James Ruse Agricultural High School

2023 **TASK 2 EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/ or calculations

Total marks: 70

Section I – 5 marks (pages 2–3)

- Attempt Questions 1–5
- Allow about 8 minutes for this section

Section II – 65 marks (pages 4–8)

- Attempt Questions 6–10
- Allow about 1 hour and 52 minutes for this section

Section I

5 marks
Attempt Questions 1 – 5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 5.

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1 Given that $\dfrac{2x-1}{(x+1)(x-2)} = \dfrac{\alpha}{x+1} + \dfrac{\beta}{x-2}$, then

A. $\alpha = -1$ and $\beta = 1$

B. $\alpha = 1$ and $\beta = 1$

C. $\alpha = -1$ and $\beta = -1$

D. $\alpha = 1$ and $\beta = -1$

2 Which of the following definite integrals has the largest value?

A. $\displaystyle\int_0^{\pi/2} \sin x \, dx$

B. $\displaystyle\int_0^{\pi/2} \sin^2 x \, dx$

C. $\displaystyle\int_0^{\pi/2} (1 - \sin x) \, dx$

D. $\displaystyle\int_0^{\pi/2} (1 - \sin^2 x) \, dx$

3 Christian thinks he has discovered a fact about even integers. He claims:

“Every even integer greater than 2 can be written as the sum of two prime numbers.”

Which of the following provides a counterexample to Christian’s claim?

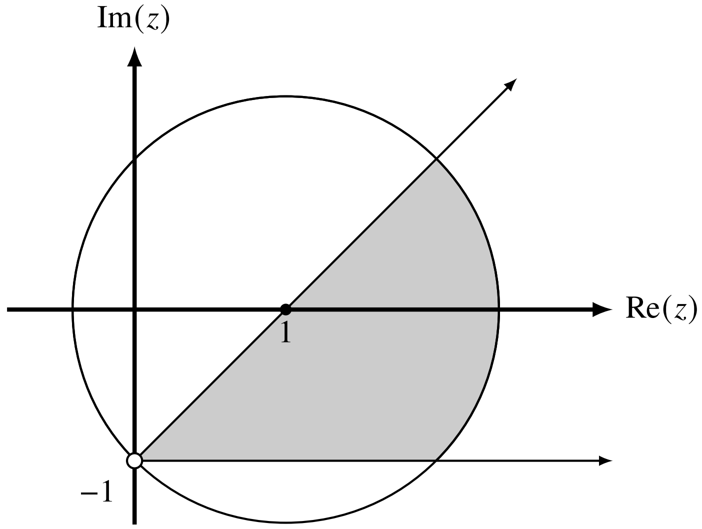
A. An even integer greater than 2 that can be written as the sum of two numbers which are not prime.

B. An odd integer greater than 2 that can be written as the sum of two prime numbers.

C. An even integer greater than 2 that cannot be written as the sum of two prime numbers.

D. An odd integer greater than 2 that cannot be written be written as the sum of two prime numbers.

4 Consider the Argand diagram below.



Which inequality defines the shaded area?

A. $|z-1| \leq \sqrt{2}$ and $0 \leq \operatorname{Arg}(z-i) \leq \dfrac{\pi}{4}$

B. $|z-1| \leq \sqrt{2}$ and $0 \leq \operatorname{Arg}(z+i) \leq \dfrac{\pi}{4}$

C. $|z-1| \leq 1$ and $0 \leq \operatorname{Arg}(z-i) \leq \dfrac{\pi}{4}$

D. $|z-1| \leq 1$ and $0 \leq \operatorname{Arg}(z+i) \leq \dfrac{\pi}{4}$

5 Consider the following two statements about two positive integers p and q:

P: The number $p^3 - q^3$ is even.

Q: The number $p + q$ is even.

Which of the following options is true?

