Student Name: ______________________

Maths class: ______________________

James Ruse Agricultural High School

2022 **YEAR 12 Task 3 Examination**

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 90 min
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 6–10, show relevant mathematical reasoning and/ or calculations

Total marks: 40

Section I – 5 marks

- Attempt Questions 1–5
- Allow about 7 minutes for this section

Section II – 60 marks

- Attempt Questions 6–10
- Allow about 83 minutes for this section

Section I

5 marks
Attempt Questions 1–5
Allow about 7 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5.

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1. Vector $\underset{\sim}{u} = 3\underset{\sim}{i} - 5\underset{\sim}{j} + 4\underset{\sim}{k}$, and $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0}$. Which of the following is the unit vector of $\underset{\sim}{v}$?

(A) $-\frac{3\sqrt{2}}{10}\underset{\sim}{i} + \frac{\sqrt{2}}{2}\underset{\sim}{j} - \frac{4\sqrt{2}}{5}\underset{\sim}{k}$

(B) $\frac{3\sqrt{2}}{10}\underset{\sim}{i} - \frac{\sqrt{2}}{2}\underset{\sim}{j} + \frac{4\sqrt{2}}{5}\underset{\sim}{k}$

(C) $-\frac{3\sqrt{2}}{10}\underset{\sim}{i} + \frac{\sqrt{2}}{2}\underset{\sim}{j} - \frac{2\sqrt{2}}{5}\underset{\sim}{k}$

(D) $\frac{3\sqrt{2}}{10}\underset{\sim}{i} - \frac{\sqrt{2}}{2}\underset{\sim}{j} + \frac{2\sqrt{2}}{5}\underset{\sim}{k}$

2. Which of the following vector equations represents a line passing through the point $A\,(-7, 1, 3)$ and is perpendicular to the vector $\underset{\sim}{d} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$?

(A) $\underset{\sim}{r} = \begin{pmatrix} -7 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

(B) $\underset{\sim}{r} = \begin{pmatrix} -7 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

(C) $\underset{\sim}{r} = \begin{pmatrix} -7 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

(D) $\underset{\sim}{r} = \begin{pmatrix} -7 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

3. How many of the following vector equations is the same as the Cartesian equation $3x + 2y = 6$?

I. $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ for $\lambda \in \mathbb{R}$

II. $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ for $\lambda \in \mathbb{R}$

III. $\underset{\sim}{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ for $\lambda \in \mathbb{R}$

IV. $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ for $\lambda \in \mathbb{R}$

(A) 0

(B) 1

(C) 2

(D) 3

4. For $3^n < n!$ where $n \in \mathbb{Z}^+$, the statement is true for

(A) $n \geq 7$

(B) $n \geq 6$

(C) $n \geq 3$

(D) All $n \in \mathbb{Z}^+$

5. If a and b are real and c and d are imaginary, which of the following inequalities will be true.

(A) $a^2c^2 + b^2d^2 \geq 2abcd$ and $a^2b^2 + c^2d^2 \geq 2abcd$

(B) $a^2c^2 + b^2d^2 \leq 2abcd$ and $a^2b^2 + c^2d^2 \geq 2abcd$

(C) $a^2c^2 + b^2d^2 \leq 2abcd$ and $a^2b^2 + c^2d^2 \leq 2abcd$

(D) $a^2c^2 + b^2d^2 \geq 2abcd$ and $a^2b^2 + c^2d^2 \leq 2abcd$

Section II begins on the next page

Section II

60 marks
Attempt Questions 6-12
Allow about 83 minutes for this section

Start each question on a new page. Extra paper is available.

In Questions 6-10, your responses should include relevant mathematical reasoning and/ or calculations.

Question 6 (12 marks) **Start a new page**.

(a)	Determine whether the point $Q\ (-3,1,4)$ lies on the line $\underset{\sim}{r} = \begin{pmatrix} 6 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$, giving reasons.	**2**
(b)	Let P, A and B be the points $(-3,1,-2)$, $(-4,2,-3)$ and $(-2,3,-1)$, respectively. (i) Find $\text{proj}_{\underset{\sim}{a}}\underset{\sim}{b}$, where $\overrightarrow{PA} = \underset{\sim}{a}$, $\overrightarrow{PB} = \underset{\sim}{b}$. (ii) Find the perpendicular distance from the point B to the line PA.	 **2** **2**
(c)	The point $P(x,y,z)$ lies on the sphere of radius 1 centred at the origin O. Prove that $\lvert xy\rvert + \lvert yz\rvert + \lvert xz\rvert \leq 1$.	**3**
(d)	Use Mathematical Induction to prove that, for $n \geq 1$ $$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} < \frac{3}{2}.$$	**3**

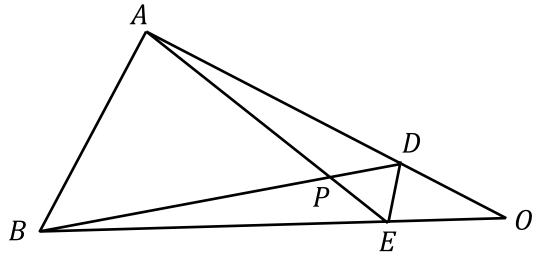
End of Question 6.

