Baulkham Hills High School

2023 **YEAR 12 ASSESSMENT TASK 1**

Mathematics Extension 2

General Instructions	• Reading time – 5 minutes • Working time – 1 hour • Write using black pen • Calculators approved by NESA may be used • Relevant formulas from the reference sheet will be provided in the answer booklet • For questions in Section II, show relevant mathematical reasoning and/or calculations
Total marks: 37	**Section I – 5 marks** (pages 2 – 3) • Attempt Questions 1 – 5 • Allow about 8 minutes for this section **Section II – 32 marks** (pages 4 – 6) • Attempt Questions 6 – 8 • Allow about 42 minutes for this section

Section I

5 marks
Attempt Questions 1 – 5
Allow about 8 minutes for this section

Use the multiple-choice answer page in the writing booklet for Questions 1 – 5.

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1 What is the value of $i+i^2+i^3+\cdots+i^{2023}$?

A. i

B. $-1+i$

C. -1

D. 0

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2 Multiplying by a complex number z causes a clockwise rotation of $\frac{\pi}{6}$ radians, which of the following represents z?

A. $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

B. $\frac{1}{2}-\frac{\sqrt{3}}{2}i$

C. $\frac{\sqrt{3}}{2}+\frac{1}{2}i$

D. $\frac{\sqrt{3}}{2}-\frac{1}{2}i$

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3 Given $z=3e^{i\theta}$ and $w=7e^{i\phi}$, which of the following is a possible value of $|z+w|$?

A. 1

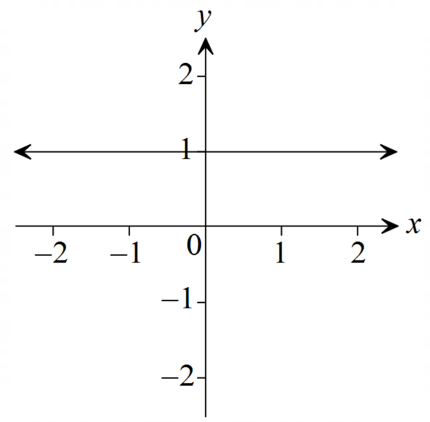
B. 3

C. 7

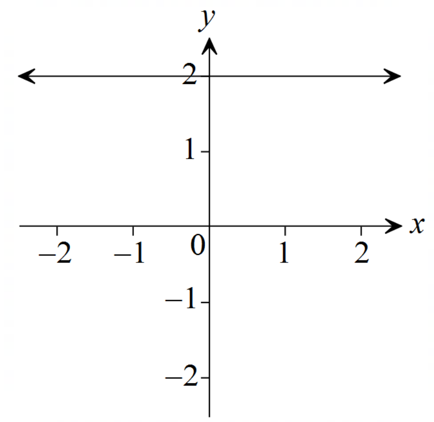
D. 11

4 Which of the following diagrams shows the subset of the complex plane satisfied by the relation $\left(z-\overline{z}\right)^2 = -4$?

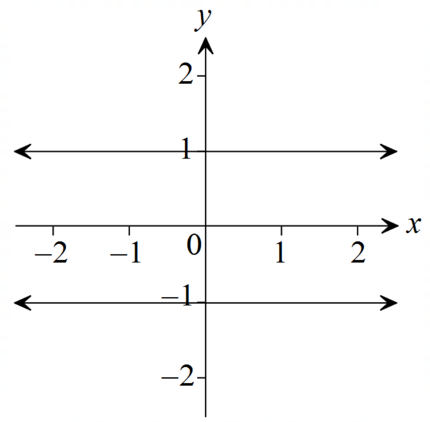
A.



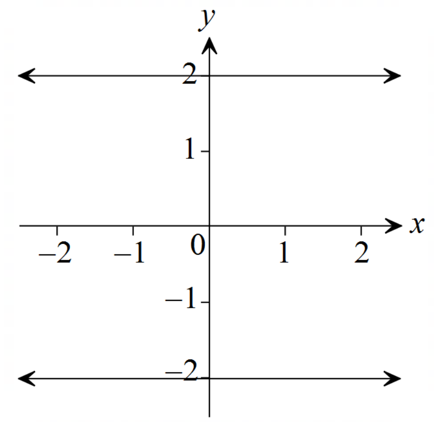
B.