A. P is necessary and sufficient for Q.

B. P is necessary but not sufficient for Q.

C. P is sufficient but not necessary for Q.

D. P is neither necessary nor sufficient for Q.

Section II

65 marks
Attempt Questions 6 – 10
Allow about 1 hour and 52 minutes for this section

Write your answers on the paper supplied. Start each question on a new page.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (13 marks) Start this question on a new page

(a) Find the least positive integer k such that $\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)$ is a solution of $z^k = 1$. **1**

(b) Let m and n be integers. Show that if the complex number w is a solution of $z^n = 1$, then so is w^m. **2**

(c) Find the following integrals:

(i) $\displaystyle\int \frac{1}{\sqrt{u}+1}du$ **2**

(ii) $\displaystyle\int x\cos(x)\sin(x)dx$ **3**

(d) Consider the following statement:

***P*: "If I am tall and athletic, I am good at basketball and volleyball."**

(i) Write the contrapositive of P as an English sentence. **1**

(ii) Copy and complete the following sentence for the negation of P: **1**

"I am tall and athletic, ... "

(e) Prove, by contrapositive, that if $\frac{mn}{2}$ is an integer, then m or n must be even. **3**

End of Question 6

Question 7 (11 marks) Start this question on a new page

(a) Let $z = \dfrac{3 + 4i}{5}$ and $w = \dfrac{5 + 12i}{13}$.

(i) Find zw and $z\overline{w}$, expressing your answer in $x + iy$ form. **2**

(ii) Hence, find two distinct ways of writing 65^2 in the form $a^2 + b^2$, where $a, b \in \mathbb{Z}$ and $0 < a < b < 65$. **1**

(b) By using a suitable substitution, or otherwise, evaluate **3**

$$\int_0^{\pi/2} \frac{d\theta}{2 + \cos\theta}.$$

(c) Suppose that b and c are odd integers. Show that the quadratic equation $x^2 - bx + c = 0$ cannot have repeated real roots. **3**

(d) Prove that there is only one pair of primes (p, q) such that $p^3 - q^3$ is also a prime number. **2**

End of Question 7

Question 8 (13 marks) Start this question on a new page

(a) Let $1, w$ and w^2 be the three cube roots of unity.

(i) Find the value of $(1+2w+3w^2)(1+2w^2+3w)$. **2**

(ii) If the equations $x^3-1=0$ and $px^5+qx+r=0$ have a common root, evaluate **2**

$$(p+q+r)(pw^2+qw+r)(pw+qw^2+r).$$

(b) (i) Find $\displaystyle\int \frac{x}{1+2x+x^2}dx$. **2**

(ii) Hence, or otherwise, find the exact volume generated when the region bounded by the curve $y=\dfrac{e^x}{1+e^x}$, the lines $x=-\ln 3$, $x=\ln 3$ and the x-axis is rotated about the x-axis. **3**

(c) (i) Find constants A and B such that **2**

$$4\cos x-8\sin x=A(-2\sin x-\cos x)+B(2\cos x-\sin x).$$

(ii) Hence, find $\displaystyle\int \frac{4\cos x-8\sin x}{2\cos x-\sin x}dx$. **2**

End of Question 8

Question 9 (14 marks) Start this question on a new page

(a) Evaluate $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos^3 x}{\sin^2 x}\,dx$. **3**

(b) (i) Show that $(1-\sqrt{t})^{n-1}\sqrt{t} = (1-\sqrt{t})^{n-1} - (1-\sqrt{t})^n$. **1**

(ii) Define **4**

$$I_n = \int_0^1 (1-\sqrt{t})^n\,dt \quad \text{for } t \geq 0.$$

Show that

$$I_n = \frac{n}{n+2} I_{n-1} \quad \text{for } n \geq 1.$$

(iii) Hence, show that $\displaystyle\int_0^1 (1-\sqrt{t})^n\,dt = \frac{2n!}{(n+2)!}$. **2**

(c) Prove that the square of every prime number greater than 3 is one more than a multiple of 24. **4**

End of Question 9

Question 10 (14 marks) Start this question on a new page

(a) It is given that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (Do NOT prove this).

(i) Find the roots of $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$, expressing your answers in the form $x = \tan\alpha$. **3**

(ii) Hence, find the exact value of **1**

$$\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right) + \tan^2\left(\frac{5\pi}{16}\right) + \tan^2\left(\frac{7\pi}{16}\right).$$

(b) It is given that $w = r(\cos\theta + i\sin\theta)$ is a solution of the equation $z^n + az + 1 = 0$, where $\sin\theta \neq 0$ and $n > 2$ is an integer.