Question 7 (12 marks) **Start a new page**.

(a) Determine whether it is possible to find numbers p and q which satisfy the vector equation **3**

$$p\begin{pmatrix}2\\-4\\3\end{pmatrix}-q\begin{pmatrix}-3\\7\\2\end{pmatrix}=\begin{pmatrix}-2\\2\\8\end{pmatrix}.$$

(b) In ΔABO shown below, let $\overrightarrow{OD}=\underset{\sim}{a}$, $\overrightarrow{OE}=\underset{\sim}{b}$.



It is given that $\overrightarrow{DA}=3\overrightarrow{OD}$ and $\overrightarrow{EB}=4\overrightarrow{OE}$.

(i) Prove that $\overrightarrow{AB}=5\underset{\sim}{b}-4\underset{\sim}{a}$. **1**

(ii) If $\left|\overrightarrow{AB}\right|=\left|\overrightarrow{AD}\right|$ and $\left|\overrightarrow{OD}\right|=\left|\overrightarrow{OE}\right|$, use vector methods to find the value of $\cos(\angle AOB)$. **3**

AE and BD intersect at the point P

(iii) Find a vector equation of line AE. **1**

(iv) Find a vector equation of line BD. **1**

(v) Find $\overrightarrow{OP}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **3**

End of Question 7.

Question 8 (12 marks) **Start a new page**.

(a)	(i) Prove for integers $n > r > 1$, $\binom{n}{r} < n\binom{n-1}{r-1}$.	**2**
	(ii) Given that $$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$ show that for $a, b > 0$ and integers $n > 1$, $(a+b)^n - a^n < nb(a+b)^{n-1}$.	**2**
(b)	Suppose $0 \le t \le \frac{1}{\sqrt{2}}$.	
	(i) Show that $0 \le \frac{2t^2}{1-t^2} \le 4t^2$.	**2**
	(ii) Hence show that $0 \le \frac{1}{1+t} + \frac{1}{1-t} - 2 \le 4t^2$.	**2**
	(iii) By integrating the result in part (ii) from $t = 0$ to $t = x$ (where $0 \le x \le \frac{1}{\sqrt{2}}$), show that $0 \le \log_e\left(\frac{1+x}{1-x}\right) - 2x \le \frac{4x^3}{3}$.	**2**
	(iv) Hence show that for $0 \le x \le \frac{1}{\sqrt{2}}$, $1 \le \left(\frac{1+x}{1-x}\right)e^{-2x} \le e^{\frac{4x^3}{3}}$.	**2**

End of Question 8.

Question 9 (12 marks) **Start a new page**.

(a)	Let $a + b + c = 1$ where a, b, and c are positive real numbers. Find the smallest possible value of $\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right)$.	**3**
(b)	(i) Let a be a positive real number. Show that $a + \frac{1}{a} \geq 2$.	**1**
	(ii) If $a^n + \frac{1}{a^n} \geq a^{n-1} + \frac{1}{a^{n-1}}$ for all positive integers $n \geq 1$, prove $a^{n+1} + \frac{1}{a^{n+1}} \geq a^n + \frac{1}{a^n}$.	**2**
	(iii)Let n be a positive integer and $a_1, a_2, \dots, a_n$ be n positive real number. Prove by induction that $(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$.	**4**
	(iv)Hence show that $\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta \geq 9$.	**2**

End of Question 9.

Question 10 (12 marks) **Start a new page**.

