BAULKHAM HILLS HIGH SCHOOL

2024

HSC Assessment Task 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using non-erasable black pen.
- Calculators approved by NESA may be used.
- Marks may be deducted for careless or badly arranged work.
- The Mathematics Reference Sheet is provided at the back of this paper.
- Answer all questions on the appropriate page of your answer booklet.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.
- If extra paper is used, it must be clearly labelled with your NESA number and the question being answered, and stapled inside your answer booklet.

Total marks – 39
Exam consists of 7 pages.

Section I: Pages 2 - 4
5 marks
Attempt Questions 1 – 5
Allow about 8 minutes for this section

Section II: Pages 5 - 7
34 marks
Attempt Questions 6 – 8
Allow about 52 minutes for this section

Section I: **Marks**

5 marks

Attempt Questions 1 - 5

Use the multiple-choice answer sheet in your booklet.

1. Which of the following is equivalent to $\sqrt{i^3}$? **1**

(A) $e^{\frac{i\pi}{4}}$

(B) $e^{\frac{i\pi}{2}}$

(C) $e^{\frac{i3\pi}{4}}$

(D) $e^{\frac{-i\pi}{2}}$

2. The polynomial $P(z)$ has real coefficients, and $z = 3 + i$ is a root of $P(z)$. Which of the following quadratic polynomials must be a factor of $P(z)$? **1**

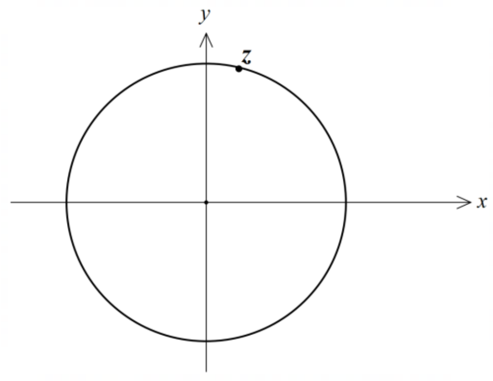
(A) $z^2 - 6z + 10$

(B) $z^2 + 6z + 10$

(C) $z^2 - 6z + 8$

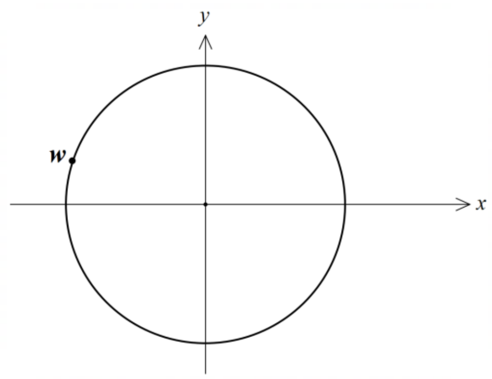
(D) $z^2 + 6z + 8$

3. The Argand diagram below shows the complex number location for the complex number z. **1**

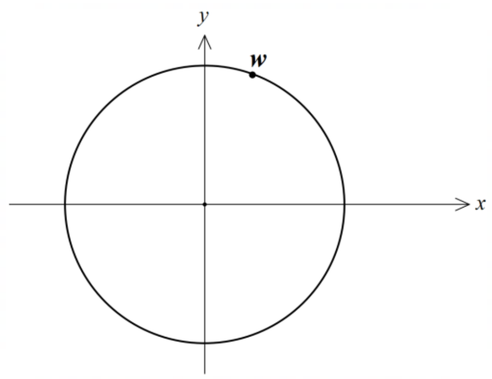


Which Argand diagram best represents $w = \dfrac{z^2}{i|z|}$?