(i) Show that **1**

$$r^n\cos(n\theta) + ar\cos\theta + 1 = 0 \qquad \text{and} \qquad r^n\sin(n\theta) + ar\sin(\theta) = 0.$$

(ii) Hence, or otherwise, show that **2**

$$|w| \geq \sqrt[n]{\frac{1}{n-1}}.$$

You may assume that $|\sin(kx)| \leq k|\sin(x)|$ for all positive integers k.

(c) Suppose $f(x)$ is a function which satisfies the following: **4**

- $f''(x)$ is continuous
- $f(\pi) = 1$
- $\displaystyle\int_0^{\pi} \big(f(x) + f''(x)\big)\sin x \, dx = 2$

By using integration by parts, or otherwise, find $f(0)$.

(d) Prove that if n is a composite positive integer, then n divides $(n-1)!$ except in the case where $n = 4$. **3**

End of Question 10

End of Examination.

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MC Answers:
1B 2A 3C 4D 5A

Question 6:

(a) Find the least positive integer k such that $\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)$ is a solution of $z^k = 1$.

$$\left[\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)\right]^k = 1$$

$$\therefore \cos\frac{4k\pi}{7} + i\sin\frac{4k\pi}{7} = 1$$

$$\therefore \frac{4k\pi}{7} = n\pi, \quad n \in \mathbb{Z}$$

$$\therefore k = \frac{7n}{4}, \text{ so } n = 4 \text{ and } \boxed{k = 7}.$$

1 mark

(b) Let m and n be integers. Show that if the complex number w is a solution of $z^n = 1$, then so is w^m.

suppose w is a solution of $z^n = 1 \therefore w^n = 1$

$$(w^m)^n = (w^n)^m = 1^m = 1$$

$\therefore w^m$ is a solution of $z^n = 1$.

1 mark for sub w^m into z^n.

1 mark for showing that $(w^m)^n = 1$

(c) Find the following Integrals:

(i) $\int \frac{1}{\sqrt{u}+1} du$

Let $x = \sqrt{u} + 1$

$$dx = \frac{1}{2\sqrt{u}} du$$

$$du = 2\sqrt{u}\, dx$$
$$= 2(x-1)\, dx$$

$$\int \frac{1}{\sqrt{u}+1} du = \int \frac{2(x-1)}{x} dx = 2\int \left(1 - \frac{1}{x}\right) dx \checkmark$$

$$= 2x - 2\ln|x| + C$$

$$= 2(\sqrt{u}+1) - 2\ln(\sqrt{u}+1) + C \checkmark$$

Note: Final answer must be in terms of u.

(ii) $\int x\cos(x)\sin(x)\,dx = \frac{1}{2}\int x \sin 2x \, dx$

let $u = x$ and $dv = \sin 2x \, dx$

$du = dx$ and $v = \frac{-1}{2}\cos 2x$

$$\int x \cos x \sin x \, dx = \frac{1}{2}\left[\frac{-1}{2} x \cos 2x - \int \left(\frac{-1}{2}\cos 2x\right) dx\right]$$

$$= \frac{-1}{4} x \cos 2x + \frac{1}{4}\int \cos 2x \, dx$$

$$= \frac{-1}{4} x \cos 2x + \frac{1}{8}\sin 2x + C$$

Note: you can not write $dv = \sin 2x$ without dx

(d) Consider the following statement:

P: " If I am tall and athletic, I am good at basketball and volleyball."

(i) Write the contrapositive of P as an English Sentence.

If I am not good at either basketball or volleyball then I am either not tall or not athletic

(ii) Copy and complete the following sentence for the negation of P.

"I am tall and athletic, and I am bad at either basketball or volleyball.

Note: $P \Rightarrow Q \equiv \neg(P \wedge \neg Q)$

$\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$

(c) Prove, by contrapositive, that if $\frac{mn}{2}$ is an integer, then m and n must be even.

