \tan x}{1 + \tan x}\, dx = \int \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\, dx$		
$= \int \tan\left(\frac{\pi}{4} - x\right) dx$	1	
$= \int \frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right)}\, dx$		
$= \ln\left\lvert \cos\left(\frac{\pi}{4} - x\right)\right\rvert + C$	1	

MATHEMATICS Extension 2: Question 6

Suggested Solutions	Marks	Marker's Comments
method 3		Students who used this method did not progress very far.

$$\int \frac{1-\tan x}{1+\tan x}\,dx$$

let $t = \tan\frac{x}{2}$



$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1}{2}\left(\sqrt{t^2+1}\right)^2$$

$$\frac{dt}{dx} = \frac{t^2+1}{2}$$

$$dx = \frac{2}{t^2+1}\,dt$$

$$\int \frac{1-\tan x}{1+\tan x}\,dx = \int \frac{1-\frac{2t}{1-t^2}}{1+\frac{2t}{1-t^2}}\left(\frac{2}{t^2+1}\right)dt$$

$$= 2\int \frac{1-t^2-2t}{(1-t^2+2t)(t^2+1)}\,dt$$

$$= 2\int \frac{t^2+2t-1}{(t^2-2t-1)(t^2+1)}\,dt$$

$$\frac{t^2+2t-1}{(t^2-2t-1)(t^2+1)} = \frac{At+B}{t^2-2t-1} + \frac{Ct+D}{t^2+1} \qquad 1$$

$$t^2+2t-1 = (At+B)(t^2+1) + (Ct+D)(t^2-2t-1)$$

$t=0$: $-1 = B - D$

$\therefore B = D - 1$

$t=1$: $2 = (A+B)(2) + (C+D)(-2)$

$1 = A + B - C - D$

$1 = A + (D-1) - C - D$

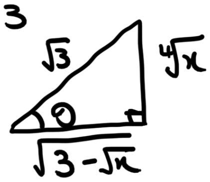
$2 = A - C$

$\therefore A = 2 + C$

MATHEMATICS Extension 2: Question 6

Suggested Solutions	Marks	Marker's Comments
$t=-1: \quad -2 = (-A+B)(2) + (-C+D)(2)$		
$-1 = -A+B-C+D$		
$-1 = -(2+C) + (D-1) - C + D$		
$-1 = -2 - C + D - 1 - C + D$		
$2 = -2C + 2D$		
$\therefore 1 = D - C \Rightarrow D = 1 + C$		
$t=2: \quad 7 = (2A+B)(5) + (2C+D)(-1)$		
$7 = 10A + 5B - 2C - D$		
$7 = 10(2+C) + 5(D-1) - 2C - (1+C)$		
$7 = 20 + 10C + 5(1+C-1) - 2C - 1 - C$		
$-12 = 12C$		
$\therefore C = -1, \ D = 0$		
$A = 1, \ B = -1$		
$\therefore 2\int \frac{t^2+2t-1}{(t^2-2t-1)(t^2+1)}\,dt$		
$= 2\int \frac{t-1}{t^2-2t-1} + \frac{-t}{t^2+1}\,dt$		
$= 2\int \frac{1}{2}\frac{2t-2}{t^2-2t-1} - \frac{1}{2}\frac{2t}{t^2+1}\,dt$		
$= \int \frac{2t-2}{t^2-2t-1} - \frac{2t}{t^2+1}\,dt$		
$= \ln\lvert t^2-2t-1\rvert - \ln\lvert t^2+1\rvert + C$		
$= \ln\lvert \tan^2\frac{x}{2} - 2\tan\frac{x}{2} - 1\rvert$ $- \ln\lvert \tan^2\frac{x}{2} + 1\rvert + C$	1	
ii) $\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$	1	
$= \sin^{-1}(x-1) + C$	1	

Suggested Solutions	Marks	Marker's Comments
iii) Method 1		
$\int \sqrt{3-\sqrt{x}}\,dx$ let $u = 3-\sqrt{x}$, $\frac{du}{dx} = -\frac{1}{2\sqrt{x}}$		
$= \int \sqrt{u}\left(-2(3-u)\right)du$ $-2\sqrt{x}\,du = dx$, $-2(3-u)\,du = dx$	1	correct substitution
$= -2\int \left(3\sqrt{u} - u\sqrt{u}\right)du$		
$= -2\left(\frac{3u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right) + C$	1	
$= -4u^{3/2} + \frac{4}{5}u^{\frac{5}{2}} + C$		
$= -4(3-\sqrt{x})^{3/2} + \frac{4}{5}(3-\sqrt{x})^{\frac{5}{2}} + C$	1	
method 2		
$\int \sqrt{3-\sqrt{x}}\,dx$ let $u = \sqrt{x}$, $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$		
$= \int \sqrt{3-u}\,(2u)\,du$ $2u\,du = dx$		
$= 2\int u\sqrt{3-u}\,du$ $x = u$, $x' = 1$ $y' = \sqrt{3-u}$, $y = -\frac{2}{3}(3-u)^{3/2}$ $= 2\left[-\frac{2}{3}u(3-u)^{3/2} + \frac{2}{3}\int (3-u)^{3/2}du\right]$	1	correct substitution and using integration by parts.
$= -\frac{4}{3}u(3-u)^{\frac{3}{2}} - \left(\frac{4}{3}\right)\left(\frac{2}{5}\right)(3-u)^{\frac{5}{2}} + C$	1	
$= -\frac{4}{3}\sqrt{x}(3-\sqrt{x})^{\frac{3}{2}} - \frac{8}{15}(3-\sqrt{x})^{\frac{5}{2}} + C$	1	

