SHORE

Exam Number:

Year 12
Mathematics Extension 2
Mid-Year Examination
2022

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black or blue pen
- Draw diagrams using pencil
- Board approved calculators may be used
- Answer Questions 1 - 10 on the Multiple Choice answer sheet provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Start each of Questions 11 – 16 in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- A NESA reference sheet is provided.
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A"

Total marks – 100

Section I Pages 2 - 6

10 marks

- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II Pages 7 - 15

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Note: Any time you have remaining should be spent revising your answers.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks

Attempt Questions 1–1

Allow about 15 minutes for this section

Use the Multiple Choice Answer Sheet for Questions 1–10.

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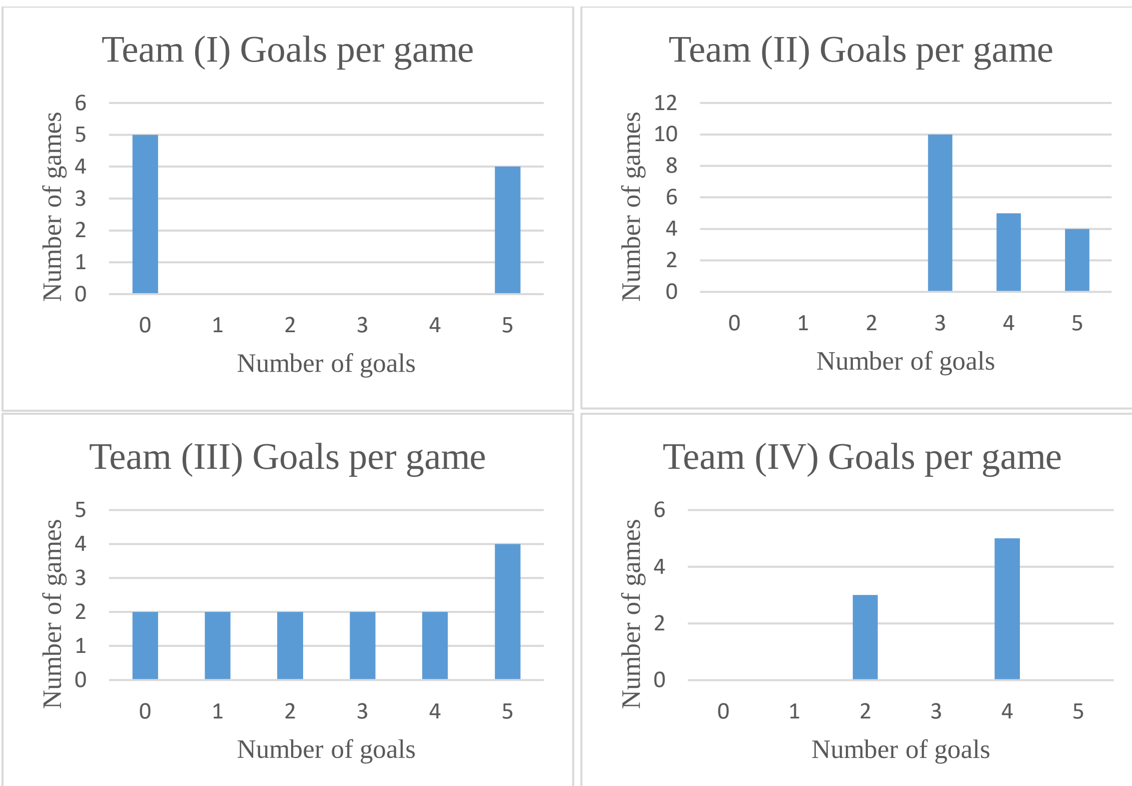
1 The bar chart shows the number of goals scored by four different teams. Which statement is true about the different statistical measures?