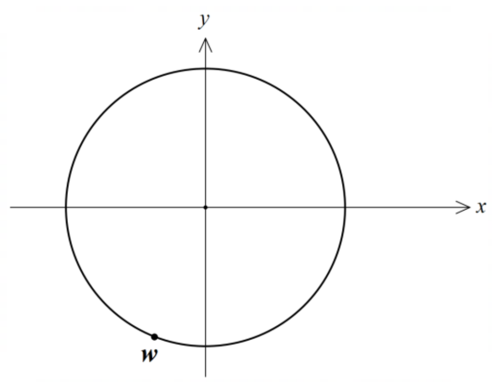
(A)



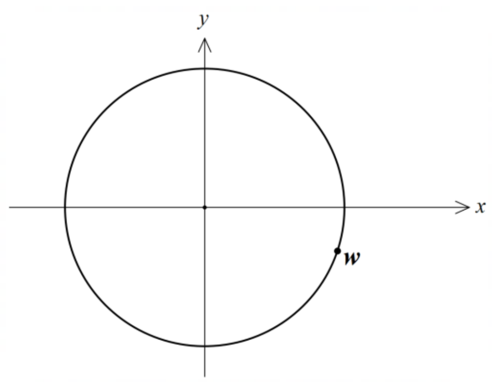
(B)



(C)



(D)



4. Which of the following expression is equivalent to $(-1+i)^n$ where n is a positive integer? **1**

(A) $\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)$

(B) $\left(\sqrt{2}\right)^{-n}\left(\cos\frac{-n\pi}{4}+i\sin\frac{-n\pi}{4}\right)$

(C) $\left(\sqrt{2}\right)^n\left(\cos\frac{3n\pi}{4}+i\sin\frac{3n\pi}{4}\right)$

(D) $\left(\sqrt{2}\right)^{-n}\left(\cos\frac{3n\pi}{4}+i\sin\frac{3n\pi}{4}\right)$

5. If $z^2+z+1=0$, where z is complex number then what is the value of **1**

$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\ldots+\left(z^6+\frac{1}{z^6}\right)^2$?

(A) 12

(B) 10

(C) 14

(D) 8

End of Section I

Section II: 34 marks

Attempt Questions 6 - 8

Answer each question on the appropriate page of the answer booklet, showing all necessary working

Question 6 (11 marks)

a) Let $w = 1 + iz$, where $w, z \in \mathbb{C}$. Express each of the following in the form of $x + iy$.

(i) w if $z = 2 + i$ 1

(ii) z if $w = 2 - i$ 1

b) Given that $z = (b + i)^2$, where b is real and positive. Find the value of b if $\arg(z) = \dfrac{\pi}{3}$. 3

c) Solve the equation $z^2 - 5z + (7 + i) = 0$. 3

d) (i) On an Argand diagram shade in the region determined by the inequalities 2

$|z - \bar{z}| \le 4$ and $\dfrac{7\pi}{6} \le \arg(z) \le \dfrac{5\pi}{4}$.

(ii) Let z_0 be the complex number of maximum modulus satisfying the above inequalities. Express z_0 in the form of $x + iy$. 1

End of Question 6

Question 7 (12 marks)

a) Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0$, where $z \in \mathbb{C}$. Two of the roots of this equation are $(a+ib)$ and $(b+ia)$, where $a, b \in \mathbb{Z}$. Find the possible values of a. 3

b) Sketch the locus of the point z on an Argand diagram, given $\arg\left(\dfrac{z-2}{z-2+4i}\right) = \dfrac{\pi}{2}$. 2

c) Prove that if $|z| = 1$, then $\dfrac{z-1}{z+1}$ is a purely imaginary number. 2

d) (i) Show that $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$. 1

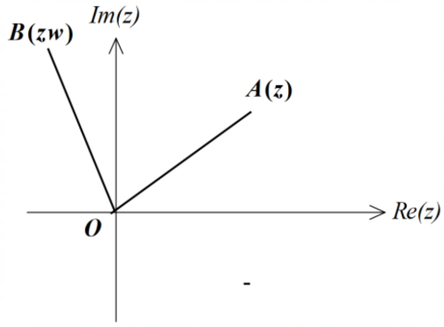
(ii) If z is a non-real root of $z^5 + 1 = 0$, prove that $z^4 + z^2 + 1 = z^3 + z$. 1