(a)	The function f is defined by $f(x) = \frac{1}{2}\left(1 - \sqrt{1-x}\right)$ for $0 < x < 1$. Given that $f_1 = f$ and $f_n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ compositions}}$, prove by Mathematical Induction that for $0 < \theta < \frac{\pi}{2}$ and positive integers n $$f_n(\sin^2\theta) = \sin^2\left(\frac{\theta}{2^n}\right).$$	**5**
(b)	In three-dimensional space, the origin is O. The sphere S has its centre at the point C, where $\overrightarrow{OC} = \underset{\sim}{c}$. The radius of the sphere is r, and the position vector of a point on the sphere is $\underset{\sim}{m}$. It is given that $\left\lvert \underset{\sim}{m} - \underset{\sim}{c} \right\rvert = r$. The line ℓ is tangent to the sphere S at the point $\underset{\sim}{p}$ and is parallel to the vector $\underset{\sim}{n}$.	
	(i) State a vector equation of the line ℓ.	**1**
	(ii) Prove that $(\underset{\sim}{p} - \underset{\sim}{c}) \cdot \underset{\sim}{n} = 0$, giving reasons.	**2**
	(iii) Show that $\lambda = \dfrac{\sqrt{r^2 - \lvert \underset{\sim}{p} - \underset{\sim}{c} \rvert^2}}{\lvert \underset{\sim}{n} \rvert}$.	**4**

END of PAPER

Multiple choice:

1. C
2. D
3. B
4. A
5. B

<table>
<tr><th colspan="3">Year 12 Extension 2: Q6</th></tr>
<tr><th>Suggested Solutions</th><th>Marks Awarded</th><th>Marker's Comments</th></tr>
<tr><td>

(a)

The point

$$Q(-3,1,4)$$

lies on the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -5 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

iff $\exists \lambda$ such that

$$\begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -5 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

By row 1,

$$-3 = 6 + 3\lambda \to \lambda = -3$$

We check this parameter value with row 2 to find

$$1 \neq 1 + (-3)(3)$$

Hence the point Q does **NOT** lies on the line.

</td><td>1
First mark for identifying what had to be done (i.e. parameter consistency check).

1
Final mark for showing at least two equations were inconsistent relative to λ and its conseq-uence.</td><td></td></tr>
<tr><td>

(b) Let

$$P = (-3,1,-2), \qquad A = (-4,2,-3), \qquad B = (-2,3,-1)$$

(i) We have:

$$\overrightarrow{PA} = \begin{bmatrix} -4-(-3) \\ 2-1 \\ -3-(-2) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \coloneqq \boldsymbol{a}$$

and

$$\overrightarrow{PB} = \begin{bmatrix} -2-(-3) \\ 3-1 \\ -1-(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \coloneqq \boldsymbol{b}$$

Then

$$\text{proj}_{\boldsymbol{a}} \boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|^2} \boldsymbol{b}$$

Now $\boldsymbol{a} \cdot \boldsymbol{b} = (-1)(1) + (1)(2) + (-1)(1) = 0$, which means

$$\text{proj}_{\boldsymbol{a}} \boldsymbol{b} = 0\boldsymbol{b} = \boldsymbol{0}$$

The projection is the zero vector.

</td><td>1
First mark for obtaining both vectors correctly.

1
Final mark for showing the projection was the zero vector (not scalar).</td><td></td></tr>
</table>

(ii) Since $\overrightarrow{PB} \cdot \overrightarrow{PA} = 0$, the line segment PB must be perpendicular to the line through PA. Hence the perpendicular distance is simply the magnitude of $\boldsymbol{b}$: $$\lvert\boldsymbol{b}\rvert = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$$ units.	1 First mark for justification (i.e. the line segments are at right angles as a result of the dot product being 0. 1 Final mark for correct distance.	
(c) Consider that $$0 \leq (\lvert x\rvert - \lvert y\rvert)^2 + (\lvert y\rvert - \lvert z\rvert)^2 + (\lvert z\rvert - \lvert x\rvert)^2$$ $$= 2(x^2 + y^2 + z^2) - 2(\lvert x\rvert\lvert y\rvert + \lvert y\rvert\lvert z\rvert + \lvert z\rvert\lvert x\rvert)$$ $$= 2 \cdot 1 - 2(\lvert xy\rvert + \lvert yz\rvert + \lvert zx\rvert)$$ where the last line follows as $x^2 + y^2 + z^2 = 1$. Rearranging, we have $$\lvert xy\rvert + \lvert yz\rvert + \lvert zx\rvert \leq 1$$	1 First mark for appropriate setup. 1 One mark for correct use of the restriction $x^2 + y^2 + z^2 = 1$. 1 Final mark for correct conclusion.	If non-standard inequalities (i.e. those not listed in the syllabus) were used without proof, even if true, full marks were not awarded.

