Student Name: ____________________

Mathematics Class: ____________________

James Ruse Agricultural High School

2024 **YEAR 12 Task 2 HSC Examination**

Mathematics Extension 2

General Instructions:

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black pen.
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- In Questions 6–10, show relevant mathematical reasoning and/ or calculations.

Total marks: 70

Section I – 5 marks.

- Attempt Questions 1–5.
- Allow about 5 minutes for this section.

Section II – 65 marks.

- Attempt Questions 6–10.
- Allow about 1 hour and 55 minutes for this section.

Start your answer to each question on a separate piece of paper.

Section I

5 marks.

Attempt Questions 1-5.

Allow about 5 minutes for this section.

Write your answers on the multiple-choice answer sheet provided.

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1. The distance between the two complex numbers z and $-\bar{z}$ is given by which of the following?

   (A) $2\,|\mathrm{Re}(z)|$
   (B) $2\,|\mathrm{Im}(z)|$
   (C) $4|z|^2$
   (D) $|\mathrm{Re}(z)| + 2\,|\mathrm{Im}(z)|$

2. Let the statement P: "Nancy and Garry are smart" be a true statement. P is a counterexample to which of the following statements?

   (A) There exist smart people with 5 letters in their names.
   (B) All smart people have 5 letters in their names.
   (C) Everyone with 5 letters in their names are smart.
   (D) None of the above

3. Consider the following two statements:

   I. $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R} : \sqrt{x^2 + y^2} = x + y.$

   II. $\forall x \in [0, \pi]: y = \cos x \Leftrightarrow x = \cos^{-1} y.$

   Which of the following is the case?

   (A) Both I and II are true.
   (B) I is true and II is false.
   (C) I is false and II is true.
   (D) Both I and II are false .

4. To which of the following does the integral $\int 2\sin 3x \cos 2x \, dx$ evaluate?

(A) $-\frac{\cos 5x}{5} + \cos x + c$

(B) $-\frac{\cos 5x}{10} + \frac{\cos x}{2} + c$

(C) $-\frac{\cos 5x}{5} - \cos x + c$

(D) $-\frac{\cos 5x}{10} - \frac{\cos x}{2} + c$

5. Let $(1-i)^n = 1 + ai$, where a is a non-zero real constant. To which of the following does the imaginary part of $(1-i)^{2n+2}$ simplify?

(A) $4a$

(B) $2(1-a^2)i$

(C) $1-a^2$

(D) $2(a^2-1)$

Section II begins on the next page.

Section II

65 marks.

Attempt Questions 6-10.

Allow about 1 hour and 55 minutes for this section.

Start each question on a new page. Extra paper is available.

In Questions 6-10, your responses should include relevant mathematical reasoning and/ or calculations.

Question 6 (13 marks). Start a new page.

(a) Write the converse of the statement, "If my legs are long, then I can jump very far". **1**

(b) Consider the statement:

'If $n^2 - 3n + 2$ is odd, then n is odd.'

i. Write the contrapositive of the statement. **1**

ii. Hence or otherwise, prove that the statement is true. **2**

(c) Prove, by contradiction, that $\log_3 7$ is irrational. **2**

(d) Define the complex numbers $z = 1 - i$ and $\omega = 2\left(\cos\frac{\pi}{6} + i\ \sin\frac{\pi}{6}\right)$.

i. Express $\frac{\omega}{z}$ in exponential form. **2**

ii. Hence, or otherwise, express $\left(\frac{\omega}{z}\right)^8$ in the form $a + i\,b$, where a and b are real numbers. **2**

(e) Evaluate: **3**

$$\int_{-1}^{1} \frac{\sqrt{2+x}}{\sqrt{2-x}}\,dx$$

Question 7 (13 marks). Start a new page.