C.



D.



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5 If z and w are two non-zero complex numbers such that $|z| = |w|$ and their principal arguments sum to π, that is, $\operatorname{Arg}(z) + \operatorname{Arg}(w) = \pi$, which of the following statements is correct?

A. $z = \overline{w}$

B. $z = -\overline{w}$

C. $z = w$

D. $z = -w$

End of Section I

Section II

32 marks
Attempt Questions 6 – 8
Allow about 52 minutes for this section

Answer each question in the appropriate section of the writing booklet. Extra writing paper is available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 marks) Use the Question 6 section of the writing booklet.

(a) (i) Find the square root(s) of $21-20i$ in real-imaginary form. **2**

(ii) Hence solve the quadratic equation $4iz^2+(3+2i)z+(2+i)=0$, expressing your answers with real denominators where applicable. **2**

(b) Sketch the subset of the complex plane containing complex numbers z such that $\arg\left(1-\frac{2}{z}\right)=\frac{\pi}{4}$ and state its Cartesian equation and any restrictions. **3**

(c) (i) Express $i-1$ in exponential form. **2**

(ii) Hence, for what value(s) of k will $(i-1)^k$ be purely real? **2**

End of Question 6

Question 7 (11 marks) Use the Question 7 section of the writing booklet.

(a) Consider the complex numbers z and w such that $z = 1 + \sqrt{3}i$ and $\dfrac{w}{z} = 2 + 2i$.

(i) By first expressing z and $\dfrac{w}{z}$ in modulus-argument form, find the complex number w in modulus-argument form. **2**

(ii) Express w in the form $x + iy$, where x and y are real numbers, and hence show that the exact value of $\tan\dfrac{7\pi}{12}$ is $-2-\sqrt{3}$. **3**

(b) (i) Find the five fifth roots of unity in modulus-argument form. **2**

(ii) Hence show that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$. **2**

(iii) Using the result in part (ii), find the exact value of $\cos\dfrac{2\pi}{5}$. **2**

End of Question 7

Question 8 (10 marks) Use the Question 8 section of the writing booklet.

(a) It is given that $5+2i$ is a zero of $P(z)=2z^3-8z^2-62z+d$, where d is a real number.

(i) Find the other two zeros of $P(z)$. **2**

(ii) Hence, or otherwise, express $P(z)$ as a product of real linear and irreducible quadratic factors over $\mathbb{R}$. **2**

(iii) Find the value of d. **1**

(b) (i) A complex number z satisfies the property $\left|\operatorname{Arg}(z)\right| \leq \frac{\pi}{6}$, where $\operatorname{Arg}(z)$ is the principal argument of z. Sketch the entire subset of the complex plane in which z lies. **2**

(ii) For any given real constant k, suppose $|z-k|=r$ describes the largest circle of radius r (for that value of k) that also satisfies the conditions on z from (i). On a separate complex plane, draw a neat sketch of a potential construction, show that $k=2r$ and hence state, in terms of r, the range of values that $|z|$ can take on. **3**