(d) We need to show that $$S(n):\quad 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}<2$$ is the case $\forall n \in \mathbb{N}$. Consider now that $$1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}=1+\frac{1}{2}\left(1+\frac{1}{2}+\cdots\frac{1}{2^n}\right)$$ $$<1+\frac{1}{2}(2) \quad \text{if } S(n) \text{ is true}$$ $$=2$$ Hence, if the claim is true for case n, it follows it's true for case $n+1$. Now, for $n=1$, we have $$\text{LHS}=1+\frac{1}{2}=\frac{3}{2}<2=\text{RHS}$$ Since the claim is true for $n=1$, and since it is true for $n+1$ if true for n, it follows that the claim is true for all integers $n \geq 1$.	1 First mark for testing of a base case. 1 Second mark for correct use of hypothesis in the proof (which includes **stating** its use when used). 1 Final mark for correct, logical conclusion.	Many students used sum of geometric series: $1+\frac{1}{2}+\cdots+\frac{1}{2^n}$ $=2-\frac{1}{2^n}$ as a starting point, but if the induction proof had been worked properly, this path would have led nowhere helpful. If the sum is used, then you're forced to take as hypothesis $$2-\frac{1}{2^n}<2$$ as true. Next step would be to show that if this is true, then $$1+\frac{1}{2}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}$$ $$=\left(2-\frac{1}{2^n}\right)+\frac{1}{2^{n+1}}$$ $$<2+\frac{1}{2^{n+1}} \quad (\text{by hyp.})$$ which isn't less than 2. Using the geometric sum doesn't help. It also makes the fact that the sum is less than 2 so obvious that working with it gets in the way.

MATHEMATICS Extension 2: Question……7…

Suggested Solutions	Marks	Marker's Comments
a) $2p + 3q = -2 \quad - ①$ $-4p - 7q = 2 \quad - ②$ $3p - 2q = 8 \quad - ③$	1	show the equation for the z component as well.
from ①: $q = \dfrac{-2-2p}{3}$		
sub. q into ②: $-4p - 7\left(\dfrac{-2-2p}{3}\right) = 2$ $-12p + 14 + 14p = 6$ $2p = -8$ $p = -4$ $\therefore q = 2$	1	there are other values for p and q
sub. $p=-4$ and $q=2$ into ③ $LHS = 3(-4) - 2(2)$ $= -16 \neq 8$ $\therefore$ no possible p and q	1	need to have conclusion
b) i) $\vec{AB} = \vec{AO} + \vec{OB}$ $= \vec{AD} + \vec{DO} + \vec{OE} + \vec{EB}$ $= -3\underset{\sim}{a} - \underset{\sim}{a} + \underset{\sim}{b} + 4\underset{\sim}{b}$ $= 5\underset{\sim}{b} - 4\underset{\sim}{a}$	1	
ii) $\lvert\vec{AB}\rvert = \lvert\vec{AD}\rvert$ $\lvert 5\underset{\sim}{b} - 4\underset{\sim}{a}\rvert = \lvert 3\underset{\sim}{a}\rvert$ $(5\underset{\sim}{b} - 4\underset{\sim}{a})^2 = (3\underset{\sim}{a})^2$ $25\underset{\sim}{b}\cdot\underset{\sim}{b} + 16\underset{\sim}{a}\cdot\underset{\sim}{a} - 40\underset{\sim}{a}\cdot\underset{\sim}{b} = 9\underset{\sim}{a}\cdot\underset{\sim}{a}$ $25\lvert\underset{\sim}{b}\rvert^2 + 16\lvert\underset{\sim}{a}\rvert^2 - 40\underset{\sim}{a}\cdot\underset{\sim}{b} = 9\lvert\underset{\sim}{a}\rvert^2$	1	Starting with $\cos\theta = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert}$ leads nowhere and hence received 0 marks