The contrapositive is:

If m and n are odd, then $\frac{mn}{2}$ is not an integer

suppose $m = 2k+1$ and $n = 2j+1$, $k, j \in \mathbb{Z}$

$$\frac{mn}{2} = \frac{(2k+1)(2j+1)}{2}$$

$$= \frac{4kj + 2(k+j) + 1}{2}$$

$$= 2kj + k + j + \frac{1}{2}$$

$$= M + \frac{1}{2}, \text{ where } M = 2kj + k + j \in \mathbb{Z}$$

but $\frac{1}{2} \notin \mathbb{Z}$ $\therefore M + \frac{1}{2} \notin \mathbb{Z}$ $\therefore \frac{mn}{2} \notin \mathbb{Z}$.

Note: you need to justify your answer by saying clearly that $\frac{1}{2}$ is not an integer.

Note: you need to write the contrapositive statement

Question 7

a)

i. $zw = \left(\frac{3+4i}{5}\right)\left(\frac{5+12i}{13}\right)$

$= \frac{(3+4i)(5+12i)}{65}$

$= \frac{15+36i+20i-48}{65}$

$= \frac{-33+56i}{65}$

$= -\frac{33}{65}+\frac{56i}{65}$ 1 mark

$z\overline{w} = \left(\frac{3+4i}{5}\right)\left(\frac{5-12i}{13}\right)$

$= \frac{(3+4i)(5-12i)}{65}$

$= \frac{15-36i+20i+48}{65}$

$= \frac{63-16i}{65}$

$= \frac{63}{65}-\frac{16i}{65}$ 1 mark

ii. $|z| = |\bar{z}| = |w| = |\overline{w}| = 1$

$\therefore |zw| = 1 \rightarrow \frac{1}{65}|-33+56i| = 1$

$\therefore \sqrt{(-33)^2+56^2} = 65$

$\therefore 33^2+56^2 = 65^2$

Similarly, $|z\overline{w}| = 1 \rightarrow \frac{1}{65}|63-16i| = 1$

$\therefore \sqrt{(63)^2+(-16)^2} = 65$

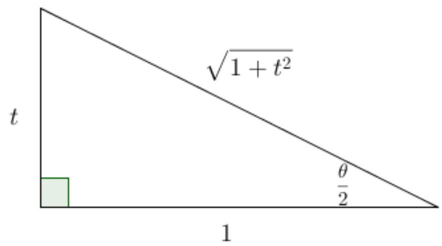
$\therefore 63^2+16^2 = 65^2$

1 mark for answer that shows the acknowledgement of $|zw| = 1$ and $|z\overline{w}| = 1$

However, the sums MUST be made up of positive integers b)

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$$

$$\text{Let } t = \tan\left(\frac{\theta}{2}\right)$$



$$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)$$

$$\frac{d\theta}{dt} = 2\cos^2\left(\frac{\theta}{2}\right)$$

$$d\theta = \frac{2}{1+t^2}dt$$

$$\text{When } \theta = \frac{\pi}{2}, t = 1$$

$$\theta = 0, t = 0$$

1 mark for finding ALL necessary information for the substitution.

$$\therefore I = \int_0^1 \frac{\frac{2}{1+t^2}}{2+\left(\frac{1-t^2}{1+t^2}\right)}dt$$

$$= \int_0^1 \frac{2}{t^2+3}dt$$

$$= \frac{2}{\sqrt{3}}\int_0^1 \frac{\sqrt{3}}{t^2+3}dt$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_0^1$$

1 mark for correctly integrating into an inverse tan function.

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right)$$

$$= \frac{2}{\sqrt{3}} \times \frac{\pi}{6}$$

$$= \frac{\pi}{3\sqrt{3}}$$

1 mark for answer.

c)

Assume that the quadratic equation $x^2 - bx + c = 0$ has repeated real roots, and

$b = 2m + 1$ and $c = 2n + 1, m, n \in Z$.

$\Delta = 0 \rightarrow (-b)^2 - 4(1)(c) = 0$ 1 mark for recognising $\Delta = 0$ in the assumption

$b^2 - 4c = 0$

$(2m + 1)^2 - 4(2n + 1) = 0$ 1 mark for substitution odd expressions into discriminant

$\therefore (2m + 1)^2 = 4(2n + 1)$

Which is a contradiction since:

- LHS is an odd number since the square of an odd number must be odd.