(A) (I): highest standard deviation, (II): highest median, (III): highest mode

(B) (I): highest median, (II): highest mode, (III): highest standard deviation

(C) (I): highest standard deviation, (II): highest mean, (IV): highest median

(D) (II): highest mode, (III): highest standard deviation, (IV): highest median

2 Which is the fully simplified form of $\sqrt{\dfrac{64^x}{4^x}}$?

(A) 2^x

(B) 4^x

(C) $6^{\frac{x}{2}}$

(D) $8^{\frac{x}{2}}$

3 What is the natural domain of $y=\sin^{-1}\left(\dfrac{x}{2}-1\right)+\dfrac{1}{\sqrt{2-x}}$?

(A) $x\in(-\infty,2]$

(B) $x\in[0,4]$

(C) $x\in[0,2)$

(D) $x\in[-\pi+2,2)$

4 A polynomial $P(z)$ has real coefficients and degree *n*. Given that $P(\alpha)=0$, $P'(\alpha)=0$ and $P'(\beta)=0$, where neither α nor β are real, what is the smallest possible value of *n*?

(A) 3

(B) 4

(C) 5

(D) 6

5 What is the negation of the statement:

"Every Thursday, both the general rubbish bin and the recycling bin get emptied by the council."

(A) Some Thursdays, the general rubbish bin and the recycling bin don't get emptied by the council.

(B) Some Thursdays, the general rubbish bin or the recycling bin don't get emptied by the council.

(C) Every Thursday, the general rubbish bin and the recycling bin don't get emptied by the council.

(D) Every Thursday, the general rubbish bin or the recycling bin don't get emptied by the council.

6 Points A and B lie on the line $y = mx + c$. Point C doesn't lie on the line.
Which expression gives the (shortest) distance from C to the line $y = mx + c$?

(A) $\left|\overrightarrow{AC} - \text{proj}_{\overrightarrow{AB}}\overrightarrow{AC}\right|$

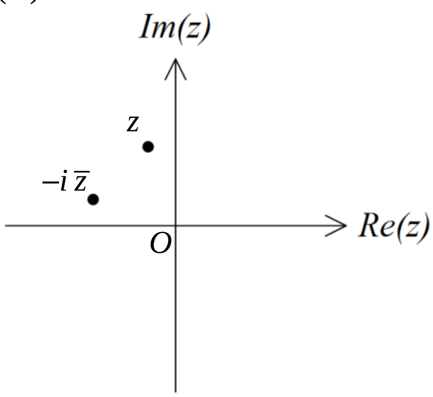
(B) $\left|\overrightarrow{AB} - \text{proj}_{\overrightarrow{AB}}\overrightarrow{AC}\right|$

(C) $\left|\overrightarrow{AC} - \text{proj}_{\overrightarrow{AC}}\overrightarrow{AB}\right|$

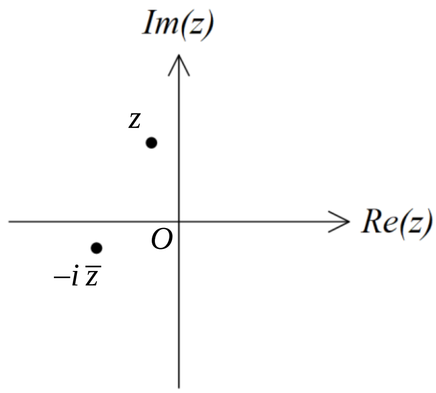
(D) $\left|\overrightarrow{AB} - \text{proj}_{\overrightarrow{AC}}\overrightarrow{AB}\right|$

7 Which Argand diagram could show the points represented by the complex numbers z and $-i\,\overline{z}$ on the complex plane?

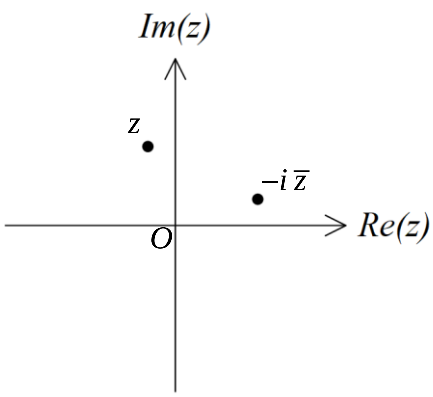
(A)



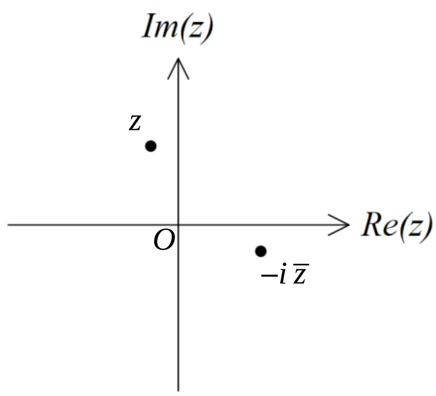
(B)