(iii) Hence or otherwise, show that $\cos\dfrac{\pi}{5} + \cos\dfrac{3\pi}{5} = \dfrac{1}{2}$. 3

End of Question 7

Question 8 (11 marks)

a) On the Argand diagram below, the complex numbers z and zw are represented by the points A and B respectively, where $z = re^{i\theta}$ and $w = e^{\frac{i\pi}{3}}$, and $r > 0$.



(i) Explain why OAB is an equilateral triangle. 2

(ii) Write the complex number $(z - zw)$ in exponential form in terms of r and θ. 2

b) (i) For a complex number z, where n is a positive integer and $|z| = 1$, show that 1

$z^n + \overline{z}^n = 2\cos n\theta$.

(ii) Given that $\theta \neq (2k+1)\pi$, where $k \in \mathbb{Z}$, prove that $\dfrac{e^{i\theta} - 1}{e^{i\theta} + 1} = i\tan\left(\dfrac{\theta}{2}\right)$. 2

c) Let z_1, z_2, z_3 and z_4 are four complex numbers such that:

$|z_1| = |z_2| = |z_3| = 1, \quad |z_4| = 3 \quad$ and $\quad z_1 + z_2 + z_3 = 0$

(i) Show that $|z_4 - z_1|^2 = 10 - (z_4\overline{z}_1 + \overline{z}_4 z_1)$. 2

(ii) Hence show that $|z_4 - z_1|^2 + |z_4 - z_2|^2 + |z_4 - z_3|^2 = 30$. 2

END OF EXAMINATION

Year-12 Assessment Task-1, 2025 Mathematics Extension-2

Marking Guidelines

Section - I

Multiple Choice (1 - 5)

1) C   2) A   3) B   4) C   5) A

Sample Solutions:

1) $i = e^{i\pi/2}$

$i^3 = e^{i3\pi/2}$

$\sqrt{i^3} = e^{i3\pi/4}$ (C)

2) $3+i$ and $3-i$ are roots

Sum of roots $= 6$

Product of roots $= (3+i)(3-i) = 10$

$\therefore$ $z^2 - 6z + 10$ is a factor of $P(z)$.   (A)

3) Let $z = re^{i\theta}$

$w = \dfrac{z^2}{i|z|} = \dfrac{r^2 e^{i2\theta}}{r \cdot e^{i\pi/2}} = re^{i(2\theta - \pi/2)}$

$\therefore$ w has same modulus as z.   (B)

$\arg w$ is in first quadrant. ($\because$ θ is in first quadrant, 2θ is in second quadrant, $(2\theta - \frac{\pi}{2})$ will be in first quadrant)

4) $(-1+i)^n = \left(\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right)^n$

$= \sqrt{2}^n\left(\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right)$ (C)

5) Since $z^2+z+1=0 \Rightarrow \frac{z^3-1}{z-1}=0$, $z \neq 1$

$\therefore$ $z = \omega, \omega^2$, where ω is a cube root of unity.

(Also $\omega^3 = 1$, $\frac{1}{\omega} = \omega^2$ or $\frac{1}{\omega^2} = \omega$)

Let $z = \omega$

$$\therefore \left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 + \cdots + \left(z^6+\frac{1}{z^6}\right)^2$$

$$= \left(\omega+\frac{1}{\omega}\right)^2 + \left(\omega^2+\frac{1}{\omega^2}\right)^2 + \left(\omega^3+\frac{1}{\omega^3}\right)^2 + \left(\omega^4+\frac{1}{\omega^4}\right)^2 + \left(\omega^5+\frac{1}{\omega^5}\right)^2 + \left(\omega^6+\frac{1}{\omega^6}\right)^2$$