Suggested Solutions	Marks	Marker's Comments
method 3		
$\int \sqrt{3-\sqrt{x}}\,dx$ let $x = 9\sin^4\theta$, $dx = 36\sin^3\theta\cos\theta\,d\theta$		
$= \int \sqrt{3-\sqrt{9\sin^4\theta}} \times 36\sin^3\theta\cos\theta\,d\theta$		
$= \int \sqrt{3-3\sin^2\theta} \times 36\sin^3\theta\cos\theta\,d\theta$		
$= 36\sqrt{3}\int \sin^3\theta\cos^2\theta\,d\theta$	1	
$= 36\sqrt{3}\int \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta$		
$= 36\sqrt{3}\int (\sin\theta\cos^2\theta - \sin\theta\cos^4\theta)\,d\theta$		
$= 36\sqrt{3}\left(-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right) + C$	1	
$x = 9\sin^4\theta$ $\frac{x}{9} = \sin^4\theta$ $\frac{\sqrt[4]{x}}{\sqrt{3}} = \sin\theta$		
		
$= 36\sqrt{3}\left(-\frac{1}{3}\left(\frac{\sqrt{3-\sqrt{x}}}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{3-\sqrt{x}}}{\sqrt{3}}\right)^5\right) + C$	1	

MATHEMATICS Ext 2: Question 7		
Suggested Solutions	**Marks**	**Marker's Comments**

a)

Let $z = re^{i\theta}$

$z^2 = a+i \Rightarrow r^2e^{2i\theta} = a+i$
$= |a+i|e^{i\,\mathrm{Arg}(a+i)}$

$\therefore r^2 = |a+i|$
$= \sqrt{a^2+1}$
$= A$
$\therefore r = \sqrt{A} \quad (r>0)$

$\therefore 2\theta = \mathrm{Arg}(a+i) + 2n\pi \quad (n \in \mathbb{Z})$

$= \alpha + 2n\pi$

$\therefore \theta = \frac{\alpha}{2} + n\pi$

when $n=0$, $\theta = \frac{\alpha}{2}$
$n=1$, $\theta = \frac{\alpha}{2} + \pi$ or $n=-1$, $\theta = \frac{\alpha}{2} - \pi$

$\therefore z = \sqrt{A}e^{\frac{i\alpha}{2}}$ or $\sqrt{A}e^{i(\frac{\alpha}{2}+\pi)}$ if $\alpha < 0$ — 1

or

$\sqrt{A}e^{i(\frac{\alpha}{2}-\pi)}$ if $\alpha \geq 0$

Note: answers such as $\pm\sqrt{A}e^{i(\frac{\alpha}{2})}$ is NOT accepted. as Euler form means constant up the front must be positive!

or

as the question said "write down", just the answer without working will suffice.

Year 12 Task 2 MATHEMATICS Ext2 Question 7

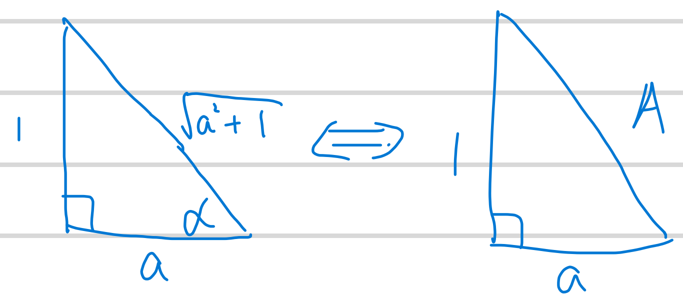
Suggested Solutions	Marks	Marker's Comments

b) let $\sqrt{A}e^{i\frac{\alpha}{2}}$ and $\sqrt{A}e^{i(\frac{\alpha}{2}+\pi)}$

$$= \pm\sqrt{A}e^{i\frac{\alpha}{2}}$$

$$= \pm\sqrt{A}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

$$\alpha = \tan^{-1}\left(\frac{1}{a}\right)$$



1

$$\cos^2\frac{\alpha}{2} = \frac{1}{2}(1+\cos\alpha)$$

$$= \frac{1}{2}\left(1+\frac{a}{A}\right)$$

$$\therefore \cos\frac{\alpha}{2} = \frac{1}{\sqrt{2}}\sqrt{\frac{A+a}{A}} \quad \text{Since } 0 < \frac{\alpha}{2} < \frac{\pi}{2}$$

1

$$\sin^2\frac{\alpha}{2} = \frac{1}{2}(1-\cos\alpha)$$

$$= \frac{1}{2}\left(1-\frac{a}{A}\right)$$

$$\sin\frac{\alpha}{2} = \frac{1}{\sqrt{2}}\sqrt{\frac{A-a}{A}}$$

Year 12 Task 2 MATHEMATICS Ext 2 Question 7

Suggested Solutions	Marks	Marker's Comments
$\therefore \pm\sqrt{A}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$		
$= \pm\sqrt{A}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{A+a}{A}} + i\frac{1}{\sqrt{2}}\sqrt{\frac{A-a}{A}}\right)$		
$= \pm\frac{1}{\sqrt{2}}\left(\sqrt{A}\times\sqrt{\frac{A+a}{A}} + \sqrt{A}\sqrt{\frac{A-a}{A}}\right)$		
$= \pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right)$	1	