(C)



(D)



8 Which of the following statements is NOT true?

(A) $\displaystyle\int_{-\pi}^{\pi} \sin x \cos 3x\, dx = 0$

(B) $\displaystyle\int_{-\pi}^{\pi} \sin(\sin x)\, dx = 0$

(C) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(2x - \frac{\pi}{2}\right) dx = 0$

(D) $\displaystyle\int_{-\pi}^{\pi} \sin\left|x - \frac{\pi}{2}\right| dx = 0$

9 At the start of each round in an eight-player card game, four players are randomly assigned as 'Evil' and the other four as 'Good'.

What is the probability that after two separate rounds of the game, there are exactly 3 players who were assigned as 'Evil' in both rounds?

(A) $\frac{1}{25}$

(B) $\frac{2}{35}$

(C) $\frac{1}{5}$

(D) $\frac{8}{35}$

10 A discrete probability distribution is given below.

x	0	1	2	3
$P(x)$	$2a$	b^2	$3a - 0.45$	$2b - 4a$

After many trials, "0" occurs 290 times, and "3" occurs 145 times. Which is the best estimate of how many times "1" occurs?

(A) 143

(B) 181

(C) 205

(D) 238

End of Section I

Section II

90 marks

Attempt Questions 11 - 16

Allow about 2 hours 45 minutes for this section

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Answer question in a SEPARATE writing booklet.

(a) Write the complex number $\omega = -3 + \sqrt{3}\,i$ in mod-arg form. **2**

(b) Given that $(m+ni)(3-2i) = \dfrac{6-4i}{1-i}$, where m and n are real, determine the values of m and n. **3**

(c) A charity plants trees each year. The amount they plant increases by 600 each year. If they plant 3200 in the first year, how many years will it take for them to have planted more than 80 000 trees in total? **3**

(d) (i) Find the derivative of $\sec x + \tan x$. **2**

(ii) Hence find $\displaystyle\int \sec x \times \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$. **1**

(e) (i) Find the values of a and b such that $(a+ib)^2 = -27-36i$, where $a, b \in \mathbb{Z}$. **2**

(ii) Hence, or otherwise, solve: **2**

$$z^2 + 4iz + 3 + 7i = z.$$

Question 12 (15 marks) Answer question in a SEPARATE writing booklet.

(a) Prove that this statement is false by counterexample: **1**

"It is impossible to divide the inside of a square into exactly 7 different regions by drawing exactly three straight line segments where each end of the line segment connects to the square."

(b) Given that $\int_{-2}^{b} |2x+2|-2 \ dx = 0$, evaluate b. **3**

(c) Solve $\sin\left(\theta+\frac{\pi}{3}\right)+\sin\left(\theta-\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$, for $0<\theta<2\pi$. **3**

(d) Prove by contradiction that no prime numbers are square numbers. **2**
(a prime number is a positive integer that has exactly two positive integer divisors).

(e) The complex number z satisfies these three conditions:

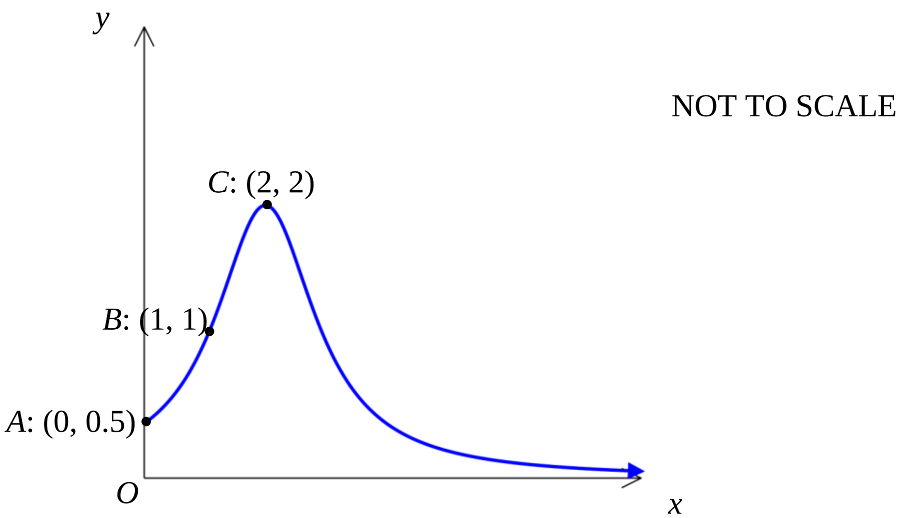
$\text{Re}(z)>1$, $\text{Arg}(z-i)\geq\frac{\pi}{4}$ and $|z-i|\leq|z-2-9i|$

