The Scots College

2023

HSC Year 12 Assessment Task 2

Monday 27th March 2023

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- For questions in Section II, show all relevant mathematical reasonings and/or calculations.
- Answer each question at the start of a NEW writing booklet.

Total marks: 70

- **Section I – 5 marks** (pages 3–4)
  - Attempt Questions 1–5
  - Allow about 8 minutes for this section
  - Use the multiple choice answer sheet provided

- **Section II – 65 marks** (pages 5–8)
  - Attempt Questions 6 - 9
  - Allow about 1 hour and 52 minutes for this section

Areas of Assessment	Marks possible
Complex Numbers	27
Proofs	21
Vectors	22
Total:	70

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Section 1

5 marks
Attempt Questions 1–5.
Allow about 8 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 5.

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1 Which of these correctly expresses $z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ in the standard form $a + ib$?

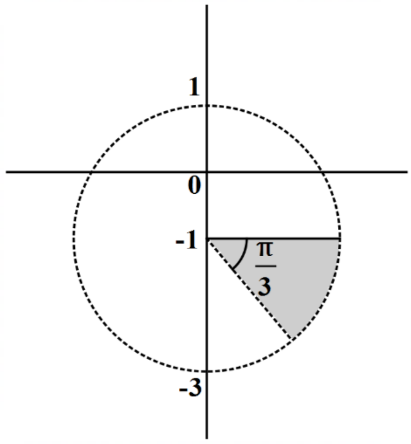
A. $z = 1 + i\sqrt{3}$

B. $z = 1 - i\sqrt{3}$

C. $z = -1 + i\sqrt{3}$

D. $z = -\sqrt{3} + i$

2 Which of the following best represents the shaded region in the Argand diagram below?



A. $\left\{z : \frac{-\pi}{3} < Arg(z+i) \le 0 \quad and \quad |z-i| < 2\right\}$

B. $\left\{z : \frac{-\pi}{3} < Arg(z+i) \le 0 \quad and \quad |z+i| < 2\right\}$

C. $\left\{z : \frac{-\pi}{3} < Arg(z-i) \le 0 \quad and \quad |z+i| < 2\right\}$

D. $\left\{z : 0 < Arg(z+i) \le \frac{\pi}{3} \quad and \quad |z+i| < 2\right\}$

3 Consider the statement: 'If I have the Covid Virus, you also have the Covid Virus.'
Which of the following is logically equivalent to the statement?

A. If you have the Covid Virus, I also have the Covid Virus.

B. If I do not have the Covid Virus, you do not have the Covid Virus.

C. If you do not have the Covid Virus, I do not have the Covid Virus.

D. You will have the Covid Virus only if I have the Covid Virus.

4 Which of the following represents a pair of lines that are NOT parallel?

A. $y = 3x - 4$ and $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 12 \end{bmatrix}$

B. $\boldsymbol{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $2x + 3y = 8$.

C. $\frac{x+10}{5} = y - 7 = \frac{z+3}{4}$ and $\boldsymbol{x} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 10 \\ 2 \\ 8 \end{bmatrix}$

D. $\frac{x+10}{5} = \frac{z+3}{-2}, y = -5$ and $\boldsymbol{x} = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} + \lambda \begin{bmatrix} 10 \\ 0 \\ -4 \end{bmatrix}$

5 The position vectors of three points are given as $A \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$, $B \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$ and $C \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.
Which of the following correctly identifies the position vector of the point *D*, such that *ABCD* is a parallelogram?

A. $\begin{bmatrix} -6 \\ -1 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

C. $\begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$

D. $\begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$

Section II

65 marks
Attempt Questions 6–9.
Allow about 1 hour and 52 minutes for this section.

Answer each question on a NEW writing booklet.

In Questions 6-9, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 marks) Start a NEW writing booklet.

(a) For what values of n is $(1+i)^n + (1-i)^n = 0$? [3]

(b) Use proof by contradiction to prove that if $7n+7$ is odd, then n is even. [2]

(c) Let P be the proposition:
'If $n^2 - 6n + 5$ is even, then n is odd'

(i) Write down the contrapositive of the proposition P. [1]

(ii) Hence, prove that the proposition P is true. [2]

(d) Find the square roots of $\sqrt{3}+i$. [2]

(e) Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 7 \\ 6 \\ -1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ be three position vectors.