(a) Let the point P represent the complex number z on the Argand plane, where z satisfies $|z-1| = 1$ and $0 < \arg(z) < \frac{\pi}{2}\cdot$

i. Plot the locus of P on an Argand diagram. **2**

ii. Find the value of the real number k such that $\arg(z-1) = k\arg(z^2 - z)$. **3**

(b) Let $z = e^{i\theta}$ be a complex number of modulus 1.

i. Show that $\frac{z^2+1}{z} = 2\cos\theta$. **1**

ii. Hence show that $z - z^3 + z^5 - z^7 + z^9 = \frac{1+\cos 10\theta}{2\cos\theta} + i\,\frac{\sin 10\theta}{2\cos\theta}\cdot$ **3**

iii. Hence write down a simplified expression for

$$\cos\theta - \cos 3\theta + \cos 5\theta - \cos 7\theta + \cos 9\theta$$ **1**

(c) Evaluate $\displaystyle\int_{-\frac{\pi}{2}}^{0} \cos^2 x \sin^2 x \, dx$. **3**

Question 8 (13 marks). Start a new page.

(a) Prove that the difference between the factorial of any two consecutive integers is divisible by the square of the smaller integer. **2**

(b) Evaluate $\displaystyle\int_{0}^{\frac{\pi}{3}} \sec^5 x \tan x \; dx$. **2**

(c) Evaluate $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{1}{5+4\sin x}\, dx$. **3**

(d) i. Find the values of A, B, C and D such that:

$$\frac{4x^3 - 13x^2 + 14x - 21}{(x^2+3)(x-1)^2} = \frac{Ax+B}{x^2+3} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$ **3**

ii. Hence find $\displaystyle\int \frac{4x^3 - 13x^2 + 14x - 21}{(x^2+3)(x-1)^2}\, dx$. **3**

Question 9 (13 marks). Start a new page.

(a) Let z_1, z_2, z and w be complex numbers.

i. By considering the relations $|z_1| = |(z_1 + z_2) + (-z_2)|$ and $|z_1 + z_2| \leq |z_1| + |z_2|$, show that $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$. **2**

ii. Hence, or otherwise, show that $|z| \geq \sqrt{5} - 2$ if $\left|z + \frac{1}{z}\right| = 4$. **4**

(b) i. Use the substitution $u = \cos\theta$, and partial fraction decomposition, to show that: **3**

$$\int \frac{\sin\theta}{1 - \cos^2\theta}\, d\theta = \frac{1}{2}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right| + c$$

ii. For $0 < \theta \leq \frac{\pi}{2}, -\infty < x \leq 0$, define $e^x = \sin\theta$. Using the chain rule, differentiate both sides of $e^x = \sin\theta$ with respect to x, to show that: **1**

$$dx = \frac{\cos\theta}{\sin\theta}\, d\theta$$

iii. Hence find $\int \frac{\sqrt{1-e^x}}{\sqrt{1+e^x}}\, dx$, leaving your answer in terms of θ. **3**

Question 10 (13 marks). Start a new page.

(a) Let $p \geq 2$ be a natural number. Prove, by use of the contrapositive, that the following statement is true: **3**

'If p is prime, then $p = 2$ or p is odd.'

(b) i. Find real numbers a and b such that:

$$z^4 + z^3 + z^2 + z + 1 = (z^2 + az + 1)(z^2 + bz + 1)$$ **2**

ii. Hence, find the value of $\cos\frac{2\pi}{5}$ given that $z = e^{\frac{2i\pi}{5}}$ is a root of the equation $z^4 + z^3 + z^2 + z + 1 = 0$. **3**

(c) Let $I_n = \int_0^a x^n\sqrt{a^2 - x^2}\, dx;\ a \in \mathbb{R}^+, n \in \mathbb{N}_0$ (i.e. $n = 0,1,2 \ldots$).

i. Using integration by parts, prove that $I_n = \frac{a^2(n-1)}{n+2} I_{n-2}$ for $n \geq 2$. **3**

ii. Hence evaluate I_6. **2**

END OF EXAMINATION

Student Number Solutions

MC Answers:
1A 2B 3A 4A 5D

Question 6

a) $\boxed{\text{If}}$ I can jump very far $\boxed{\text{then}}$ my legs are long.