- RHS is an even number since it is divisible by 4, hence divisible by 2.

$\therefore x^2 - bx + c = 0$ cannot have real repeated roots if b and c are both odd.

1 mark for arriving to a contradiction and conclusion.

Alternatively,

Let $f(x) = x^2 - bx + c$ and the repeated root for $f(x) = 0$ be α.

Since $f(x)$ is monic and b and c are integers, α must be an integer. 1 mark

Repeated root $\rightarrow f'(\alpha) = 0$ 1 mark

$\therefore 2\alpha - b = 0$

$b = 2\alpha$

$\therefore b$ is even, which is a contradiction as b is defined to be odd.

$\therefore x^2 - bx + c = 0$ cannot have real repeated roots if b and c are both odd. 1 mark

Student MUST explain why the root of the equation has to be an integer before using that as a premise in your working.

d)

$p^3 - q^3 = (p-q)(p^2 + pq + q^2)$

$p^3 - q^3$ is prime $\rightarrow$ $(p - q = 1$ and $p^2 + pq + q^2 > 1)$ or $(p - q > 1$ and $p^2 + pq + q^2 = 1)$

p and q are prime $\rightarrow$ $p^2 + pq + q^2 \neq 1$ as $p, q \geq 2$

$\therefore p^2 + pq + q^2 > 1$ and $p - q = 1$

Since there are only 2 prime numbers that have a difference of 1, namely 2 and 3, then, there is only one pair of prime (p, q) such that $p^3 - q^3$ is also a prime.

1 mark for getting the implication for if $p^3 - q^3$ is prime.

i.e. $(p - q = 1$ or $p^2 + pq + q^2 = 1)$

1 mark for final answer.

MATHEMATICS EXT 2: Question 8

Suggested Solutions	Marks	Marker's Comments
a i) Note that $1+\omega+\omega^2=0$ & $\omega^3=1$.		
$(1+2\omega+3\omega^2)(1+2\omega^2+3\omega)$		
$=1+2\omega^2+3\omega+2\omega+4\omega^3+6\omega^2+3\omega^2+6\omega^4+9\omega^3$		
$=1+5\omega+11\omega^2+13\omega^3+6\omega^4$		
$=14+11\omega+11\omega^2$, since $\omega^3=1$	1	Using $\omega^3=1$ with reasonable progress.
$=3+11(1+\omega+\omega^2)$		
$=3+11\times 0$		
$=3$	1	Final answer
ii) Since $x^3-1=0$ & $px^5+qx+r=0$ share a common root, the root must be either 1, ω or ω^2.		
Case 1: 1 is the common root		
Then $p(1)^5+q(1)+r=0$	1	Fully correct working.
$p+q+r=0$		
$\Rightarrow (p+q+r)(p\omega^2+q\omega+r)(p\omega+q\omega^2+r)=0$		
Case 2: ω is the common root		Most students falsely claimed that $z=1$ was THE root, or that $z=\omega$ was THE root. This reduces the generality and thus is an incomplete solution.
Then $p\omega^5+q\omega+r=0$		
$p\omega^2+q\omega+r=0$, since $\omega^3=1$		
$\Rightarrow (p+q+r)(p\omega^2+q\omega+r)(p\omega+q\omega^2+r)=0$		
Case 3: ω^2 is the common root		
Then $p\omega^{10}+q\omega^2+r=0$		
$p\omega+q\omega^2+r=0$, since $\omega^3=1$		
$\Rightarrow (p+q+r)(p\omega^2+q\omega+r)(p\omega+q\omega^2+r)=0$		
In either case, the product is always equal to 0.	1	