MATHEMATICS Extension 2: Question……7…

Suggested Solutions	Marks	Marker's Comments
as $\lvert\overrightarrow{OD}\rvert = \lvert\overrightarrow{OE}\rvert$		
$\lvert \underset{\sim}{a} \rvert = \lvert \underset{\sim}{b} \rvert$		
$\therefore \lvert \underset{\sim}{a} \rvert^2 = \lvert \underset{\sim}{b} \rvert^2$		
$\therefore 25\lvert \underset{\sim}{a} \rvert^2 + 16\lvert \underset{\sim}{a} \rvert^2 - 40\underset{\sim}{a}\cdot\underset{\sim}{b} = 9\lvert \underset{\sim}{a} \rvert^2$		
$32\lvert \underset{\sim}{a} \rvert^2 = 40\underset{\sim}{a}\cdot\underset{\sim}{b}$	1	
$\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert \underset{\sim}{a} \rvert^2} = \dfrac{32}{40}$		
$\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert \underset{\sim}{a} \rvert \lvert \underset{\sim}{b} \rvert} = \dfrac{4}{5}$		
$\therefore \cos\angle AOB = \dfrac{4}{5}$	1	
iii) line AE: $\underset{\sim}{r} = \underset{\sim}{b} + \lambda(\underset{\sim}{b} - 4\underset{\sim}{a})$, $\lambda \in \mathbb{R}$	1	
or $\underset{\sim}{r} = 4\underset{\sim}{a} + \lambda(\underset{\sim}{b} - 4\underset{\sim}{a})$		
iv) line BD: $\underset{\sim}{r} = \underset{\sim}{a} + \mu(\underset{\sim}{a} - 5\underset{\sim}{b})$, $\mu \in \mathbb{R}$	1	
or $\underset{\sim}{r} = 5\underset{\sim}{b} + \mu(\underset{\sim}{a} - 5\underset{\sim}{b})$		
v) P is the intersection of AE and BD		
$\underset{\sim}{b} + \lambda(\underset{\sim}{b} - 4\underset{\sim}{a}) = \underset{\sim}{a} + \mu(\underset{\sim}{a} - 5\underset{\sim}{b})$		
$-4\lambda\underset{\sim}{a} + (1+\lambda)\underset{\sim}{b} = (1+\mu)\underset{\sim}{a} - 5\mu\underset{\sim}{b}$		
equating coefficients:		
$-4\lambda = 1 + \mu \quad - (1)$ $1 + \lambda = -5\mu$	$\}$ 1	
$\lambda = -1 - 5\mu \quad - (2)$		
sub. (2) into (1)		

MATHEMATICS Extension 2: Question…7…

Suggested Solutions	Marks	Marker's Comments
$-4(-1-5\mu) = 1+\mu$		
$4+20\mu = 1+\mu$		
$19\mu = -3$		
$\mu = -\frac{3}{19}$	1	
$\therefore \vec{OP} = \underset{\sim}{a} + \left(-\frac{3}{19}\right)(\underset{\sim}{a} - 5\underset{\sim}{b})$		
$= \frac{16}{19}\underset{\sim}{a} + \frac{15}{19}\underset{\sim}{b}$	1	

Question 8 :

(a) (i) $n > r > 1$, $\binom{n}{r} < n\binom{n-1}{r-1}$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n}{r} \times \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= \frac{n}{r}\binom{n-1}{r-1}$$ ✓

$r > 1 \therefore \frac{1}{r} < 1$ ✓

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1} < n\binom{n-1}{r-1}.$$

(ii) $$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n}b^n$$

$$(a+b)^n - a^n = \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n}b^n$$ ✓

$$\leqslant n\binom{n-1}{0}a^{n-1}b + n\binom{n-1}{1}a^{n-2}b^2 + \dots + n\binom{n-1}{n-1}b^n$$

$$= nb\left[\binom{n-1}{0}a^{n-1} + \binom{n-1}{1}a^{n-2}b + \dots + \binom{n-1}{n-1}b^{n-1}\right]$$ ✓

$$= nb\,(a+b)^{n-1}.$$

(8) $0 \le t \le \frac{1}{\sqrt{2}}$

(i) $0 \le t \le \frac{1}{\sqrt{2}} \quad \therefore \quad 0 \le t^2 \le \frac{1}{2} \quad \therefore \quad 0 \le 2t^2 \le 1$

$0 \le t^2 \le \frac{1}{2} \quad \therefore \quad -\frac{1}{2} \le -t^2 \le 0 \quad \therefore \quad \frac{1}{2} \le 1-t^2 \le 1$

$\therefore \quad 1 \le \frac{1}{1-t^2} \le 2$

$\therefore \quad 2t^2 \le \frac{2t^2}{1-t^2} \le 4t^2$

$\therefore \quad 0 \le \frac{2t^2}{1-t^2} \le 4t^2$. since $2t^2 \geq 0$

(ii) $\frac{2t^2}{1-t^2} = \frac{2t^2-2+2}{1-t^2} = \frac{2(t^2-1)}{1-t^2} + \frac{2}{1-t^2}$

$= -2 + \frac{2}{1-t^2}$

$= -2 + \frac{1}{1-t} + \frac{1}{1+t}$

$\therefore \quad 0 \le \frac{1}{1+t} + \frac{1}{1-t} - 2 \le 4t^2$

(iii) $\int_0^x 0\, dt \le \int_0^x \left(\frac{1}{1+t} + \frac{1}{1-t} - 2\right) dt \le \int_0^x 4t^2\, dt$