Must have "if" and "then" for mark.

b) i) If n is not odd then n^2-3n+2 is not odd.

i.e. If n is even then n^2-3n+2 is even.

Must have "if" and "then"; either form is fine.

ii) If n is even i.e. if $n=2k$, $k\in\mathbb{Z}$

then $n^2-3n+2 = (2k)^2-3(2k)+2$

$=2(2k^2-3k+1)$

$=2p$, $p=2k^2-3k+1\in\mathbb{Z}$

i.e. if n is even then n^2-3n+2 is even

One mark for building proof of contrapositive truth $\boxed{\text{if}}$ n is even.

$\therefore$ Contrapositive is also true

i.e. if n^2-3n+2 is odd then n is odd.

need to refer to original statement suitably for full marks.

c) $\boxed{\text{If } \log_3 7 \text{ is rational}}$ i.e. if $\log_3 7=\frac{a}{b}$, $a,b\in\mathbb{N}$ with $GCD(a,b)=1$

then: $b\log_3 7=a$

$\log_3 7^b=a$

$7^b=3^a$

one mark logical proof structure.

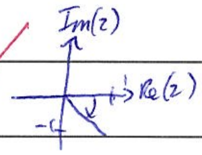
one mark algebraic manipulation.

but this contradicts the Fundamental theorem of Arithmetic

$\therefore$ By contradiction $\log_3 7$ is irrational

d) i) $z=1-i$

$=\sqrt{2}e^{-i\frac{\pi}{4}}$ ✓



First mark for z in exponential form.

$w=2e^{i\frac{\pi}{6}}$

$\therefore \frac{w}{z}=\frac{2e^{i\frac{\pi}{6}}}{\sqrt{2}e^{-i\frac{\pi}{4}}}=\sqrt{2}e^{i\frac{5\pi}{12}}$ ✓ Correct answer.

Student Number Solutions

d) ii) $\left(\frac{w}{z}\right)^8 = \left(\sqrt{2}\, e^{i\frac{5\pi}{12}}\right)^8$

$= (\sqrt{2})^8 e^{i\cancel{\frac{40\pi}{12}}\frac{10\pi}{3}}$ ✓ Progress

$= 16 e^{i(2\pi + \frac{4\pi}{3})}$

$= 16 \cancel{e^{i2\pi}}^{1} . e^{i\frac{4\pi}{3}}$

$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= 16\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= -8 + i(-8\sqrt{3})$ ✓ answer in a+ib form.

e) $\int_{-1}^{1} \frac{\sqrt{2+x}}{\sqrt{2-x}}\,dx = \int_{-1}^{1} \frac{2+x}{\sqrt{4-x^2}}\,dx$ ✓ rationalising numerator

$= \int_{-1}^{1} \frac{2}{\sqrt{4-x^2}}\,dx + \int_{-1}^{1} -\frac{1}{2}(-2x)(4-x^2)^{-\frac{1}{2}}\,dx$

$= \left[2\sin^{-1}\frac{x}{2} - (4-x^2)^{\frac{1}{2}}\right]_{-1}^{1}$ ✓ antiderivative