MATHEMATICS EXT 2: Question 8

Suggested Solutions	Marks	Marker's Comments
bi, $\displaystyle\int \frac{x}{1+2x+x^2}\,dx$		
$\displaystyle = \int \frac{x}{(1+x)^2}\,dx$		
$\displaystyle = \int\left(\frac{x+1}{(1+x)^2} - \frac{1}{(1+x)^2}\right)dx$	1	Algebraic manipulation
$\displaystyle = \int\left(\frac{1}{1+x} - (1+x)^{-2}\right)dx$		
$\displaystyle = \ln\lvert 1+x\rvert + \frac{1}{1+x} + C$	1	Final answer → MUST have absolute values.
ii, $\displaystyle V = \pi\int_{-\ln 3}^{\ln 3}\left(\frac{e^x}{1+e^x}\right)^2 dx$		
$\displaystyle = \pi\int_{-\ln 3}^{\ln 3}\frac{e^x \cdot e^x}{1+2e^x+e^{2x}}\,dx$ let $u = e^x$ $du = e^x\,dx$ $x = \ln 3 \rightarrow u = 3$ $x = -\ln 3 \rightarrow u = \frac{1}{3}$	1	Recognising $u = e^x$ sub or equivalent
$\displaystyle = \pi\int_{1/3}^{3}\frac{u}{1+2u+u^2}\,du$		
$\displaystyle = \pi\left[\ln\lvert 1+u\rvert + \frac{1}{1+u}\right]_{1/3}^{3}$	1	Using part (i)
$\displaystyle = \pi\left(\ln 4 + \frac{1}{4} - \left(\ln\left(\frac{4}{3}\right) + \frac{3}{4}\right)\right)\ u^3$		
$\displaystyle = \pi\left(\ln 3 - \frac{1}{2}\right)\ u^3$	1	Final answer

MATHEMATICS EXT 2: Question 8

Suggested Solutions	Marks	Marker's Comments
c i) $4\cos x - 8\sin x = (2B - A)\cos x + (-2A - B)\sin x$		
$\Rightarrow \begin{cases} 4 = 2B - A \\ -8 = -2A - B \end{cases}$		
Solving simultaneously:		
$A = \frac{12}{5} \qquad B = \frac{16}{5}$	2	One for each.
ii) $\int \frac{4\cos x - 8\sin x}{2\cos x - \sin x}\,dx$		
$= \int \frac{\frac{12}{5}(-2\sin x - \cos x) + \frac{16}{5}(2\cos x - \sin x)}{2\cos x - \sin x}\,dx$	1	Using part (i)
$= \frac{12}{5}\ln\lvert 2\cos x - \sin x\rvert + \frac{16}{5}x + C$	1	Final answer.

MATHEMATICS Extension 2: Question 9

Suggested Solutions	Marks	Marker's Comments
a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^2 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x(1-\sin^2 x)}{\sin^2 x}\,dx$	1	
$\displaystyle = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} - \cos x\,dx$		
$\displaystyle = \left[\frac{-1}{\sin x} - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$	1	
$\displaystyle = \frac{-1}{\sin\frac{\pi}{2}} - \sin\frac{\pi}{2} - \left(-\frac{1}{\sin\frac{\pi}{6}} - \sin\frac{\pi}{6}\right)$		
$= -1 - 1 - \left(-2 - \frac{1}{2}\right)$		
$= \frac{1}{2}$	1	
b) i. RHS $= (1-\sqrt{t})^{n-1} - (1-\sqrt{t})^{n}$		
$= (1-\sqrt{t})^{n-1}\left[1 - (1-\sqrt{t})\right]$	1	
$= (1-\sqrt{t})^{n-1}\sqrt{t} =$ LHS		
ii. $I_n = \int_0^1 (1-\sqrt{t})^n\,dt$		
$\displaystyle = \left[t(1-\sqrt{t})^n\right]_0^1 - \int_0^1 t \times \frac{-n}{2\sqrt{t}}(1-\sqrt{t})^{n-1}\,dt$	1	Recognising by parts
$\displaystyle = (1(1-1)^n - 0) + \frac{n}{2}\int_0^1 \sqrt{t}(1-\sqrt{t})^{n-1}\,dt$	1	Applying by parts correctly
$\displaystyle = \frac{n}{2}\int_0^1 (1-\sqrt{t})^{n-1} - (1-\sqrt{t})^n\,dt$	1	Many students used part i) too early on and got a statement of $I_n = I_n$ or $I_{n-1} = I_{n-1}$ No marks awarded for that.
$= \frac{n}{2}\left(I_{n-1} - I_n\right)$		
$2I_n = nI_{n-1} - nI_n$		
$I_n(2+n) = nI_{n-1}$		
$\displaystyle I_n = \frac{n}{2+n} I_{n-1}$	1	
iii) $\displaystyle I_n = \frac{n}{n+2} I_{n-1}$		
$\displaystyle = \frac{n}{n+2} \times \frac{n-1}{n+1} I_{n-2}$		
$\displaystyle = \frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n} I_{n-3}$	1	