End of Paper

YEAR 12 ASSESSMENT TASK 1 2023
MATHEMATICS EXTENSION 2
MARKING GUIDELINES

Section I

Multiple-choice Answer Key

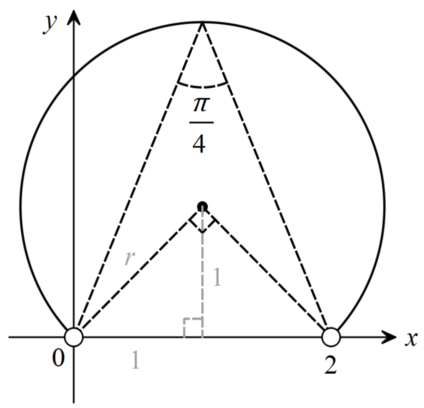
Question	Answer
1	C
2	D
3	C
4	C
5	B

Questions 1 – 5

Sample solution	
1.	$\left(i+i^2+i^3+i^4\right)+\left(i^5+i^6+i^7+i^8\right)+\cdots+\left(i^{2017}+i^{2018}+i^{2019}+i^{2020}\right)+i^{2021}+i^{2022}+i^{2023}$ $=\left(i-1-i+1\right)+\left(i-1-i+1\right)+\cdots+\left(i-1-i+1\right)+i-1-i$ $=-1$
2.	A clockwise rotation of $\frac{\pi}{6}$ radians is achieved by multiplying the complex number $\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)$. $\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)$ $=\frac{\sqrt{3}}{2}-\frac{1}{2}i$
3.	$\lvert z\rvert=3, \lvert w\rvert=7$ By the triangle inequality: $\big\lvert \lvert z\rvert-\lvert w\rvert \big\rvert \le \lvert z+w\rvert \le \lvert z\rvert+\lvert w\rvert$ $\lvert 3-7\rvert \le \lvert z+w\rvert \le 3+7$ $4\le \lvert z+w\rvert \le 10$
4.	$\left(z-\overline{z}\right)^2=-4$ $\left[2i\,\mathrm{Im}(z)\right]^2=-4$ $-4\left[\mathrm{Im}(z)\right]^2=-4$ $\left[\mathrm{Im}(z)\right]^2=1$ $\mathrm{Im}(z)=\pm 1$
5.	If $\mathrm{Arg}(z)+\mathrm{Arg}(w)=\pi$, then $\mathrm{Arg}(z)\ge 0$, otherwise $\mathrm{Arg}(w)>\pi$. Noting $\mathrm{Arg}(w)=\pi-\mathrm{Arg}(z)$ and $\lvert z\rvert=\lvert w\rvert$, w is the reflection of z in the imaginary axis, making $z=-\overline{w}$ by geometry.

Section II

Question 6

		Sample solution	Suggested marking criteria
(a)	(i)	Let $\sqrt{21-20i} \equiv x+iy$, where $x, y \in \mathbb{R}$. $21-20i = (x+iy)^2$ $21-20i = (x^2-y^2)+(2xy)i$ Comparing real and imaginary parts: $x^2-y^2=21$ $2xy=-20$ Solving simultaneously yields $x=5, y=-2$ and $x=-5, y=2$ $\therefore \sqrt{16+30i} = \pm(5-2i)$	• 2 – correct answer • 1 – correct equations in x and y, or equivalent merit
	(ii)	$4iz^2+(3+2i)z+(2+i)=0$ $\Delta = (3+2i)^2 - 4\times 4i(2+i)$ $= 9+12i+4i^2-32i-16i^2$ $= 9+12i-4-32i+16$ $= 21-20i$ $z = \dfrac{-(3+2i)\pm\sqrt{21-20i}}{2\times 4i}$ $= \dfrac{-(3+2i)\pm(5-2i)}{8i}$ $= \dfrac{2-4i}{8i}, \dfrac{-8}{8i}$ $= \dfrac{4+2i}{-8}, \dfrac{-8i}{-8}$ $= -\dfrac{1}{2}-\dfrac{1}{4}i\ ,\ i$	• 2 – correct solution • 1 – correct discriminant (separately or as part of the quadratic formula)
(b)		$\arg\left(1-\dfrac{2}{z}\right)=\dfrac{\pi}{4}$ $\arg\left(\dfrac{z-2}{z}\right)=\dfrac{\pi}{4}$ Cartesian equation: $(x-1)^2+(y-1)^2=2, y>0$ 	• 3 – correct solution • 2 – correctly sketches the required subset of the complex plane • 1 – recognises that z lies on a circular arc
(c)	(i)	$i-1=-1+i$ $=\sqrt{2}e^{i\frac{3\pi}{4}}$	• 2 – correct answer • 1 – correct modulus or argument (not necessarily principal argument)
	(ii)	$(i-1)^k = \left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^k$ $= \left(\sqrt{2}\right)^k e^{i\frac{3k\pi}{4}}$ This is purely real if $\dfrac{3k\pi}{4} = 0, \pm\pi, \pm 2\pi, \ldots$, i.e. $k = 0, \dfrac{\pm 4}{3}, \dfrac{\pm 8}{3}, \dfrac{\pm 12}{3} \ldots$ Therefore, $(i-1)^k$ is purely real if k is an integer multiple of $\dfrac{4}{3}$.	• 2 – correct solution • 1 – recognises that a complex number is purely real if its argument are integer multiples of π – correctly finds a value of k such that $(i-1)^k$ is purely real