$= \frac{2\pi}{6} - \sqrt{3} - \left(-\frac{2\pi}{6} - \sqrt{3}\right)$

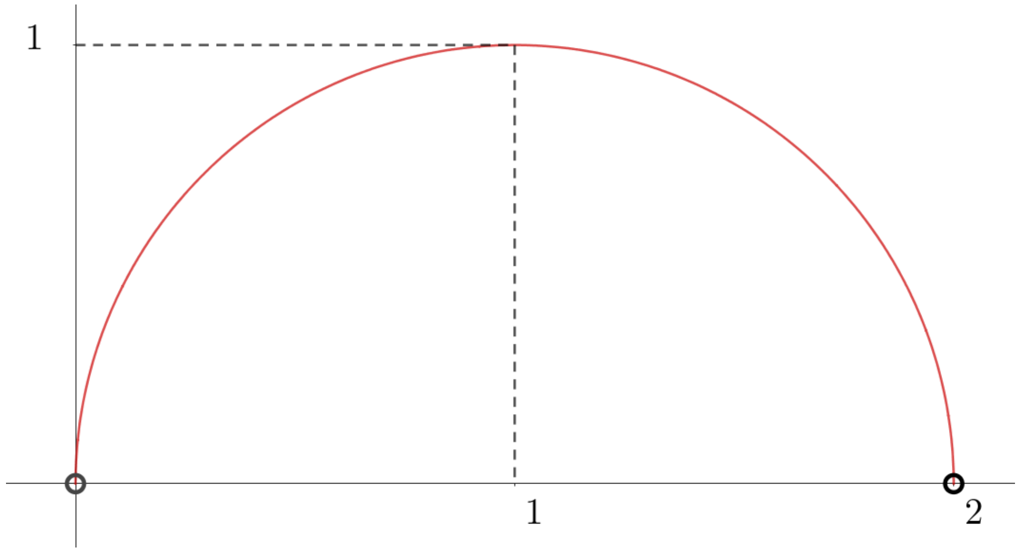
$= \frac{2\pi}{3}$ ✓ answer

If students chose not to rationalise numerator then first mark for progress e.g. substitution that is leading towards antiderivative.

Question 7

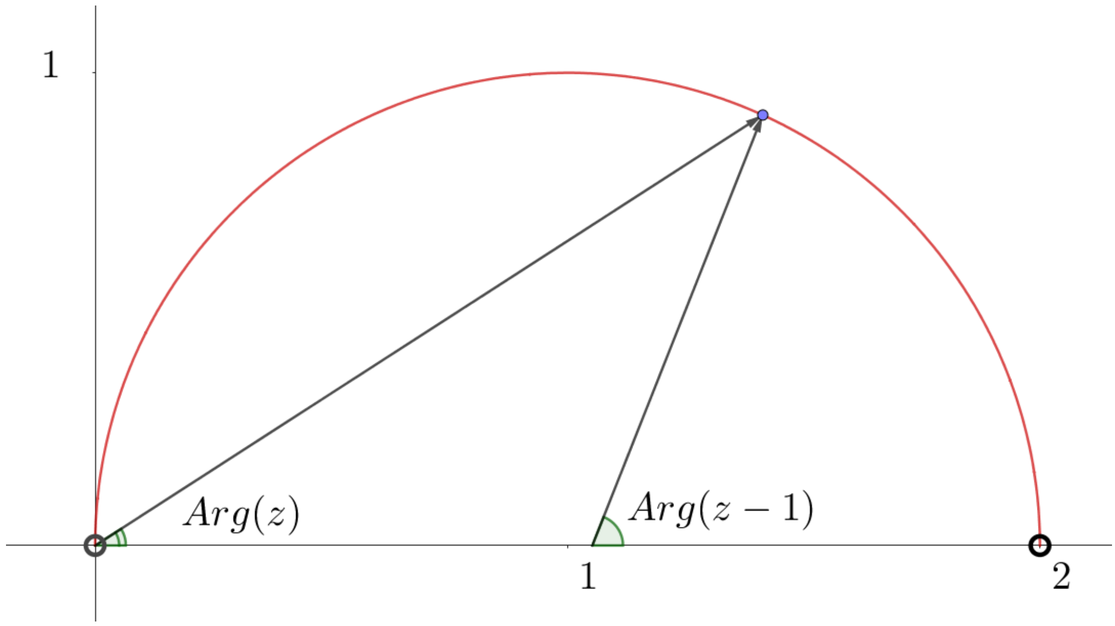
a)

i.



- 1 out of 2 if students were able to either:
  - Sketch the full correct semi-circle or
  - Recognise the locus is part of a circle (semi-circle, quadrant, full circle) AND indicate at least one open circles
- 2 out of 2 for having everything correct

ii.