MATHEMATICS Extension 2: Question 9

Suggested Solutions	Marks	Marker's Comments
$= \frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n} \times \dots \times \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times I_0$		Could use I_1 as ending of the reduction.
$I_0 = \int_0^1 dt$		
$= 1$		
$\therefore I_n = \frac{n!}{(n+2)(n+1)n \times \dots 4 \times 3} \times \frac{2}{2} \times 1$		Had to show I_0 or I_1 and then adjustment for factorial in denominator.
$= \frac{2n!}{(n+2)!}$	1	
c) Let $p > 3$ be prime.		
p must be odd (only even prime is 2)		
prove $p^2 = 24N + 1$, $N \in \mathbb{Z}$	1	
Consider:		
$p^2 - 1$		
$\therefore p = 2k + 1$ for some integer k		
$p^2 - 1 = (2k+1)^2 - 1$		
$= 4k^2 + 4k + 1 - 1$		
$= 4k(k+1)$		
Since k and $(k+1)$ are consecutive, one must be even.		
i.e. $k(k+1) = 2M$, $M \in \mathbb{Z}$		
$\therefore p^2 - 1 = 4(2M)$	1	
$= 8M$		
$(p-1) \times p \times (p+1) = 3N$, $N \in \mathbb{Z}$ as they are 3 consecutive numbers		
Since p is prime the factor of 3 lies in $(p-1)\times(p+1)$	1	
$\therefore p^2 - 1 = 8 \times 3N$, $N \in \mathbb{Z}$ as 3 and 8 are coprime		
$\therefore p^2 - 1 = 24N$	1	
$p^2 = 24N + 1$		
OR Possible primes occur at $6k+1$ or $6k-1$ (remove all numbers with factors of 2 and/or 3) for some integer k		Can also use $6k+5$
CASE 1: $(6k+1)^2 = 36k^2 + 12k + 1$		
$= 12(3k+1)k + 1$		
If k is odd then $3k$ is odd and $3k+1$ is even.		
If k is even then $3k$ is even and $3k+1$ is odd		
i.e. $k(3k+1) = 2M$, $M \in \mathbb{Z}$	1	
$\therefore$ Our squared prime can be expressed as $12(2M)+1$		

MATHEMATICS Extension 2: Question....9....

Suggested Solutions	Marks	Marker's Comments
or $24M + 1$ i.e. one greater than a multiple of 24		
CASE 2 : $(6k-1)^2 = 36k^2 - 12k + 1$		
$= 12(3k-1)k + 1$		
If k is odd then $3k$ is odd and $3k-1$ is even.		
If k is even then $3k$ is even and $3k-1$ is odd	①	
i.e. $k(3k+1) = 2N$, $N \in \mathbb{Z}$		THIS CANNOT BE DONE BY MATHEMATICAL INDUCTION.
$\therefore$ Our squared prime can be expressed as $12(2N)+1$ or $24N+1$ i.e. one greater than a multiple of 24.		