MATHEMATICS Ext 2 : Question…7…..		
Suggested Solutions	**Marks**	**Marker's Comments**

b) i

$$\frac{1}{x(x-1)(x^2+2)} = \frac{A(x-1)(x^2+2)+Bx(x^2+2)+(Cx+D)x(x-1)}{x(x-1)(x^2+2)}$$

$$\therefore 1 \equiv A(x-1)(x^2+2)+Bx(x^2+2)+x(Cx+D)(x-1)$$

Sub $x=0$, $-2A=1$,

$$\therefore A = \frac{-1}{2}$$

$x=1$, $3B=1$

$$B=\frac{1}{3}$$

Compare coefficients of x^3:

$$A+B+C=0$$
$$\therefore C=-A-B$$
$$=\frac{1}{2}-\frac{1}{3}$$
$$=\frac{1}{6}$$

Compare coefficients of x^2:

$$-A-C+D=0$$
$$D=A+C$$
$$=\frac{-1}{2}+\frac{1}{6}$$
$$=\frac{-1}{3}$$

(1) for 1 correct value

(2) for all 4 correct values

MATHEMATICS Ext 2: Question……..

Suggested Solutions	Marks	Marker's Comments
ii $\therefore \displaystyle\int \frac{dx}{x(x-1)(x^2+2)}$	1	for 2 correct integrals
$= \frac{-1}{2}\int \frac{1}{x}\,dx + \frac{1}{3}\int \frac{1}{x-1}\,dx + \frac{1}{6}\int \frac{x}{x^2+2}\,dx - \frac{1}{3}\int \frac{1}{x^2+2}\,dx$	2	for all 4
$= \frac{-1}{2}\ln\lvert x\rvert + \frac{1}{3}\ln\lvert x-1\rvert + \frac{1}{12}\ln\lvert x^2+2\rvert - \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$		Some carrier forward error marks are NOT applicable if one of your constants solved in part i) was zero.
c) i False. — 1		
Consider $n^2 = 9$ $= 9\times 1$		
$\therefore n^2$ is divisible by 9		
$n = 3$ $(n \in \mathbb{Z})$ $= 9\left(\frac{1}{3}\right)$		
$\therefore n^2$ is not divisible by 9		
$\therefore$ The statement is false by a counterexample.	1	
ii Suppose n is divisible by 9		
$\therefore n = 9k$ $(k \in \mathbb{Z})$ $n^2 = (9k)^2$ $= 81k^2$ $= 9(9k^2)$	1	stating statement is true
Since $9k^2$ is an integer, as integers 9 and k are closed under multiplication, then n^2 is divisible by 9.	1	correct prove
$\therefore$ The statement is True		

MATHEMATICS Ext 2: Question 7

Suggested Solutions	Marks	Marker's Comments
d) Assume there exists a $z = x + iy$, where $x, y \in \mathbb{R}$, such that $\lvert z \rvert - z = i$	1	for statement of assumption
$\therefore \sqrt{x^2+y^2} - x - iy = i$		
Equating real parts:		
$\sqrt{x^2+y^2} - x = 0$ $\sqrt{x^2+y^2} = x$ $x^2+y^2 = x^2$ $\therefore y^2 = 0$ $y = 0$	1	
Equating imaginary parts:		
$-y = 1$ $y = -1$		
which is a contradiction to $y = 0$		
$\therefore$ The assumption is proven to be false by contradiction.	1	
<u>Note</u>: If you started operating with LHS = RHS without stating an assumption, then you do not receive the full marks. Words such as "if", "suppose", "assume" at the START of your proof is crucial!		

Year 12 Extension 2: Q8		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
(a) (i) $\dfrac{\cot\theta + i}{\cot\theta - i} = \dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}$ $= e^{i\theta}\cdot\dfrac{1}{e^{-i\theta}}$ $= e^{i\theta}\cdot e^{i\theta}$ $= \left(e^{i\theta}\right)^2$ $= e^{2i\theta}$ de Moivre	1 Sufficient progress 1 Final result	No major issues here.
(ii) Identifying $\cot\theta = \sqrt{3}$, we have $\theta = \frac{\pi}{6} + n\pi$ for $n \in \mathbb{Z}$. Hence $\dfrac{\sqrt{3}+i}{\sqrt{3}-i} = \dfrac{\cot\theta + i}{\cot\theta - i}\Big\vert_{\theta=\frac{\pi}{6}+n\pi}$ $= e^{2i\theta}\big\vert_{\theta=\frac{\pi}{6}+n\pi}$ $= e^{2i(\pi/6+n\pi)}$ $= e^{i(\pi/3+2n\pi)}$ $\forall n \in \mathbb{Z}$	1	Principal argument was wrongly assumed by nearly all candidates. Question makes no mention of the branch in which the argument sits, so you cannot assume principal. If the question were worth more than one mark, there would have been a penalty.
(iii) We require all $z \in \mathbb{C}$ such that $z^4 = \dfrac{\sqrt{3}+i}{\sqrt{3}-i} = e^{i\left(\frac{\pi}{3}+2n\pi\right)}$ ... (1) Let $z = \|z\|e^{i\,\mathrm{ar}\ \ (z)}$. Then $z^4 = \left(\|z\|e^{i\arg(z)}\right)^4 = \|z\|^4\left(e^{i\arg(z)}\right)^4$ $= \|z\|^4 e^{i4\arg(z)}$ by de Moivre's theorem Identifying this with components of (1): $\|z\|^4 = 1 \to \|z\| = 1$ $4\arg(z) = \dfrac{\pi}{3} + 2n\pi \to \arg(z) = \dfrac{\pi}{12} + \dfrac{n\pi}{2}$ So $z = \exp\left(i\left(\dfrac{\pi}{12} + \dfrac{n\pi}{2}\right)\right)$	1 First mark for extracting general solution.	Some students applied de Moivre's theorem using a fractional power. This is incorrect; the theorem deals only with integral powers and exponentiation by non-integral powers is not part of the course. Consider: $\mathrm{cis}^4\left(\frac{\pi}{4}\right) = \mathrm{cis}(\pi) = -1$ $\mathrm{cis}^4\left(\frac{\pi}{4} + 2k\pi\right) = \mathrm{cis}(\pi + 8k\pi) = -1$ for all $k \in \mathbb{Z}$ **but (erroneously)** $\mathrm{cis}^{\frac{1}{4}}\left(\frac{\pi}{4}\right) = \mathrm{cis}\left(\frac{\pi}{16}\right)$ while