$0 \le \Big[\ln|1+t| - \ln|1-t| - 2t\Big]_0^x \le \left[\frac{4t^3}{3}\right]_0^x$

$0 \le \ln\left(\frac{1+x}{1-x}\right) - 2x \le \frac{4x^3}{3}$ $\quad \frac{1+x}{1-x} > 0$

vi) $0 \leq \log_e\left(\frac{1+x}{1-x}\right) - 2x \leq \frac{4x^3}{3}$

$$e^0 \leq e^{\log_e\left(\frac{1+x}{1-x}\right) - 2x} \leq e^{4x^3/3}$$

$$1 \leq e^{\log_e\left(\frac{1+x}{1-x}\right)} e^{-2x} \leq e^{4x^3/3}$$

$$1 \leq \left(\frac{1+x}{1-x}\right) e^{-2x} \leq e^{4x^3/3}.$$

Question 9

Sunday, 24 July 2022 10:21 pm

Let $a+b+c=1$ where a, b, and c are positive real numbers. Find the smallest possible value of $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$.	3

a) $(\sqrt{x}-\sqrt{y})^2 \geq 0$

$x - 2\sqrt{xy} + y \geq 0$

$\therefore\ x+y \geq 2\sqrt{x}$

$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$

$= \frac{(1-a)(1-b)(1-c)}{abc}$

$= \frac{(b+c)(a+c)(a+b)}{abc}$ since $a+b+c=1$ ① using $a+b+c=1$ following some helpful manipulation of $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$

$\geq \frac{2\sqrt{bc}\times 2\sqrt{ac}\times 2\sqrt{ab}}{abc}$ since $x+y \geq 2\sqrt{x}$ ① using AM-GM inequality

$= \frac{8abc}{abc}$

$= 8$

$\therefore$ Smallest possible value is 8. ① answer

① mark for using $a+b+c=1$ following some helpful manipulation of $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$

①

b) (i) Let a be a positive real number. Show that $a+\frac{1}{a} \geq 2$.

$(a-1)^2 \geq 0$ OR $\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2 \geq 0$ ①

$a^2-2a+1 \geq 0$ $\quad$ $a-2+\frac{1}{a} \geq 0$

$a^2+1 \geq 2a$ $\quad$ $a+\frac{1}{a} \geq 2$

$a+\frac{1}{a} \geq 2$

(ii) If $a^n+\frac{1}{a^n} \geq a^{n-1}+\frac{1}{a^{n-1}}$ for all positive integers $n \geq 1$, prove $a^{n+1}+\frac{1}{a^{n+1}} \geq a^n+\frac{1}{a^n}$.

Given

$a^n + \frac{1}{a^n} \geq a^{n-1} + \frac{1}{a^{n-1}}$

we can map n to $n+1$:

$a^{n+1} + \frac{1}{a^{n+1}} \geq a^{n-1} + \frac{1}{a^{n-1}}$ ②

(iii) Let n be a positive integer and $a_1, a_2, \dots, a_n$ be n positive real number. Prove by induction that $(a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right) \geq n^2$.

(iii) Prove true for $n=1$

When $n=1$

LHS $= a_1 \times \frac{1}{a_1} = 1$

RHS $= 1^2 = 1 =$ LHS

$\therefore$ True for $n=1$ ①

Assume true for $n=k$

ie assume

$(a_1+a_2+\dots+a_k)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}\right) \geq k^2$

Prove true for $n=k+1$

ie. prove

$(a_1+a_2+\dots+a_k+a_{k+1})\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}+\frac{1}{a_{k+1}}\right) \geq (k+1)^2$

LHS $= (a_1+a_2+\dots+a_k+a_{k+1})\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}+\frac{1}{a_{k+1}}\right)$

$= (a_1+a_2+\dots+a_k)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}\right) + \frac{1}{a_{k+1}}(a_1+a_2+\dots+a_k) + a_{k+1}\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}\right) + \left(\frac{1}{a_{k+1}}\right)a_{k+1}$

$\geq k^2 + \frac{1}{a_{k+1}}(a_1+a_2+\dots+a_k) + a_{k+1}\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}\right) + 1$ ① by assumption (must have correct expressions above)

$= k^2 + \sum_{j=1}^{k}\left(\frac{a_j}{a_{k+1}} + \frac{1}{\frac{a_j}{a_{k+1}}}\right) + 1$

$\geq k^2 + \sum_{j=1}^{k} 2 + 1$ (from (i)) ①

$= k^2 + 2k + 1$

$= (k+1)^2$ ①

$=$ RHS

$\therefore$ True for $n=k+1$ + conclusion

(iv) Hence show that $\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta \geq 9$.