Question 10

a, i, $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$$x^4 - 6x^2 + 1 = 4x - 4x^3$$

$$1 = \frac{4x - 4x^3}{x^4 - 6x^2 + 1}$$ ✓

let $x = \tan\theta$

$$1 = \frac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1}$$

$$1 = \tan 4\theta \quad \text{(given)}$$ ✓

$$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots$$

$$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \ldots$$

$\therefore$ Roots are $x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$ ✓

$x = \tan\left(\frac{\pi}{16} + \frac{n\pi}{4}\right)$: max ②

ii, $\tan\frac{9\pi}{16} = \tan\left(-\frac{7\pi}{16}\right)$ $\qquad$ $\tan\frac{13\pi}{16} = \tan\left(-\frac{3\pi}{16}\right)$

$= -\tan\frac{7\pi}{16}$ $\qquad$ $= -\tan\frac{3\pi}{16}$

$\therefore$ The four roots of equation in part i, can be written as

$$x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, -\tan\frac{3\pi}{16}, -\tan\frac{7\pi}{16}$$

$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16}$ ✓

$= \tan^2\frac{\pi}{16} + \left(-\tan\frac{3\pi}{16}\right)^2 + \tan^2\frac{5\pi}{16} + \left(-\tan\frac{7\pi}{16}\right)^2$

$= (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)$

$= (-4)^2 - 2(-6)$

$= 16 + 12$ ✓ CNE allow

$= 28$

'Hence' Q show the link

no mark for answer only

b, i, $w = r(\cos\theta + i\sin\theta)$ is a sol$^{\underline{n}}$ of $z^n + az + 1 = 0$

$\therefore (r \operatorname{cis}\theta)^n + a(r\operatorname{cis}\theta) + 1 = 0$ for $\operatorname{cis}\theta = \cos\theta + i\sin\theta$

allow cis

$r^n \operatorname{cis}(n\theta) + ar\operatorname{cis}\theta + 1 = 0$ ✓

Equating real part: $r^n\cos(n\theta) + ar\cos\theta + 1 = 0$

Equating imaginary part: $r^n\sin(n\theta) + ar\sin\theta = 0$

© Equating coefficients
© Equating complex part

c, $\int_0^\pi (f(x) + f''(x))\sin x \, dx$

© $\int_0^\pi \underset{u}{(f(x)+f''(x))} \underset{v'}{\sin x} \, dx$

$= \int_0^\pi f(x)\sin x\,dx + \int_0^\pi \underset{u'}{f''(x)} \underset{v}{\sin x}\,dx$

$u = f'(x) \quad v' = \cos x$

$= \int_0^\pi f(x)\sin x\,dx + \left[f'(x)\sin x\right]_0^\pi - \int_0^\pi f'(x)\cos x\,dx$ ✓ by parts

$= \int_0^\pi f(x)\sin x\,dx + 0 - \int_0^\pi \underset{u'}{f'(x)} \underset{v}{\cos x}\,dx$ ✓ eval.

$u = f(x) \quad v' = -\sin x$

$= \int_0^\pi f(x)\sin x\,dx - \left[f(x)\cos x\right]_0^\pi - \int_0^\pi f(x)\sin x\,dx$ ✓ by parts

$= \left[f(x)\cos x\right]_\pi^0$

$= f(0)\cos(0) - f(\pi)\cos(\pi)$

$= f(0) + 1$ as $f(\pi) = 1$ given

Since $\int_0^\pi (f(x) + f''(x))\sin x\,dx = 2$ given,

$f(0) + 1 = 2$

$\therefore f(0) = 1$ ✓

d, Suppose $n = ab$, where $a, b \in \mathbb{Z}$ ✱

$1 < a \leq b < n$

↳ or $2 \leq a \leq b \leq n-1$

Case I if $a \neq b$

Since a and b are distinct integers less than n, both a and b show up as different factors in $(n-1)!$

$\therefore \quad n = ab \mid (n-1)!$ ✓

Case II if $a = b = 2$

$n = 4 \nmid 3! = 6$

$\therefore \quad n \nmid (n-1)!$ for $n = 4$ ✓

Case III if $a = b > 2$

i.e. $n = a^2 > 4$

For $a > 2$, $2a < a^2$

$2a \leq a^2 - 1$

$2a \leq n - 1$

$\therefore \quad 2a \mid (n-1)!$

But, also $a \mid (n-1)!$ as $a \in \mathbb{Z}$, $a < n$

Since a and $2a$ are distinct integers less than n

$2a^2 \mid (n-1)!$

$\therefore \quad n = a^2 \mid (n-1)!$ ✓

$\therefore$ If n is a composite positive integer, then $n \mid (n-1)!$ except when $n = 4$