$\arg(z-1) = k\arg(z^2 - z)$

$= k\arg[z(z-1)]$

$= k\arg(z) + k\arg(z-1)$ 1 mark for separating using the argument property

$\arg(z-1) = 2\arg(z)$ (angle at the circumference is half the angle at the centre standing on the same arc) 1 mark for the relationship between $\arg(z-1)$ and $\arg(z)$

$\therefore 2\arg(z) = k\arg(z) + 2k\arg(z)$

$= 3k\arg(z)$

$3k = 2$

$k = \frac{2}{3}$ 1 mark for reaching final answer.

b)

i. Let $z = \cos\theta + i\sin\theta$

$$\frac{z^2+1}{z} = z + \frac{1}{z}$$
$$= (\cos\theta + i\sin\theta) + (\cos\theta + i\sin\theta)^{-1} \text{ (De Moivre's)}$$
$$= (\cos\theta + i\sin\theta) + (\cos(-\theta) + i\sin(-\theta))$$
$$= (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$
$$= 2\cos\theta \qquad \text{1 mark for getting the proof with all clear steps shown}$$

ii.

$$\text{LHS} = z - z^3 + z^5 - z^7 + z^9$$
$$= \frac{z((-z^2)^5 - 1)}{-z^2 - 1} \quad \text{(Geometric series of 5 terms with } r = -z^2\text{)} \quad \text{1 mark for simplifying series}$$
$$= \frac{z(z^{10}+1)}{z^2+1}$$
$$= \frac{z}{z^2+1} \times (z^{10}+1)$$
$$= \frac{(\cos\theta + i\sin\theta)^{10} + 1}{\cos\theta} \quad \text{1 mark writing expression in terms of } \sin\theta \text{ and } \cos\theta$$
$$= \frac{\cos 10\theta + i\sin 10\theta + 1}{\cos\theta} \quad \text{(De Moivre's)}$$
$$= \left(\frac{\cos 10\theta + 1}{\cos\theta}\right) + i\left(\frac{\sin 10\theta}{\cos\theta}\right) \quad \text{1 mark for completing the proof}$$

Alternate approach:

$$\text{RHS} = \left(\frac{1+\cos 10\theta}{2\cos\theta}\right) + i\left(\frac{\sin 10\theta}{2\cos\theta}\right) = \frac{1 + (\cos 10\theta + i\sin 10\theta)}{2\cos\theta}$$
$$= \frac{1+z^{10}}{2\cos\theta} \quad \text{1 mark for using de Moivre's Theorem and removing } \cos\theta \text{ and } i\sin\theta$$
$$= (1+z^{10}) \div \left(\frac{z^2+1}{z}\right) \quad \text{1 mark for using part a)}$$
$$= \frac{z^{11}+z}{z^2+1}$$
$$= z - z^3 + z^5 - z^7 + z^9 \quad \text{1 mark for completing the prooff}$$

$$\begin{array}{r|cccccc}
 & & z^9 & -z^7 & z^5 & -z^3 & +z \\
\hline
z^2 \quad +1 & z^{11} & & & & & +z \\
 & z^{11} & +z^9 & & & & \\
\hline
 & & -z^9 & & & & -z \\
 & & -z^9 & -z^7 & & & \\
\hline
 & & & z^7 & & & +z \\
 & & & z^7 & +z^5 & & \\
\hline
 & & & & -z^5 & & -z \\
 & & & & -z^5 & -z^3 & \\
\hline
 & & & & & z^3 & +z \\
 & & & & & z^3 & +z \\
\hline
 & & & & & & 0
\end{array}$$

iii.

$$\cos\theta - \cos 3\theta + \cos 5\theta - \cos 7\theta + \cos 9\theta = Re(z - z^3 + z^5 - z^7 + z^9)$$
$$= Re\left(\left(\frac{1+\cos 10\theta}{2\cos\theta}\right) + i\left(\frac{\sin 10\theta}{2\cos\theta}\right)\right)$$
$$= \left(\frac{1+\cos 10\theta}{2\cos\theta}\right) \qquad \text{1 mark}$$

c)

$$\cos 4x = \cos^2 2x - \sin^2 2x$$
$$= 1 - 2\sin^2 2x$$
$$\therefore \sin^2 2x = \frac{1-\cos 4x}{2}$$

$$\int_{-\frac{\pi}{2}}^{0} \cos^2 x \sin^2 x \, dx$$
$$= \int_{-\frac{\pi}{2}}^{0} (\cos x \sin x)^2 \, dx$$
$$= \int_{-\frac{\pi}{2}}^{0} \left(\frac{1}{2}\sin 2x\right)^2 \, dx$$
$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{0} (\sin^2 2x) \, dx$$