Four unique (mod 2π) solutions in the complex plane are given by four consecutive integers for n; choose $n = 0, \pm 1, 2$. Then

$$z = e^{i\left(\frac{\pi}{12}+2k\pi\right)}, e^{i\left(\frac{7\pi}{12}+2k\pi\right)}, e^{i\left(-\frac{5\pi}{12}+2k\pi\right)}, e^{i\left(\frac{13\pi}{12}+2k\pi\right)}$$

for $k \in \mathbb{Z}$.

Note, all integers n may be expressed in the form

$$n = 4j + r$$

where $j \in \mathbb{Z}$ and $r = 0, 1, 2, 3$ (Quotient-Remainder Theorem), so

$$z = \exp\left(i\left(\frac{\pi}{12} + \frac{n\pi}{2}\right)\right) = \exp\left(i\left(\frac{\pi}{12} + \frac{(4j+r)\pi}{2}\right)\right)$$

$$= \exp\left(i\left(\frac{\pi}{12} + \frac{r\pi}{2} + 2j\pi\right)\right)$$

So for $r = 0, 1, 2, 3$:

$$z = \begin{cases} \exp\left(i\left(\frac{\pi}{12} + 2j\pi\right)\right) \\ \exp\left(i\left(\frac{7\pi}{12} + 2j\pi\right)\right) \\ \exp\left(i\left(\frac{13\pi}{12} + 2j\pi\right)\right) \\ \exp\left(i\left(\frac{19\pi}{12} + 2j\pi\right)\right) \end{cases}$$

This is the full set of solutions in Euler form.

(b) (i)

There are two ways of answering this question: (1) evaluating the integral directly using *integration by parts*, (2) taking the derivative of the RHS + using the Fundamental Theorem of Calculus.

Here, the first (i.e. majority) solution is shown. The solution is presented in the format that is encouraged.

1
Second mark for identifying any legitimate four, non-equivalent solutions.

$\text{cis}^{\frac{1}{4}}\left(\frac{\pi}{4} + 2k\pi\right) = \text{cis}\left(\frac{\pi}{16} + \frac{k\pi}{2}\right) \neq \text{cis}\left(\frac{\pi}{16}\right)$ for all $k \in \mathbb{Z}$

Since sine and cosine are blind to multiples of 2π, not including the general argument in whole number cases makes no difference to the outcome, while in the second case, we start with two numbers that are the same, just expressed differently, and finish with two numbers that are not necessarily equal after raising to a fractional (i.e. non-integral) power. **Do not apply de Moivre's theorem using non-integer powers**.

$$I \coloneqq \int \sec^3 x\,dx = \int \sec x \cdot \sec^2 x\,dx$$ Let $u = \sec x \to du = \sec x \tan x\,dx,\ v' = \sec^2 x \to v = \tan x$. Then by IBP, using the accepted form $uv - \int v\,du$: $$\begin{aligned} I &= \sec x \tan x - \int \sec x \tan^2 x\,dx \\ &= \sec x \tan x - \int \sec x\,(\sec^2 x - 1)\,dx \\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\ dx \end{aligned}$$	1 First mark for correct IBP setup	
Hence $$\begin{aligned} 2I &= \sec x \tan x + \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx \\ &= \sec x \tan x + \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx \\ &= \sec x \tan x + \int \frac{(\sec x + \tan x)'}{\sec x + \tan x}\,dx \\ &= \sec x \tan x + \log\lvert \sec x + \tan x \rvert + C \end{aligned}$$	1 Second mark for sufficient progress.	Students needed to evaluate the integral of $\sec x$, not quote. Most evaluated as given here, though t-substitution could also be used.
So $$I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log\lvert\sec x + \tan x\rvert + C'$$ A t-substitution could also have been used on the integral of $\sec x$. This way is ‘discouraged’ (i.e. not explicitly identifying substituents): $$\begin{aligned} \int \sec^3 x\,dx &= \int \sec x\,d\tan x \\ &= \sec x \tan x - \int \tan^2 x \sec x\,dx \\ &= \sec x \tan x - \int \sec^3 x - \sec x\,dx \end{aligned}$$ $$\begin{aligned} 2\int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx \\ &= \sec x \tan x + \int \frac{(\tan x + \sec x)'}{\tan x + \sec x}\,dx \\ &= \sec x \tan x + \log\lvert \tan x + \sec x \rvert + C_1 \end{aligned}$$ $$\therefore \int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log\lvert \tan x + \sec x \rvert + C$$	1 Third mark for obtaining final result.	