$$= (\omega+\omega^2)^2 + (\omega^2+\omega)^2 + (1+1)^2 + (\omega+\omega^2)^2 + (\omega^2+\omega)^2 + (1+1)^2$$

$$= 4(\omega+\omega^2)^2 + 4 + 4$$

$$= 4(-1)^2 + 4 + 4$$

$$= 12 \qquad (A)$$

END OF SECTION-I

Section II

Question 6

a) i) $w = 1 + iz$ — 1 mark Correct Solution

$w = 1 + i(2+i)$

$= 1 + 2i + i^2$

$= 2i$

ii) $w = 2 - i$, $z = ?$ — 1 mark Correct Solution

$2 - i = 1 + iz$

$iz = 1 - i$

$z = \frac{1}{i} - 1$

$= -1 - i$

b) $z = (b+i)^2$ — 3 marks Correct Solution.

$= b^2 + i2b - 1$ — 2 marks Solving the quadratic equation.

$= (b^2 - 1) + i2b$

Since $\arg z = \frac{\pi}{3}$ — 1 mark finding the quadratic equation by using arg. or equivalent

$\therefore \quad \frac{2b}{b^2-1} = \tan\frac{\pi}{3}$

$\frac{2b}{b^2-1} = \sqrt{3}$

$\sqrt{3}b^2 - 2b - \sqrt{3} = 0$

$(\sqrt{3}b + 1)(b - \sqrt{3}) = 0$

$b = \sqrt{3}, -\frac{1}{\sqrt{3}}$

$b = \sqrt{3}$ ($\because b > 0$ as $\arg z$ is acute)

c) $z^2 - 5z + (7+i) = 0$

$\Delta = (-5)^2 - 4(7+i)$

$= 25 - 28 - 4i$ — 3 marks Correct Solution.

$= -3 - 4i$ — 2 marks Finding the square root of discriminant.

$\sqrt{-3-4i} = a + ib \quad , \quad a, b \in \mathbb{R}$

$-3 - 4i = a^2 - b^2 + 2iab$ — 1 mark Finding the discrimint of given equation.

$a^2 - b^2 = -3, \quad ab = -2$

Solve then $a + ib = \pm(1 - 2i)$

$\therefore \sqrt{-3-4i} = \pm(1-2i)$

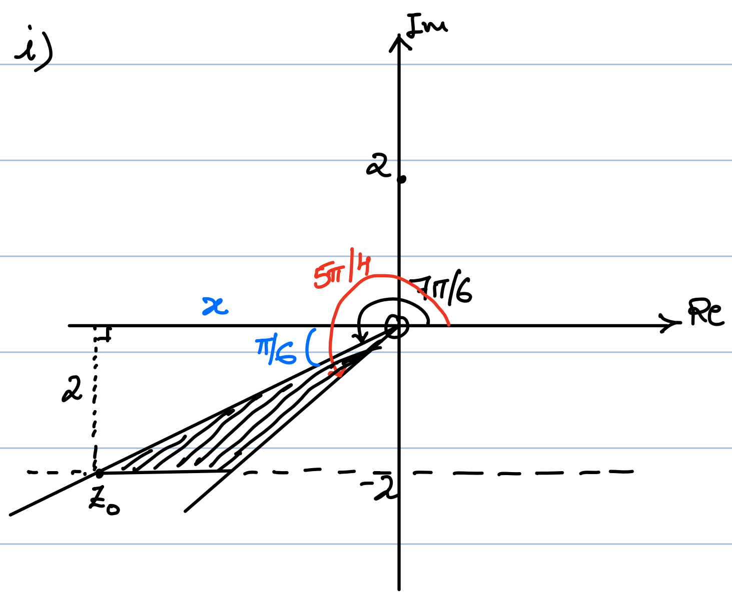
$z = \dfrac{5 \pm \sqrt{-3-4i}}{2}$

$= \dfrac{5 \pm (1-2i)}{2}$

$= \dfrac{6-2i}{2} \quad , \quad \dfrac{4+2i}{2}$

$= 3 - i \quad , \quad 2 + i$

d) i)



2 marks Correct solution including region.