1 mark for simplifying integrand into 1 trig function

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{0} \left(\frac{1-\cos 4x}{2}\right) dx$$
$$= \frac{1}{8}\int_{-\frac{\pi}{2}}^{0} (1-\cos 4x) \, dx$$
$$= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_{-\frac{\pi}{2}}^{0}$$

1 mark for integrating

$$= \frac{1}{8}\left[\left(0 - \left(\frac{1}{4}\times 0\right)\right) - \left(-\frac{\pi}{2} - \frac{1}{4}\sin(-2\pi)\right)\right]$$
$$= \frac{1}{8}\left[\frac{\pi}{2}\right]$$
$$= \frac{\pi}{16}$$

1 mark for evaluating and reaching answer

Question 8

Start here.

(a) $k \in \mathbb{Z}$, so k and $k+1$ are two consecutive integers

$(k+1)! - k!$

$= k!(k+1) - k!$ 1 mark for $(k+1)! = k!(k+1)$

$= k!(k+1-1)$

$= k!\,k$

$= (k-1)!\,k\,k$

$= (k-1)!\,k^2$ 1 mark for $(k-1)!\,k^2$

$\therefore (k+1)! - k!$ is divisible by k^2.

(b) $\int_0^{\pi/3} \sec^5 x \tan x \, dx$

let $u = \sec x \therefore du = \sec x \tan x \, dx$

For $x = 0$, $u = \sec 0 = 1$

$x = \pi/3$, $u = \sec \frac{\pi}{3} = 2$ 1 mark

$\int_0^{\pi/3} \sec^5 x \tan x \, dx = \int_0^{\pi/3} \sec^4 x \sec x \tan x \, dx$

$= \int_1^2 u^4 \, du = \left[\frac{u^5}{5}\right]_1^2 = \frac{32}{5} - \frac{1}{5} = \frac{31}{5}$

1 mark

(c) $\displaystyle\int_0^{\pi/2} \frac{1}{5+4\sin x}\,dx$

Let $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}(1+t^2)\,dx$

$x=0,\ t=\tan 0 = 0$

$dx = \frac{2\,dt}{1+t^2}$

$x=\frac{\pi}{2},\ t=\tan\frac{\pi}{4}=1$

$\sin x = \frac{2t}{1+t^2}$ 1 mark

$$\int_0^{\pi/2}\frac{1}{5+4\sin x}\,dx = \int_0^1 \frac{1}{5+4\left(\frac{2t}{1+t^2}\right)} \times \frac{2\,dt}{1+t^2}$$

$$= \int_0^1 \frac{2\,dt}{5(1+t^2)+8t} = \int_0^1 \frac{2\,dt}{5t^2+8t+5}$$

1 mark

$$= \frac{2}{5}\int_0^1 \frac{dt}{\left(t+\frac{4}{5}\right)^2+\frac{9}{25}} = \left[\frac{2}{5}\times\frac{5}{3}\tan^{-1}\left(\left(t+\frac{4}{5}\right)\times\frac{5}{3}\right)\right]_0^1$$

$$= \left[\frac{2}{3}\tan^{-1}\left(\frac{5t+4}{3}\right)\right]_0^1$$

$$= \frac{2}{3}\left(\tan^{-1}3 - \tan^{-1}\frac{4}{3}\right) \simeq 0.215$$ 1 mark

Additional writing space on back page.

Start here.