(ii) Rationalising the denominator of the left-hand integral gives $$\int \frac{dx}{x+\sqrt{1+x^2}} = \int dx \cdot \frac{x-\sqrt{1+x^2}}{x^2-(1+x^2)}$$ $$= \int \sqrt{1+x^2} - x\, dx$$ $$= \int \sqrt{1+x^2}\, dx - \frac{x^2}{2}$$	No mark awarded for rationalising the denominator	
Now, in $$J := \int \sqrt{1+x^2}\, dx$$ let $x = \tan u$ where $u \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Then $$J = \int \sqrt{1+\tan^2 u}\sec^2 u\, du$$ $$= \int \sqrt{\sec^2 u}\sec^2 u\, du$$ $$= \int \lvert\sec u\rvert \sec^2 u\, du$$ Since $-\frac{\pi}{2} < u < \frac{\pi}{2}$, it follows that $\sec u \geq 1 > 0$, so $\lvert\sec u\rvert = \sec u$. Hence,	1 First mark for correct substitution	Not many identified a domain for the substituted variable. This should be done so you can justify why future choices can be made. It should be noted that $\sqrt{\sec^2 u} = \lvert \sec u \rvert$. Students were not penalised this time. In the Trial HSC, penalties will be applied.
$$J = \int \sec^3 u\, du$$ $$= \frac{1}{2}\sec u \tan u + \frac{1}{2}\log\lvert \sec u + \tan u \rvert + C$$ by (i). Now, $x^2 = \tan^2 u \to \sec^2 u = 1 + x^2 \to \sec u = \sqrt{1+x^2}$ since $\sec u > 0$ here. Hence $$J = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\log\left\lvert x + \sqrt{1+x^2}\right\rvert + C$$	1 Second mark for correctly evaluating integral in substituted form	Candidates did not have to justify the removal of absolute value brackets in logarithm, but this should be checked in future.
Note that if the argument of the logarithm is zero, then $$x + \sqrt{1+x^2} = 0 \to -x = \sqrt{1+x^2} \to x^2 = 1 + x^2 \to 0 = 1$$ which is false. Also, if $$x + \sqrt{1+x^2} < 0$$	1 Final mark for integral returned as function of x. Must have provided adequate justification for choice of positive	

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Then

$$\sqrt{1+x^2} < -x$$

If $x > 0$, then LHS $<$ RHS while LHS positive and RHS negative (+ve $<$ -ve). This is false. Else if $x < 0$, then since $0 < a < b$ implies $a^2 < b^2$, we see after squaring,

$$1 + x^2 < x^2$$

so

$$1 < 0$$

again, false.

Since $x + \sqrt{1+x^2}$ is not ever negative nor zero, and since it must have some relation to zero, it must be positive always.

Hence

$$\int \frac{dx}{x+\sqrt{1+x^2}} = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\log\left(x+\sqrt{1+x^2}\right) - \frac{x^2}{2} + C$$

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<td>vs negative square root.</td>
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(c)

(i) Let $P(x)$ mean, $'x$ is P'.

Then for $a, b \in \mathbb{Z}$, the statement as given can be expressed as

$$\mathrm{odd}\left(a^2(b^2-2b)\right) \to \mathrm{odd}(a) \wedge \mathrm{odd}(b)$$

The contrapositive is

$$\neg\left(\mathrm{odd}(a) \wedge \mathrm{odd}(b)\right) \to \neg\mathrm{odd}\left(a^2(b^2-2b)\right)$$
$$\equiv \neg\mathrm{odd}(a) \vee \neg\mathrm{odd}(b) \to \neg\mathrm{odd}\left(a^2(b^2-2b)\right)$$

which is equivalent to

$$\mathrm{even}(a) \vee \mathrm{even}(b) \to \mathrm{even}\left(a^2(b^2-2b)\right)$$

If either a is even or b is even, then $a^2(b^2-2b)$ *is even.*

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Logical notation not necessary, but must have correct statement.</td>
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So, if at least one of a or b is even, then the given expression is even. (ii) Suppose a is even and b is any integer. Then $$a = 2k \;\; \exists k \in \mathbb{Z}$$ We have, $$a^2(b^2 - 2b) = 4k^2(b^2 - 2b)$$ $$= 2\left(2k^2(b^2 - 2b)\right)$$ Since $\mathbb{Z}$ is closed under $+, \times, -$, it follows that $2k^2(b^2 - 2b) \in \mathbb{Z}$. Hence $2 \mid a^2(b^2 - 2b)$ if a is even. Now, suppose that $b = 2k$ ($\exists k \in \mathbb{Z}$) and a is an integer. Then $$a^2(b^2 - 2b) = a^2(4k^2 - 4k)$$ $$= 2\left(a^2(2k^2 - 2k)\right)$$ Again, since $\mathbb{Z}$ is closed under the previously stated operations, $$a^2(2k^2 - 2k) \in \mathbb{Z}$$ So $$2 \mid a^2(b^2 - 2b)$$ Hence, if at least one of a or b is even, then $a^2(b^2 - 2b)$ is even. Therefore, if $a^2(b^2 - 2b)$ is odd, then both a is odd and b is odd.	1 mark awarded for each of the three cases to be tested. If answered as solutions here, 2 marks for first round (e.g. even a, b any integer), final mark for second round.	