1 mark Correct graph of any one inequality.

ii) $\tan\frac{\pi}{6} = \frac{2}{x}$ — 1 mark Correct solution

$\frac{1}{\sqrt{3}} = \frac{2}{x} \Rightarrow x = 2\sqrt{3} \qquad \therefore z_0 = -2\sqrt{3} - 2i$

Question: 7

Since $(a+ib)$ and $(b+ia)$ are roots

$\therefore$ $(a-ib)$ and $(b-ia)$ are also roots

Sum of the roots $= -4$

$\therefore$ $a+ib+a-ib+b+ia+b-ia = -4$

$\therefore$ $2a+2b = -4$

$a+b = -2$ ——— (1)

Product of the roots $= 400$

$\therefore$ $(a+ib)(a-ib)(b+ia)(b-ia) = 400$

$(a^2+b^2)(a^2+b^2) = 400$

$(a^2+b^2)^2 = 400$

$a^2+b^2 = 20 \quad (\because a^2+b^2 > 0)$

from (1) $b = -2-a$

$\therefore$ $a^2 + (-2-a)^2 = 200$

$a^2 + a^2 + 4a + 4 = 20$

$2a^2 + 4a - 16 = 0$

$a^2 + 2a - 8 = 0$

$(a+4)(a-2) = 0$

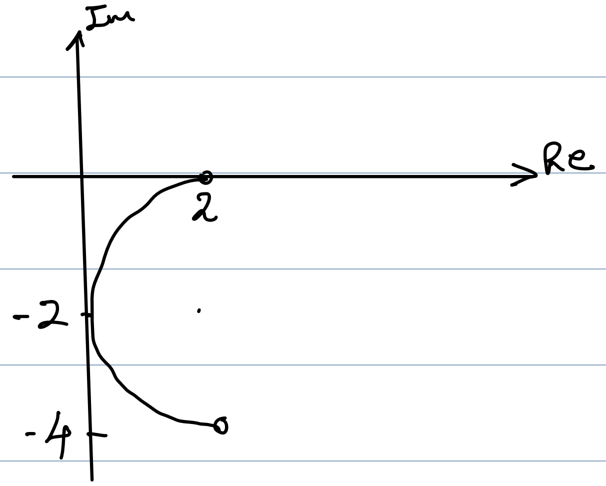
$a = -4, 2$

3 marks Correct Solution

2 marks Using Sum and Product of roots to find the equations in terms of a & b or equivalent merit

1 mark Finding the Conjugate of the given roots or four roots of given eq.

b)



2 marks correct solution

1 mark sketching the locus as semi-circle in any form.

c) Geometric Approach:

Let $w = \dfrac{z-1}{z+1}$

$wz + w = z - 1$

$w + 1 = z - wz$

$w + 1 = z(1-w)$

$z = \dfrac{w+1}{1-w}$

Since $|z| = 1$

$\therefore \left|\dfrac{w+1}{1-w}\right| = 1$

$\therefore |w+1| = |w-1|$

which represents the distance of complex number w from 1 and -1 is same. That means w lies on y-axis or imaginary axis of Argand diagram.

Therefore $w = \dfrac{z-1}{z+1}$ is purely imaginary number.

