Northern Beaches Secondary College
Manly Campus

2022

HSC Assessment 2

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- For questions in Section II, show relevant mathematical reasoning and / or and/or calculations

Total Marks: 43

Section I – 5 marks (pages 1–2)

- Attempt Questions 1–5
- Allow about 10 minutes for this section
- Answer on Multiple Choice Answer Sheet provided

Section II – 39 marks (pages 3–5)

- Attempt Questions 6–8
- Allow about 80 minutes for this section
- Answer each question in a separate booklet

Section I

5 marks

Attempt Questions 1 – 5

Allow about 10 minutes for this section

Use the multiple-choice answer sheet for questions 1 – 5.

Q1. The negation of the statement $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y = 2$

A. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y \neq 2$

B. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y \neq 2$

C. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y \neq 2$

D. $\exists x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y \neq 2$

Q2. A theorem states "If the shape is a triangle, then the sum of the interior angles is 180 degrees." The converse of this theorem is which of the following?

A. If the shape has a 180 degree angle, then it is a triangle

B. If the shape has three angles, then it is a triangle

C. If the sum of the three interior angles of the shape is 180 degrees, then the shape is a triangle

D. If the sum of the interior angles of a shape is 180 degrees, then the shape has three angles.

Q3. If ω is a complex cube root of unity, then $(1 + \omega - \omega^2)^7$ equals

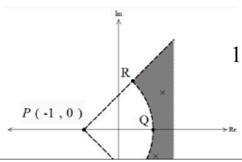
A. 128ω

B. -128ω

C. $128\omega^2$

D. $-128\omega^2$

Q4. Four points $P(-1, 0)$, $Q(1, 0)$, $R(\sqrt{2} - 1, \sqrt{2})$ and $S(\sqrt{2} - 1, -\sqrt{2})$ are given on a complex plane as shown.



The equation of the locus of the shaded region excluding the boundaries is given by

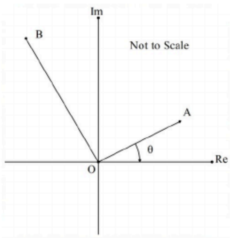
A. $|z+1| > 2$ and $|\arg(z+1)| < \dfrac{\pi}{4}$

B. $|z+1| > 2$ and $|\arg(z+1)| < \dfrac{\pi}{2}$

C. $|z-1| > 2$ and $|\arg(z-1)| < \dfrac{\pi}{4}$

D. $|z-1| > 2$ and $|\arg(z-1)| < \dfrac{\pi}{2}$

Q5. The points A and B in the diagram represent the complex numbers z_1 and z_2 respectively where $|z_1| = 1$ and $\arg(z_1) = \theta$ and $z_2 = \sqrt{3}\, iz_1$



Which of the following represents $z_2 - z_1$?

A. $2e^{i\left(\frac{2\pi}{3}+\theta\right)}$

B. $3e^{i\left(\frac{2\pi}{3}+\theta\right)}$

C. $2e^{i\left(\frac{2\pi}{3}-\theta\right)}$

D. $3e^{i\left(\frac{2\pi}{3}-\theta\right)}$

Section II marks

Attempt Questions 6 – 8

Allow about 80 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 Use a separate writing booklet. **15 Marks**

a. i. Find the square roots of $16 - 30i$ **2**

ii. Hence solve $z^2 - 2z - (15 - 30i) = 0$ **2**

b. Given the equation $z^5 - 1 = 0$

i. Determine the non-real roots of $z^5 - 1 = 0$, in terms of ω where ω is the root with the smallest positive argument.

ii. Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

iii. Hence show that $\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$

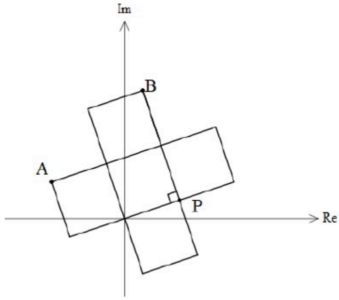
iv. Hence show that $\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$

c. A convex polygon has all interior angles less than 180 degrees. Use mathematical induction to prove that a convex *n* sided polygon has $\frac{1}{2}n(n - 3)$ diagonals for $n \geq 3$

Question 7 Use a separate writing booklet. **15 marks**

a. Convert $1+\sqrt{3}\,i$ to exponential form. **1**

b. The diagram below consists of five squares on the Argand diagram. The point P represents the complex number z. Give expression, in terms of z, for



i. $\overrightarrow{OA}$ **1**

ii. $\overrightarrow{AB}$ ` **1**

c. Prove that there are no integers m, n which satisfy $m^2 - 8n^2 = 7$ **3**

d. i. Show that $e^{n\theta i} + e^{-n\theta i} = 2\cos n\theta$ **1**

ii. Show that $(e^{i\theta} + e^{-i\theta})^3 = (e^{3\theta i} + e^{-3\theta i}) + 3(e^{i\theta} + e^{-i\theta})$ **1**

iii. Hence prove that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$ **2**

e. i. Show that $\sin\{f(x)\} + \cos\{f(x)\} = \sqrt{2}\sin\left\{f(x) + \frac{\pi}{4}\right\}$

ii. Let $y = e^x \sin x$. Use mathematical induction to show that

$$\frac{d^n y}{dx^n} = (\sqrt{2})^n e^x \sin\left(x + \frac{n\pi}{4}\right)$$

Question 8 Use a separate writing booklet. **15 marks**

a. Consider the complex numbers

$$z = 3\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) \text{ and } \omega = 5\left(\cos\frac{k\pi}{12} + i\sin\frac{k\pi}{12}\right) \text{ where } k \in \mathbb{Z}^+$$

Find the following

i) $|z\omega|$ **1**

ii) $\text{Arg}(z\omega)$ **1**

Suppose that $z\omega \in \mathbb{Z}$

iii) Find the minimum value of k **2**

iv) For the value of k found in part (iii), find the value of $z\omega$. **1**

b. Use contraposition to prove that, given n is a positive integer, if $3n + 2$ is even then n is even. **2**

c. Prove by mathematical induction that for $n \in \mathbb{Z}^+$ and $n \geq 2$. **4**

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} < \frac{n-1}{n}$$

d. Let $a, b, c \in \mathbb{R}$

i. Prove that $\frac{a}{b} + \frac{b}{a} \geq 2$ **1**

ii. Assume $0 < a, b, c < 1$ and $a + b + c = 2$.

Use substitutions $x = 1 - a$, $y = 1 - b$, $z = 1 - c$ to prove that

$$\left(\frac{a}{1-a}\right)\left(\frac{b}{1-b}\right)\left(\frac{c}{1-c}\right) \geq 8$$

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End of paper