Question 9 (14 marks) Use the Question 9 Writing Booklet

(a) (i) Show that, for $\theta > 0$, **1**

$$\tan^{-1}(\theta) + \tan^{-1}\left(\frac{1}{\theta}\right) = \frac{\pi}{2}.$$

ai, let $\alpha = \tan^{-1}(\theta)$ & $\beta = \tan^{-1}\left(\frac{1}{\theta}\right)$

$\tan(\alpha) = \theta$

$\tan(\beta) = \frac{1}{\theta}$

$\theta = \cot(\beta)$

$\theta = \tan\left(\frac{\pi}{2} - \beta\right)$

$\therefore \tan(\alpha) = \tan\left(\frac{\pi}{2} - \beta\right)$

$\alpha = \frac{\pi}{2} - \beta$

$\therefore \alpha + \beta = \frac{\pi}{2}$

✓

(ii) Define the function **4**

$$T(x) = \int_0^x \frac{\tan^{-1}(y)}{y}\, dy.$$

Using part (i), or otherwise, show that for $x > 0$

$$T(x) - T\left(\frac{1}{x}\right) = \frac{\pi}{2}\ln x.$$

ii) $T(x) - T\left(\frac{1}{x}\right) = \int_0^x \frac{\tan^{-1}(y)}{y}\,dy - \int_0^{1/x} \frac{\tan^{-1}(y)}{y}\,dy$

$= \int_{1/x}^{x} \frac{\tan^{-1}(y)}{y}\,dy$ ✓

let $u = \frac{1}{y}$

$du = -\frac{1}{y^2}\,dy$

$-\frac{1}{u^2}\,du = dy$

$= \int_x^{1/x} \frac{\tan^{-1}\left(\frac{1}{u}\right)}{1/u} \times -\frac{1}{u^2}\,du$

$= \int_{1/x}^{x} \frac{\tan^{-1}\left(\frac{1}{u}\right)}{u}\,du$

$= \int_{1/x}^{x} \frac{\tan^{-1}\left(\frac{1}{y}\right)}{y}\,dy$

$\therefore\ 2\left(T(x) - T\left(\frac{1}{x}\right)\right) = \int_{1/x}^{x} \frac{\tan^{-1}(y)}{y}\,dy + \int_{1/x}^{x} \frac{\tan^{-1}(1/y)}{y}\,dy$ ✓

$= \int_{1/x}^{x} \frac{\tan^{-1}(y) + \tan^{-1}(1/y)}{y}\,dy$

$= \int_{1/x}^{x} \frac{\pi/2}{y}\,dy$ ✓

$= \frac{\pi}{2}\left[\ln|y|\right]_{1/x}^{x}$

$2\left(T(x) - T\left(\frac{1}{x}\right)\right) = \frac{\pi}{2}\left(\ln x - \ln\left(\frac{1}{x}\right)\right)$ ✓

$= \pi \ln x$

$\therefore\ T(x) - T\left(\frac{1}{x}\right) = \frac{\pi}{2}\ln x.$

Note: (1) Students who used the Fundamental theorem of Calculus failed to justify that the constant of integration must be zero for all values of x.

(2) The majority of students used integration by parts but they failed to show that $\ln x \tan^{-1} x$ approaches zero as $x \to 0$. Also they failed to show the $\int v\,du$

part of the integration is zero.

(b) (i) Prove that the product of three consecutive numbers is always divisible by 6. 3

bi, Let the 3 numbers be $n, n+1, n+2$.

Consider $\frac{n(n+1)(n+2)}{6}$

$= \frac{n(n+1)(n+2)}{3!}$

$= \frac{(n+2)!}{3!(n-1)!}$ ✓

$= \frac{(n+2)!}{3!((n+2)-3)!}$

$= \binom{n+2}{3}$ ✓, which is an integer since all binomial coeffs. are integers

$\therefore \frac{n(n+1)(n+2)}{6} = k$ for some $k \in \mathbb{Z}$

$n(n+1)(n+2) = 6k$, $k \in \mathbb{Z}$ ✓

$\therefore$ product of 3 consecutive numbers is div. by 6.

Note: (1) Alternative methods such proof by mathematical induction is also a valid approach.

(2) students must consider the same set of consecutive numbers in their proofs.

(ii) Hence or otherwise, prove that for all natural numbers n, $\frac{n}{3}+\frac{n^2}{2}+\frac{n^3}{6}$ is a natural number. 2

$\frac{n}{3}+\frac{n^2}{2}+\frac{n^3}{6} = \frac{2n+3n^2+n^3}{6}$

$= \frac{n(2+3n+n^2)}{6}$

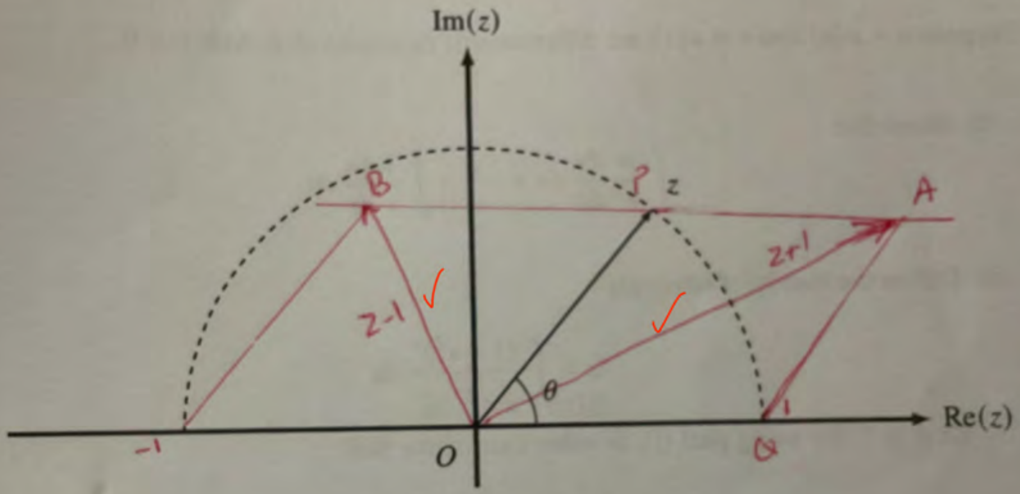
$= \frac{n(n+1)(n+2)}{6}$ ✓

$= \frac{6K}{6}$, $K \in \mathbb{N}$ (from i) ✓

$= K$

$\therefore \frac{n}{3}+\frac{n^2}{2}+\frac{n^3}{6} \in \mathbb{N}$.