2 marks Correct proof

1 mark using $|z| = 1$ and rearranging $z = \dfrac{w+1}{w-1}$

OR

Algebraic Approach:

Let $z = a + ib$, $a, b \in \mathbb{R}$

$|z| = 1 \Rightarrow a^2 + b^2 = 1$

2 marks Correct proof

$$\frac{z-1}{z+1} = \frac{a+ib-1}{a+ib+1}$$

1 mark Express $\frac{z-1}{z+1}$ in simplest form by assuming either $z = a+ib$ or $|z| = 1$

$$= \frac{(a-1)+ib}{(a+1)+ib} \times \frac{(a+1)-ib}{(a+1)-ib}$$

$$= \frac{(a^2-1)+b^2+i[b(a+1)-b(a-1)]}{(a+1)^2+b^2}$$

$$= \frac{(a^2+b^2-1)+2ib}{(a+1)^2+b^2}$$

$$= \frac{2ib}{(a+1)^2+b^2} \quad (\because a^2+b^2=1)$$

$\therefore \frac{z-1}{z+1}$ is purely imaginary.

d) i) $R.H.S = (z+1)(z^4 - z^3 + z^2 - z + 1)$

$= z(z^4 - z^3 + z^2 - z + 1) + (z^4 - z^3 + z^2 - z + 1)$

$= z^5 - z^4 + z^3 - z^2 + z + z^4 - z^3 + z^2 - z + 1$

$= z^2 + 1$

$= L.H.S$

1 mark Correct proof.

ii) Since z is complex root of $z^5+1=0$

$\therefore (z+1)(z^4-z^3+z^2-z+1)=0$

$\Rightarrow z^4-z^3+z^2-z+1=0 \quad (\because z\neq -1)$

$\Rightarrow z^4+z^2+1=z^3+z$

Hence proved.

1 mark Correct proof.

iii) If $z^5=-1$

then $z=\text{Cis}\left(\pm\frac{\pi}{5}\right), \text{Cis}\left(\pm\frac{3\pi}{5}\right), 1$

Let $z=\text{Cis}\frac{\pi}{5}$

$1+z^2+z^4=z+z^3 \quad$ (from (ii))

$(\div z^2) \quad \frac{1}{z^2}+1+z^2=\frac{1}{z}+z$

$\left(\frac{1}{z^2}+z^2\right)+1=\left(\frac{1}{z}+z\right)$

$2\cos\frac{2\pi}{5}+1=2\cos\frac{\pi}{5}$

$-2\cos\frac{3\pi}{5}+1=2\cos\frac{\pi}{5}$

$2\cos\frac{\pi}{5}+2\cos\frac{3\pi}{5}=1$

$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$

Hence Shown

3 marks Correct solution.

2 marks Using part (ii) to express it in $\cos\frac{n\pi}{5}$ form or equivalent merit.

1 mark Listing all the five roots of z.

Question-8

a) i) $|w| = 1$ and $\arg w = \frac{\pi}{3}$ $(60°)$

$\therefore |zw| = |z|$ and $\arg(zw) = \arg z + \frac{\pi}{3}$

$\therefore \Delta OAB$ is equilateral. (2 sides are equal and included angle is $60°$)

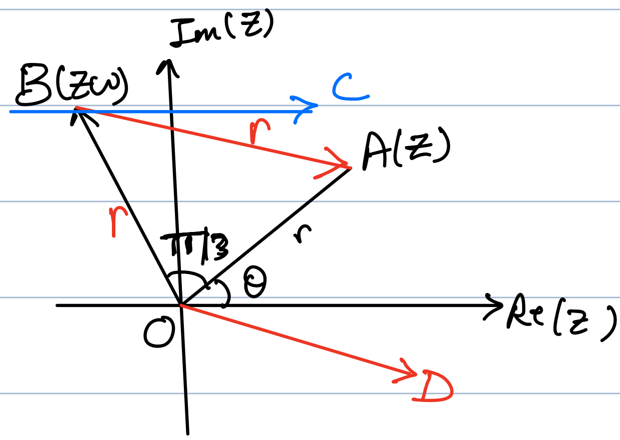
2 marks Correct solution.