(i) Sketch the region defined by the above conditions. **3**

(ii) Show that the largest possible value of $|z|$ is $\frac{\sqrt{773}}{5}$. **3**

Question 13 (15 marks) Answer question in a SEPARATE writing booklet.

(a) The velocity-time graph of $y = f(x)$ is shown on the axes below. The y-intercept is at A: (0, 0.5), B: (1, 1) is an inflection point and C: (2, 2) is the maximum.



(i) Sketch $y = \frac{1}{2} f\left(\frac{1}{2}x\right)$. Show the transformed coordinates. **2**

(ii) Sketch $y = \dfrac{1}{f(x)}$. Show the transformed coordinates. **2**

(iii) Sketch $y = f'(x)$. Indicate A, B and C. **2**

(iv) Sketch $y = \int_0^x f(t)\,dt$. Indicate A, B and C. **1**

Question 13 continues on page 10

(b) A family borrows \$600 000 to buy a house. The loan is to be repaid in equal monthly instalments with monthly repayments of \$$M$. The interest is 6% per annum and is compounded monthly.

The amount owing after n months, \$$A_n$, is given by:

$$A_n = Pr^n - M\left(1 + r + r^2 + ... + r^{n-1}\right) \quad \text{(DO NOT prove this)}$$

where $r = 1.005$ and \$$P$ is the amount borrowed.

(i) The loan is to be repaid over 30 years. Given that \$$P$ = \$600 000, show that the monthly repayment is \$3597 to the nearest dollar. **2**

(ii) Show that the balance owing after 18 years is \$369 000 to the nearest thousand dollars. **1**

After 18 years the family borrows an extra amount, so that the family then owes a total of \$440 000. The monthly repayment remains \$3597 and the interest rate remains the same.

(iii) How long will it take to repay the \$440 000? **2**

(c) Given that a, b, and c are real, positive numbers:

(i) Prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$. **1**

(ii) Hence, or otherwise, prove that $a^2(b+c) + b^2(c+a) + c^2(a+b) \geq 6abc$. **2**

End of Question 13

Question 14 (15 marks) Answer question in a SEPARATE writing booklet.

(a) Use Mathematical induction to prove that for all integers $n \geq 2$, **3**

$$\sum_{j=1}^{n} \frac{1}{j^2} < 2 - \frac{1}{n}.$$

(b) Find all the solutions of $\left|\text{cis}\, 2\theta + \sqrt{2}i\right| = \sqrt{5}$ satisfying $-\pi < \theta \leq \pi$. **3**

(c) Suppose that $x, y \in \mathbb{R}$. Prove by contrapositive that: **2**

If $y^3 + yx^2 \leq x^3 + xy^2$ then $y \leq x$.

(d) The complex number $z = a + bi$, where a and b are integers, satisfies $|z| \leq 3$.

(i) Show, using a diagram, that there are 29 possible values that z can take. **2**

If z takes one of these values,

(ii) Evaluate $P(a > b \mid b \geq 0)$. **2**

(e) By using calculus, or otherwise, prove that $2x^2 - 2 \geq \log_e x$ for all $x \geq 1$. **3**

Question 15 (15 marks) Answer question in a SEPARATE writing booklet.

(a) In an experiment, two particles, P_1 and P_2 are projected from the origin on level ground and act only under the force of gravity. The equation of motion for P_1 is given by:

$$\underset{\sim}{r}_1 = 15t_1\underset{\sim}{i} + \left(35t_1 - 5t_1^{\,2}\right)\underset{\sim}{j}$$

where t_1 is the time after P_1 is projected.