(c) A complex number z lies on the upper half of a unit semi-circle.

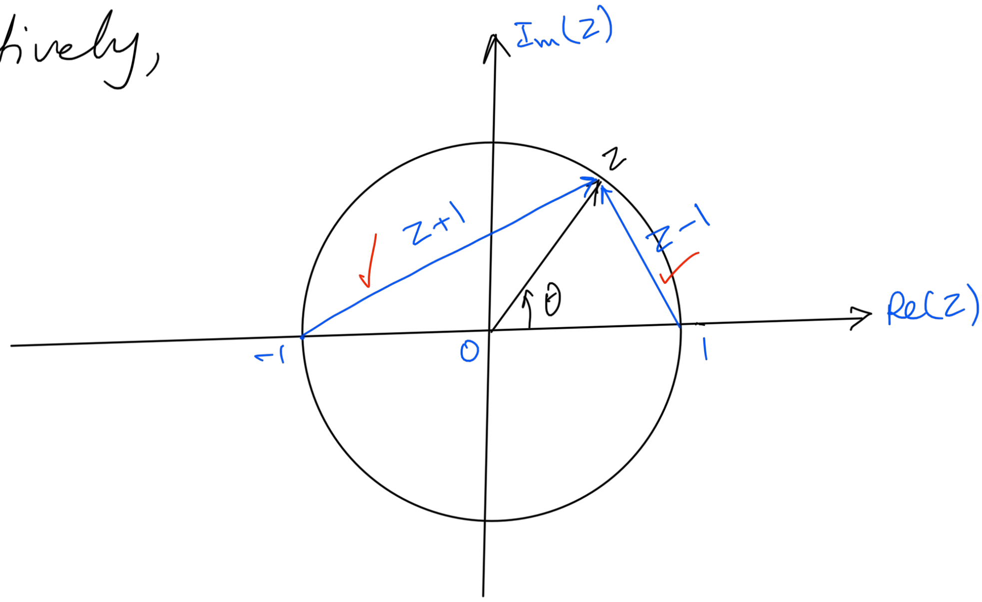


−1 mark if point B lies on the unit circle.

Copy the diagram neatly onto your page.

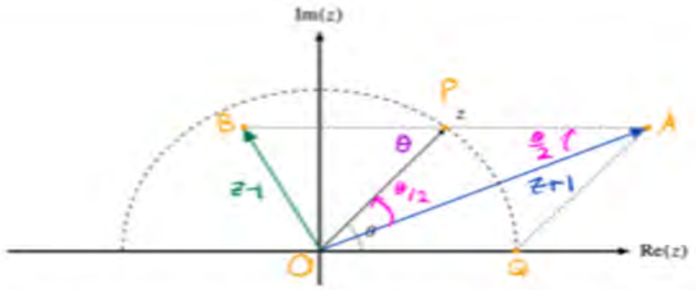
(i) Sketch the vectors $z + 1$ and $z - 1$ on the same diagram. **2**

Alternatively,



(*)

(ii) Using your diagram, or otherwise, prove that $\operatorname{Arg}\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$. **2** ci,



ii, Consider the labelling O, A, B, P, Q as above.

Note that OPAQ is a rhombus since all side lengths are 1.

$\therefore$ OA bisects $\angle POQ$

$\therefore \operatorname{Arg}(z) - \operatorname{Arg}(z+1) = \frac{\theta}{2}$ ①

Also, $OP = BP = 1 \Rightarrow \triangle OPB$ is isosceles

$\angle BPO = \angle POQ = \theta$ (alternate $\angle$'s)

$\therefore \angle BOP = \frac{\pi - \theta}{2}$ (angle sum of $\triangle$)

$\therefore \arg(z-1) - \arg(z) = \frac{\pi-\theta}{2}$ ②

① + ②: $\operatorname{Arg}(z-1) - \operatorname{Arg}(z+1) = \frac{\pi}{2}$

$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$

Alternatively, considering the diagram (*) you can notice easily that the angle between $z+1$ and $z-1$ is $\frac{\pi}{2}$ (angle in semi-circle).

- The two marks will be awarded for correct proof and explanation.

(iii) Hence, prove that $\operatorname{Arg}(z) = 2\tan^{-1}\left|\frac{z-1}{z+1}\right|$.

iii, From diagram, $\tan\frac{\theta}{2} = \frac{|z-1|}{|z+1|}$

$\frac{\theta}{2} = \tan^{-1}\left|\frac{z-1}{z+1}\right|$

$\theta = 2\tan^{-1}\left|\frac{z-1}{z+1}\right|$

$\therefore \operatorname{Arg}(z) = 2\tan^{-1}\left|\frac{z-1}{z+1}\right|$

Note: The majority of students need to revise the properties of special quadrilaterals.