1 mark Either show $|zw| = |z|$ or $|w| = 1$ and its arg $\frac{\pi}{3}$.

ii) $\overrightarrow{BA} = z - zw$

$\overrightarrow{OD} = \overrightarrow{BA}$

$= z - zw$



$\overrightarrow{OD} = \overrightarrow{OA} \cdot e^{-i\pi/3}$

$= z \cdot e^{-i\pi/3}$

$= re^{i\theta} \cdot e^{-i\pi/3}$

$= re^{i(\theta - \pi/3)}$

2 marks Correct solutions

1 mark Attempt to find the argument of $(z - zw)$ or equivalent merit.

b) $L.H.S = \dfrac{e^{i\theta}-1}{e^{i\theta}+1}$

$$= \frac{(e^{i\theta}-1) \div e^{i\theta/2}}{(e^{i\theta}+1) \div e^{i\theta/2}}$$

$$= \frac{e^{i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2} + e^{-i\theta/2}}$$

$$= \frac{2i\sin\theta/2}{2\cos\theta/2}$$

$$= i\tan\theta/2$$

$$= R.H.S$$

3 marks Correct Solution

2 marks Express $e^{i\theta}$ as $\text{Cis}\,\theta$ to rationalise & simplifying to $\dfrac{i\sin\theta}{1+\cos\theta}$ or in terms of $\dfrac{e^{i\theta/2}-e^{-i\theta/2}}{e^{i\theta/2}+e^{-i\theta/2}}$

1 mark Expressing $e^{i\theta}$ as $\text{Cis}\,\theta$ and rationalising den. or dividing Num and den. by $e^{i\theta/2}$

OR

$$\frac{e^{i\theta}-1}{e^{i\theta}+1} = \frac{(\cos\theta-1)+i\sin\theta}{(\cos\theta+1)+i\sin\theta}$$

$$= \frac{(\cos\theta-1)+i\sin\theta}{(\cos\theta+1)+i\sin\theta} \times \frac{(\cos\theta+1)-i\sin\theta}{(\cos\theta+1)-i\sin\theta}$$

$$= \frac{(\cos^2\theta-1)+\sin^2\theta + i\left[\sin\theta(\cos\theta+1)-\sin\theta(\cos\theta-1)\right]}{(\cos\theta+1)^2+\sin^2\theta}$$

$$= \frac{2i\sin\theta}{\cos^2\theta+2\cos\theta+1+\sin^2\theta}$$

$$= \frac{2i\sin\theta}{2(1+\cos\theta)} = \frac{2i\sin\theta/2\,\cos\theta/2}{2\cos^2\theta/2}$$

$$= i\tan\theta/2$$

c) i) $L.H.S = |z - z_1|^2$

$= (z - z_1)\overline{(z - z_1)}$ $\quad (\because |z|^2 = z\bar{z})$

$= (z - z_1)(\bar{z} - \bar{z}_1)$

$= z\bar{z} - z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1$

$= |z|^2 - (z\bar{z}_1 + z_1\bar{z}) + |z_1|^2$

$= 9 - (z\bar{z}_1 + z_1\bar{z}) + 1$

$= 10 - (z\bar{z}_1 + z_1\bar{z})$

2 marks correct solution

1 mark Expressing $|z|^2 = z\bar{z}$ and expanding or equivalent merit.

ii) $|z - z_1|^2 = 10 - (z\bar{z}_1 + \bar{z}z_1)$ (using (i))

Similarly $|z - z_2|^2 = 10 - (z\bar{z}_2 + \bar{z}z_2)$

& $|z - z_3|^2 = 10 - (z\bar{z}_3 + \bar{z}z_3)$

Adding them.

$|z - z_1|^2 + |z - z_2|^2 + |z - z_3|^2 =$

$= 30 - [z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) + \bar{z}(z_1 + z_2 + z_3)]$

$= 30 - [z(\overline{z_1 + z_2 + z_3}) + \bar{z}(0)]$

$= 30 - [z(0) + 0]$

$= 30$

2 marks Correct solution

1 mark Using part (i) to Express the L.H.S in terms of $z, \bar{z}, z_1, z_2$ & z_3.

END OF EXAM