Let $a_1 = \cos^2\theta$, $a_2 = \sin^2\theta$, $a_3 = \tan^2\theta$ ①

$(\sin^2\theta+\cos^2\theta+\tan^2\theta)(\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta) \geq 3^2$ from (iii) need to state this

$(1+\tan^2\theta)(\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta) \geq 9$

$\sec^2\theta\,(\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta) \geq 9$

$\therefore \operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta \geq 9\cos^2\theta$ ①

Year 12 Task 3 MATHEMATICS Ext 2 Question 10

Suggested Solutions	Marks	Marker's Comments
a) Test for $n=1$		
$f_1(\sin^2\theta) = \frac{1}{2}\left(1-\sqrt{1-\sin^2\theta}\right)$		
$= \frac{1}{2}\left(1-\sqrt{\cos^2\theta}\right)$		
$= \frac{1}{2}\left(1-\lvert\cos\theta\rvert\right)$		
$= \frac{1}{2}(1-\cos\theta)$ (since $0<\theta<\frac{\pi}{2}$)	1	Explain why the absolute value is not needed
$= \frac{1}{2}\left(1-\left(1-2\sin^2\frac{\theta}{2}\right)\right)$		
$= \sin^2\frac{\theta}{2}$		
$\therefore$ The statement is true for $n=1$	1	using appropriate trig identity without skipping steps
Assume the statement is true for $n=k$, $k\in\mathbb{Z}$, $k\geq 1$		
$\therefore f_k(\sin^2\theta) = \sin^2\left(\frac{\theta}{2^k}\right)$	1	correct assumption
Test for $n=k+1$, RTP: $f_{k+1}(\sin^2\theta) = \sin^2\left(\frac{\theta}{2^{k+1}}\right)$		
$LHS = f_{k+1}(\sin^2\theta)$		
$= f\circ f_k(\sin^2\theta)$		
$= f\left(\sin^2\left(\frac{\theta}{2^k}\right)\right)$ (By assumption)	1	Substitution with the phrase "assumption"
$= \frac{1}{2}\left(1-\sqrt{1-\sin^2\left(\frac{\theta}{2^k}\right)}\right)$		
$= \frac{1}{2}\left(1-\sqrt{\cos^2\left(\frac{\theta}{2^k}\right)}\right)$		
$= \frac{1}{2}\left(1-\left\lvert\cos\left(\frac{\theta}{2^k}\right)\right\rvert\right)$		
$= \frac{1}{2}\left(1-\cos\left(\frac{\theta}{2^k}\right)\right)$ $\left(0<\frac{\theta}{2^k}<\frac{\pi}{2}\right)$		
$= \frac{1}{2}\left(1-\left(1-2\sin^2\left(\frac{\theta}{2^{k+1}}\right)\right)\right)$		
$= \frac{1}{2}\left(2\sin^2\left(\frac{\theta}{2^{k+1}}\right)\right)$		
$= \sin^2\left(\frac{\theta}{2^{k+1}}\right)$		
$\therefore$ If the statement is true for $n=k$, then it is true for $n=k+1$,	1	Finish the proof off correctly with reason.
$\therefore$ The statement is true by the principle of mathematical induction		

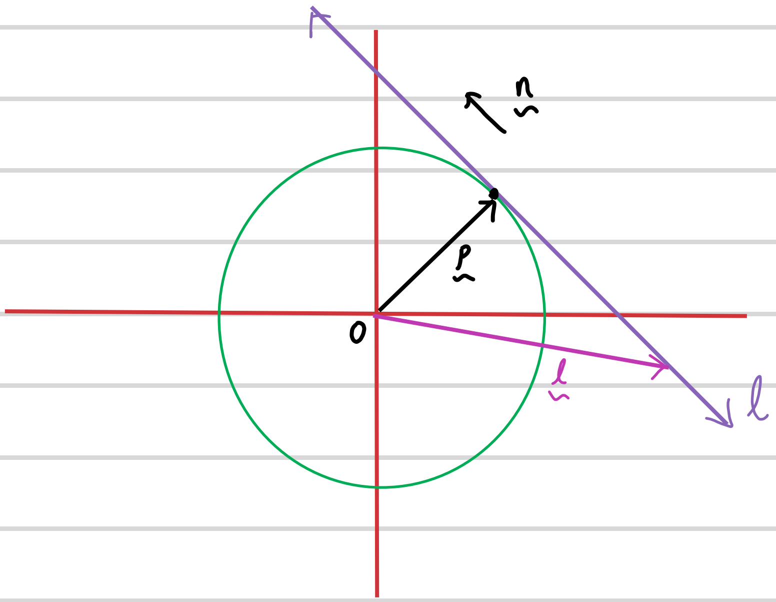
Year 12 Task 3 MATHEMATICS Ext2 Question 10