Question 10a)

Wednesday, 16 March 2022 6:32 pm

f, $\int_{\pi/3}^{\pi/2} \frac{1}{2-\cos x}\,dx$

$= \int_{1/\sqrt{3}}^{1} \frac{1}{2-\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt$

let $t = \tan\frac{x}{2}$

$\frac{2dt}{1+t^2} = dx$

$x = \pi/2 \to t = 1$

$x = \pi/3 \to t = \frac{1}{\sqrt{3}}$

algebra very poor

$= \int_{1/\sqrt{3}}^{1} \frac{2}{2+2t^2-1+t^2}\,dt$

$= \int_{1/\sqrt{3}}^{1} \frac{2}{1+3t^2}\,dt$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}t)\right]_{1/\sqrt{3}}^{1}$

$= \frac{2}{\sqrt{3}}\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}(1)\right)$

$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \frac{\pi}{6\sqrt{3}} = \frac{\pi\sqrt{3}}{18}$

1m bounds

1m substitution

1m resut

- Use the reference sheet
- No mixed pronumerals

Question 10:

a, Consider $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$= \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$$

Integrate both sides w.r.t. x

$$\int \frac{d}{dx}\left(\frac{u}{v}\right)dx = \int \frac{1}{v}\frac{du}{dx}\,dx - \int \frac{u}{v^2}\frac{dv}{dx}\,dx$$

$$\frac{u}{v} = \int \frac{1}{v}\frac{du}{dx}\,dx - \int \frac{u}{v^2}\frac{dv}{dx}\,dx$$

$$\therefore \int \frac{u}{v^2}\frac{dv}{dx}\,dx = -\frac{u}{v} + \int \frac{1}{v}\frac{du}{dx}\,dx$$

(i) $u = u(x)$ $\quad v = v(x)$

$$\int \frac{u}{v^2} \frac{dv}{dx} \, dx$$

Let $U = u$ $\quad V' = \frac{1}{v^2}$

$U' = \frac{du}{dx}$ $\quad V = \frac{-1}{v}$

$$= \frac{-u}{v} - \int -\frac{1}{v} \frac{du}{dx} \, dx$$

$$= \frac{-u}{v} + \int \frac{1}{v} \frac{du}{dx} \, dx$$

ii) $I_n = \int \frac{(1+x^2)^n}{x^2} \, dx$

Let $u = (1+x^2)^n$ and $v = x$

$\therefore \frac{du}{dx} = n(1+x^2)^{n-1} 2x$ and $\frac{dv}{dx} = 1$

Then $I_n = \int \frac{u}{v^2} \frac{dv}{dx} \, dx$

$$= -\frac{u}{v} + \int \frac{1}{v} \frac{du}{dx} \, dx \quad \text{, from (i)}$$

$$= \frac{-(1+x^2)^n}{x} + \int \frac{1}{x} \times 2nx (1+x^2)^{n-1} \, dx$$

$$= -\frac{(1+x^2)^n}{x} + 2n \int (1+x^2)^{n-1} \, dx$$

$$= -\frac{(1+x^2)^n}{x} + 2n \int \frac{x^2 (1+x^2)^{n-1}}{x^2} \, dx$$

$$= -\frac{(1+x^2)^n}{x} + 2n \int \frac{[(1+x^2) - 1](1+x^2)^{n-1}}{x^2} \, dx$$

$$= -\frac{(1+x^2)^n}{x} + 2n \int \frac{(1+x^2)^n}{x^2} \, dx - 2n \int \frac{(1+x^2)^{n-1}}{x^2} \, dx$$

$$I_n = -\frac{(1+x^2)^n}{x} + 2n I_n - 2n I_{n-1}$$

$$(2n-1) I_n = \frac{(1+x^2)^n}{x} + 2n I_{n-1}$$

$$\therefore I_n = \frac{1}{2n-1} \frac{(1+x^2)^n}{x} + \frac{2n}{2n-1} I_{n-1}$$

Question 10c)

Wednesday, 16 March 2022 6:34 pm

bi, RTP: $e^{i\frac{2\pi k}{n}}$ is primitive $\Rightarrow$ k and n are coprime

Let $\omega = e^{i\frac{2\pi k}{n}}$ be a primitive n^{th} root of unity.

Suppose, for contradiction, that k and n are not coprime.
Then k and n have a common factor, say, d.

Thus $k = ad$ & $n = bd$, where $a, b, d \in \mathbb{Z}^+$ and $1 < a < k$, $1 < b < n$ (1)

many students said $\frac{k}{n} = p, p \in \mathbb{Z}$

$\therefore \frac{k}{n} = \frac{a}{b}$

$\Rightarrow \omega = e^{i\frac{2\pi a}{b}}$

$\Rightarrow \omega^b = 1$ (1)

Which contradicts the fact that ω is a primitive n^{th} root of unity, since $b < n$

So our assumption that k and n are not coprime is false. (1) statement

$\therefore$ k and n are coprime

iii, 1 and 5 are the only numbers less than 6 which are coprime with 6.

$\therefore$ primitive 6th roots of unity $= e^{i\frac{2\pi}{6}}, e^{i\frac{10\pi}{6}}$

$= e^{i\pi/3}, e^{i\frac{5\pi}{3}}$

$= e^{i\pi/3}, e^{-i\pi/3}$ (1)

Many students ignored the definition given and just used roots of unity

iv, $Q_6(x) = (x - e^{i\pi/3})(x - e^{-i\pi/3})$ (1)

$= x^2 - 2\cos(\frac{\pi}{3})x + 1$

$\therefore Q_6(x) = x^2 - x + 1$ (1)

* Many students did not write a polynomial in x. 0m

* −1 for including $(x-1)$
Use the "hint"