Question 7

Sample solution			Suggested marking criteria
(a)	(i)	$z = 1+\sqrt{3}i$ $= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ $\frac{w}{z} = 2+2i$ $= 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ $z\times\frac{w}{z} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\times 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ $w = 4\sqrt{2}\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\right]$ $= 4\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$	• 2 – correct solution • 1 – correctly expresses either z or $\frac{z}{w}$ in modulus-argument form
	(ii)	$\frac{w}{z} = 2+2i$ $w = (2+2i)z$ $= (2+2i)\left(1+\sqrt{3}i\right)$ $= 2+2\sqrt{3}i+2i+2\sqrt{3}i^2$ $= \left(2-2\sqrt{3}\right)+\left(2+2\sqrt{3}\right)i$ $\therefore \left(2-2\sqrt{3}\right)+\left(2+2\sqrt{3}\right)i = 4\sqrt{2}\cos\frac{7\pi}{12} + i\,4\sqrt{2}\sin\frac{7\pi}{12}$ Comparing real and imaginary parts: $4\sqrt{2}\sin\frac{7\pi}{12} = 2+2\sqrt{3}$ $4\sqrt{2}\cos\frac{4\pi}{12} = 2-2\sqrt{3}$ Dividing through yields: $\frac{\sin\left(\frac{7\pi}{12}\right)}{\cos\left(\frac{7\pi}{12}\right)} = \frac{2+2\sqrt{3}}{2-2\sqrt{3}}$ $\tan\frac{7\pi}{12} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$ $= \frac{\left(1+\sqrt{3}\right)^2}{\left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right)}$ $= \frac{1+2\sqrt{3}+3}{1^2-\left(\sqrt{3}\right)^2}$ $= \frac{4+2\sqrt{3}}{-2}$ $= -2-\sqrt{3}$	• 3 – correct solution • 2 – attempts to express $\sin\frac{7\pi}{12}$ and $\cos\frac{7\pi}{12}$ as exact values using appropriate technique(s) • 1 – correctly expresses w in the rectangular form

Question 7 (continued)

Sample solution		Suggested marking criteria
(b)	(i) Let $z = r\operatorname{cis}\theta$: $z^5 = 1$ $(r\operatorname{cis}\theta)^5 = 1\operatorname{cis}(k2\pi)$, where k is an integer $r^5\operatorname{cis}(5\theta) = 1\operatorname{cis}(k2\pi)$ Equating moduli and arguments: $r^5 = 1$ $r = 1$ $5\theta = k2\pi$, where $k = 0, \pm 1, \pm 2$ $\theta = \dfrac{k2\pi}{5}$ $= 0, \pm\dfrac{2\pi}{5}, \pm\dfrac{4\pi}{5}$ Therefore the five fifth roots of unity are $1, \operatorname{cis}\left(\pm\dfrac{2\pi}{5}\right), \operatorname{cis}\left(\pm\dfrac{4\pi}{5}\right)$.	• 2 – correct solution • 1 – correctly applies de Moivre's theorem to find one non-real fifth root of unity
	(ii) $\sum\alpha = 0$ $1 + \operatorname{cis}\left(\dfrac{2\pi}{5}\right) + \operatorname{cis}\left(-\dfrac{2\pi}{5}\right) + \operatorname{cis}\left(\dfrac{4\pi}{5}\right) + \operatorname{cis}\left(-\dfrac{4\pi}{5}\right) = 0$ $1 + \operatorname{cis}\left(\dfrac{2\pi}{5}\right) + \overline{\operatorname{cis}\left(\dfrac{2\pi}{5}\right)} + \operatorname{cis}\left(\dfrac{4\pi}{5}\right) + \overline{\operatorname{cis}\left(\dfrac{4\pi}{5}\right)} = 0$ $1 + 2\operatorname{Re}\left[\operatorname{cis}\left(\dfrac{2\pi}{5}\right)\right] + 2\operatorname{Re}\left[\operatorname{cis}\left(\dfrac{4\pi}{5}\right)\right] = 0$ $1 + 2\cos\dfrac{2\pi}{5} + 2\cos\dfrac{4\pi}{5} = 0$ $2\left(\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5}\right) = -1$ $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$	• 2 – correct proof • 1 – attempts to use properties of polynomials to find the exact value of $\cos\frac{2\pi}{5}$ or $\cos\frac{4\pi}{5}$ (or their sum)
	(iii) $\cos\dfrac{2\pi}{5} + 2\cos^2\dfrac{2\pi}{5} - 1 = -\dfrac{1}{2}$ $4\cos^2\dfrac{2\pi}{5} + 2\cos\dfrac{2\pi}{5} - 1 = 0$ $4X^2 + 2X - 1 = 0 \quad \left(\text{Let } X = \cos\dfrac{2\pi}{5}\right)$ $X = \dfrac{-2 \pm \sqrt{2^2 - 4\times 4\times -1}}{2\times 4}$ $= \dfrac{-2 \pm \sqrt{20}}{8}$ $= \dfrac{-2 \pm 2\sqrt{5}}{8}$ $= \dfrac{-1 \pm \sqrt{5}}{4}$ Since $\cos\dfrac{2\pi}{5} > 0$, $\cos\dfrac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}$.	• 2 – correct solution • 1 – forms an appropriate quadratics equation and solves it