(i) Calculate the time taken for P_1 to reach its maximum height. **1**

(ii) Particle P_2 is projected some time after P_1. Given that its initial velocity is $30\underset{\sim}{i} + 40\underset{\sim}{j}$ ms^{-1}, and taking the acceleration due to gravity g as 10 ms^{-2}, derive the displacement vector $\underset{\sim}{r}_2$ for P_2 in terms of t_2, the time after P_2 is projected. **3**

(iii) Particle P_2 is projected exactly 2 seconds after P_1 is projected. Show that the particles collide and find the time at which they do so. **3**

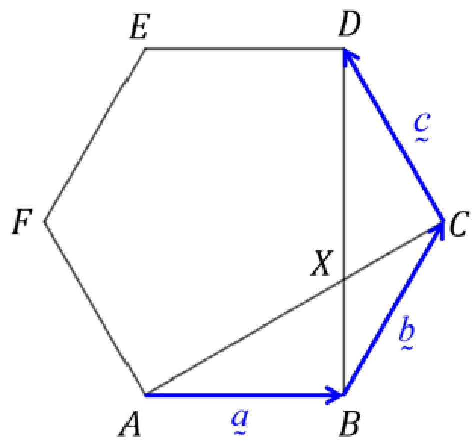
(iv) The experiment is repeated, but now P_2 is projected 2.5 seconds after P_1, at the same angle but with a larger initial speed. What initial speed must P_2 have so that the particles still collide? **3**

Question 15 continues on page 13

(b) Let *ABCDEF* be a regular hexagon with sides of length 1 unit.

The diagonals *AC* and *BD* meet at *X*.

Let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$ and $\underset{\sim}{c} = \overrightarrow{CD}$.



(i) Determine the values of $\underset{\sim}{a} \cdot \underset{\sim}{b}$, $\underset{\sim}{b} \cdot \underset{\sim}{c}$ and $\underset{\sim}{a} \cdot \underset{\sim}{c}$. **1**

(ii) There exist scalars λ and μ such that $\overrightarrow{AX} = \lambda \overrightarrow{AC}$ and $\overrightarrow{BX} = \mu \overrightarrow{BD}$. **1**
Find an equation linking λ, μ, $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

(iii) By considering different dot products of your equation in (ii), use this and your results from (i) to find the values of λ and μ. **3**

End of Question 15

Question 16 (15 marks) Answer question in a SEPARATE writing booklet.

(a) The concentration of a drug in a body is $F(t)$, where t is the time in hours after the drug is taken.

Initially the concentration of the drug is zero. The rate of change of concentration of the drug is given by

$$F'(t) = 50e^{-0.5t} - 0.4F(t).$$

(i) Use the product rule to show that: **2**

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = 50e^{-0.1t}.$$

(ii) Hence, or otherwise, show that $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$. **2**

(iii) The concentration of the drug increases to a maximum. For what value of t does this maximum occur? **2**

(b) The relations $y^2 = \dfrac{4(1+x)}{x}$ and $y = x$ are plotted on the same set of axes. **3**

By considering the stationary points of an appropriate equation, prove that they intersect exactly once.

(DO NOT find the point of intersection)

Question 16 continues on page 15

(c) Consider the general cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$.

(i) Find the x value of the point of inflection of $P(x)$. **1**

(ii) Explain why the polynomial **1**

$$Q(x) = a\left(x - \frac{b}{3a}\right)^3 + b\left(x - \frac{b}{3a}\right)^2 + c\left(x - \frac{b}{3a}\right) + d$$

must have its point of inflection on the y axis.

(iii) For $d = a\left(\frac{b}{3a}\right)^3 - b\left(\frac{b}{3a}\right)^2 + c\left(\frac{b}{3a}\right)$, prove that the x intercepts of $y = Q(x)$ **2**

are at $x = 0$, $x = \dfrac{\pm\sqrt{b^2 - 3ac}}{\sqrt{3}\,a}$.

(iv) Hence solve the cubic equation: **2**

$$x^3 + x^2 - x + \left(\frac{1}{27} - \frac{1}{9} - \frac{1}{3}\right) = 0.$$

End of paper