(i) Find $\boldsymbol{u} + 2\boldsymbol{v} - 3\boldsymbol{w}$ [2]

(ii) Show that $\boldsymbol{u}.(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}.\boldsymbol{v} + \boldsymbol{u}.\boldsymbol{w}$ [3]

Question 7 (17 marks) Start a NEW writing booklet

(a) (i) By considering the case where a positive integer k is even $(k = 2m)$ and the case where k is odd $(k = 2m + 1)$, where m is a positive integer, show that $k^2 + k$ is always even. [2]

(ii) Using the result in part (i), prove, by mathematical induction, that for all positive integral values of n, that $n^3 + 5n$ is divisible by 6. [3]

(b) If $z = \cos\theta + i\sin\theta$, show that $\sin\theta = \dfrac{z^2 - 1}{2iz}$ [2]

(c) (i) Prove that $a^2 + b^2 \geq 2ab$ for all $a,b > 0$. [1]

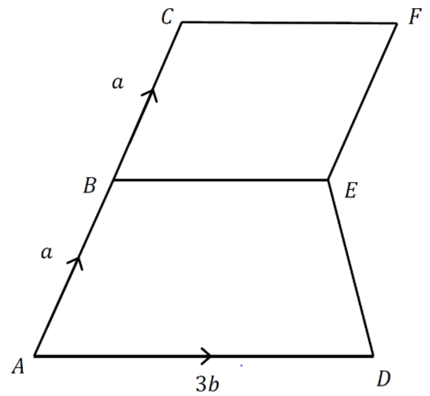
(ii) Hence, deduce that when a and b are positive, [1]

$\frac{a}{b} + \frac{b}{a} \geq 2.$

(iii) Using the result in parts (i) and (ii) above, prove that when a, b, c are all positive, [2]

$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6.$

(d) On the diagram given below (not drawn to scale), $\overrightarrow{AB} = \overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{AD} = 3\mathbf{b}$, AC is a straight line, $CBEF$ is a parallelogram, and $AD : BE : CF = 3 : 2 : 2$.



(i) Find, in terms of **a** and **b**, expressions for vectors $\overrightarrow{DC}$ and $\overrightarrow{FD}$. [3]

(ii) DE is extended upwards so that it meets CFat the point X. Find the ratio of $CX : XF$. [3]

Question 8 (16 marks) Start a NEW writing booklet

(a) Simplify $\left(1+i\sqrt{3}\right)^4\left(\sqrt{3}-i\right)^3$ [3]

(b) (i) By expanding $\left(\cos\theta+i\sin\theta\right)^6$ using binomial expansion and De Moivre's theorem, show that [4]

$\tan 6\theta = -\left(\frac{6t^5-20t^3+6t}{t^6-15t^4+15t^2-1}\right)$, where $t = \tan\theta$

(ii) Hence, or otherwise, solve [3]

$$x^6+6x^5-15x^4-20x^3+15x^2+6x=1$$

(c) If p, q are positive real numbers and $p+q=1$, show that $(px+qy)^2 \le px^2+qy^2$, for all real values of x and y. [3]

(d) A line whose parametric vector equation is $\mathbf{r} = \begin{bmatrix}2\\0\\3\end{bmatrix} + \lambda\begin{bmatrix}-1\\2\\3\end{bmatrix}$ intersects with a sphere $(x-1)^2+(y+2)^2+(z+3)^2 = 31$. [4]

Find the position vector of the midpoint of the points of intersection.

Question 9 (17 marks) Start a NEW writing booklet

(a) The points A, B and C have position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, respectively.

(i) Find the magnitude of the angle ABC to the nearest minute. [2]

(ii) Find the area of the triangle ABC using the formula $Area = \frac{1}{2}ab\sin C$. [1]

(iii) Find the position vector of the point D, such that BD is perpendicular to AC. [2]

(b) Let $z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

(i) Show that z is a root of $x^5 - 1 = 0$. [1]

(ii) Show that $1 + z + z^2 + z^3 + z^4 = 0$. [2]

(iii) Prove that $a = z + \frac{1}{z}$ is a root of the equation $a^2 + a - 1 = 0$. [2]

(iv) Hence, show that $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$. [3]

(c) A sequence of real numbers is defined as follows: [3]

$x_1 = 6$

$x_n = 3n(n+1) + x_{n-1}\,; n = 2, 3, 4, \dots \dots$

Prove by mathematical induction that $x_n < (n+1)^3$ for all positive integers n.

End of Paper

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