Suggested Solutions	Marks	Marker's Comments
b) i. l : $\underset{\sim}{l} = \underset{\sim}{p} + \lambda \underset{\sim}{n}$	1	
ii Since $\underset{\sim}{p} - \underset{\sim}{c}$ represents the vector from the centre to $\underset{\sim}{p}$, then it is perpendicular to $\underset{\sim}{l}$.		
(radius is perpendicular to the tangent at the point of contact)	1	circle geometry property
Since $\underset{\sim}{l}$ has direction vector $\underset{\sim}{n}$, then $\underset{\sim}{p} - \underset{\sim}{c}$ is perpendicular to $\underset{\sim}{n}$.		
$\therefore (\underset{\sim}{p} - \underset{\sim}{c}) \cdot \underset{\sim}{n} = 0$	1	connect perpendicular vectors to dot product being 0.
iii Since $\underset{\sim}{p}$ lies on $\underset{\sim}{l}$ and the surface of the sphere:		
Solving $\underset{\sim}{l} = \underset{\sim}{p} + \lambda \underset{\sim}{n}$ and $\lvert \underset{\sim}{m} - \underset{\sim}{c} \rvert = r$ simultaneously,		
$\lvert (\underset{\sim}{l} + \lambda \underset{\sim}{n}) - \underset{\sim}{c} \rvert = r$ (substitute $\underset{\sim}{l}$ into $\underset{\sim}{m}$)	1	Attempt to solve simultaneously
$\lvert (\underset{\sim}{p} - \underset{\sim}{c}) + \lambda \underset{\sim}{n} \rvert^2 = r^2$		
$((\underset{\sim}{p} - \underset{\sim}{c}) + \lambda \underset{\sim}{n}) \cdot ((\underset{\sim}{p} - \underset{\sim}{c}) + \lambda \underset{\sim}{n}) = r^2$		
$\lvert \underset{\sim}{p} - \underset{\sim}{c} \rvert^2 + 2\lambda (\underset{\sim}{p} - \underset{\sim}{c}) \cdot \underset{\sim}{n} + \lambda^2 \lvert \underset{\sim}{n} \rvert^2 = r^2$	1	Expansion
$\therefore \lvert \underset{\sim}{p} - \underset{\sim}{c} \rvert^2 + \lambda^2 \lvert \underset{\sim}{n} \rvert^2 = r^2$ ($(\underset{\sim}{p} - \underset{\sim}{c}) \cdot \underset{\sim}{n} = 0$ was proven in ii).	1	use of property in part ii
$\therefore \lambda^2 = \dfrac{r^2 - \lvert \underset{\sim}{p} - \underset{\sim}{c} \rvert^2}{\lvert \underset{\sim}{n} \rvert^2}$		
$\lambda = \dfrac{\sqrt{r^2 - \lvert \underset{\sim}{p} - \underset{\sim}{c} \rvert^2}}{\lvert \underset{\sim}{n} \rvert}$		
$= 0$ (since $\lvert \underset{\sim}{p} - \underset{\sim}{c} \rvert = r$)	1	Answer
or		
Since $\underset{\sim}{p}$ lies on $\underset{\sim}{l} = \underset{\sim}{p} + \lambda \underset{\sim}{n}$ (from i)		
then $\underset{\sim}{p} = \underset{\sim}{p} + \lambda \underset{\sim}{n}$		
$\therefore \lambda \underset{\sim}{n} = \underset{\sim}{0}$		
$\lambda = 0$ (since $\underset{\sim}{n} \neq \underset{\sim}{0}$)	4	

Year 12 Task 3 MATHEMATICS Ext 2 Question 10

Suggested Solutions	Marks	Marker's Comments

Some notes about lines:



If ℓ has equation $\ell : \underset{\sim}{l} = \underset{\sim}{p} + \lambda \underset{\sim}{n}$

$\therefore \underset{\sim}{p} \perp \ell$ NOT $\underset{\sim}{p} \perp \underset{\sim}{l}$!!!

ℓ is the tangent, $\underset{\sim}{l}$ is the variable positional vector from O to any point on the line ℓ.