Question 8

Sample solution			Suggested marking criteria
(a)	(i)	Since the coefficients of $P(z)$ are real, therefore complex roots occur in conjugate pairs, making $5-2i$ another zero of $P(z)$. Let the third root (which must be real) be k. $\sum \alpha = -\frac{(-8)}{2}$ $(5+2i)+(5-2i)+k=4$ $10+k=4$ $k=-6$ Therefore, the three zeros of $P(z)$ are -6 and $5 \pm 2i$.	• 2 – correct solution • 1 – states that $5-2i$ is also a root of $P(z)$
	(ii)	$P(z)=2[z-(5+2i)][z-(5-2i)][z-(-6)]$ $=2(z-5-2i)(z-5+2i)(z+6)$ $=2[(z-5)-2i][(z-5)+2i](z+6)$ $=2\left[(z-5)^2-(2i)^2\right](z+6)$ $=2\left(z^2-10z+25+4\right)(z+6)$ $=2\left(z^2-10z+29\right)(z+6)$	• 2 – correct factorisation • 1 – expresses $P(z)$ as a product of linear factors using the zeros found in part (i)
	(iii)	By comparing constant terms: $d=2\times 29\times 6$ $=348$ Or by product of roots: $\sum \alpha\beta\gamma = -\frac{d}{2}$ $-6(5+2i)(5-2i)=-\frac{d}{2}$ $d=12\left[5^2-(2i)^2\right]$ $=12\times(25+4)$ $=348$	• 1 – correct solution

Question 8 (continued)

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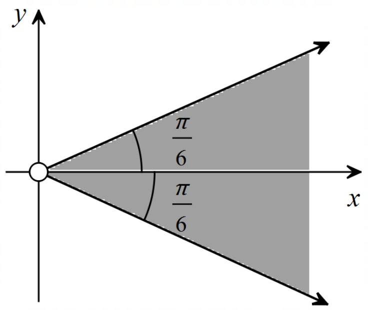
<tr><th colspan="2">Sample solution</th><th>Suggested marking criteria</th></tr>
<tr><td>(b)</td><td>(i)



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- 2 – correct sketch
- 1 – correct rays
  – recognises a correct portion of the subset (e.g. the top half)

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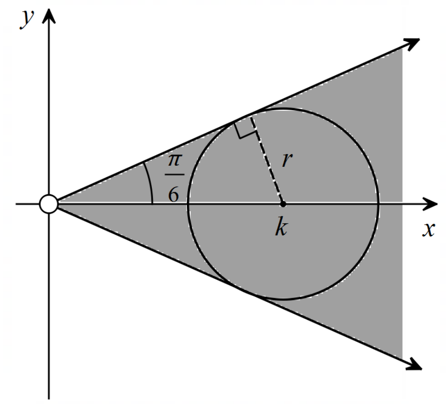
<tr><td></td><td>(ii)



Using right-angled trigonometry,

$$\sin\frac{\pi}{6} = \frac{r}{k}$$

$$\frac{1}{2} = \frac{r}{k}$$

$$k = 2r$$

$$k - r \le |z| \le k + r$$

$$r \le |z| \le 3r$$

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- 3 – correct solution
- 2 – shows $k = 2r$ and attempts to find the maximum (or minimum) value of $|z|$
- 1 – shows $k = 2r$

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