(d) (i)

$$\frac{4x^3-13x^2+14x-21}{(x^2+3)(x-1)^2} = \frac{Ax+B}{x^2+3} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Multiply by $(x-1)^2$ then set $x=1$

$$\frac{4(1)^3-13(1)^2+14(1)-21}{(1^2+3)} = \frac{A(1)+B}{(1)^2+3}(1-1)^2 + C(1-1) + D$$

$$\boxed{D = -4}$$

For $x=0$, $\frac{-21}{3} = \frac{B}{3} - C + D$

$$\boxed{3C - B = 9} \quad (1)$$

For $x=-1$, $\frac{-4-13-14-21}{4\times 4} = \frac{-A+B}{4} - \frac{C}{2} + \frac{D}{4}$

$$-13 = -A + B - 2C + D$$

$$\boxed{A - B + 2C = 9} \quad (2)$$

For $x=2$

$$\frac{4(8)-13(4)+14(2)-21}{7} = \frac{2A+B}{7} + C + D$$

$$\frac{-13}{7} = \frac{2A+B}{7} + C + D$$

$$\boxed{2A + B + 7C = 15} \quad (3)$$

$(2) - (1) \therefore A - C = 0 \therefore A = C$

$(3) \quad 9A + B = 15$

$(1) \quad 3A - B = 9$

$12A = 24 \quad \therefore \boxed{A = 2}$

$\therefore \boxed{C = 2}$

$B = 3A - 9$
$= 6 - 9$
$\boxed{B = -3}$

3 marks for all corrects
2 marks for one error
1 mark for one or two correct

$$\frac{4x^3 - 13x^2 + 14x - 21}{(x^2+3)(x-1)^2} = \frac{2x - 3}{x^2+3} + \frac{2}{x-1} - \frac{4}{(x-1)^2}$$

$$\int \frac{4x^3 - 13x^2 + 14x - 21}{(x^2+3)(x-1)^2}\,dx = \int \frac{2x\,dx}{x^2+3} - 3\int \frac{dx}{x^2+3} + 2\int \frac{dx}{x-1} - 4\int \frac{dx}{(x-1)^2}$$

$$= \ln(x^2+3) - 3 \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + 2\ln|x-1| + \frac{4}{x-1} + C$$

$$= \ln(x^2+3) - \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + 2\ln|x-1| + \frac{4}{x-1} + C$$

3 marks for all corrects
2 marks for one error
1 mark for one or two correct.

Additional writing space on back page.

Alternative solution for (i)

$4x^3-13x^2+14x-21 = (Ax+B)(x^2-2x+1)+C(x^2+3)(x-1)+D(x^2+3)$

$= Ax^3-2Ax^2+Ax+Bx^2-2Bx+B+$

$Cx^3-Cx^2+3Cx-3C+Dx^2+3D$

$=(A+C)x^3-(2A-B+C-D)x^2+$

$(A-2B+3C)x+B-3C+3D$

For $x=1$, $4-13+14-21=4D$ $\therefore D=-4$

Equating the coefficients, we obtain:

$A+C=4$

$2A-B+C-D=13$

$A-2B+3C=14$

$B-3C+3D=-21$

$A+C=4$ (1)

$2A-B+C=9$ (2)

$A-2B+3C=14$ (3)

$B-3C=-9$ (4)

(2) + (4) $2A-2C=0$ $\therefore$ $A=C$

(1) $\therefore$ $2A=4$ $\therefore$ $A=C=2$

(4) $\therefore$ $B=-9+36=-3$

$A=2,\ B=-3,\ C=2,\ D=-4$

DDT = Do not do that

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

MATHEMATICS Extension 2: Question 9

Suggested Solutions	Marks	Marker's Comments
a) i) $\lvert z\rvert = \lvert z+w+(-w)\rvert$		
$\leq \lvert z+w\rvert + \lvert -w\rvert$		
$\therefore \lvert z\rvert \leq \lvert z+w\rvert + \lvert w\rvert$ as $\lvert -w\rvert = \lvert w\rvert$		
$\therefore \lvert z+w\rvert \geq \lvert z\rvert - \lvert w\rvert \quad - (1)$	1	
also $\lvert w\rvert = \lvert w+z+(-z)\rvert$		
$\lvert w\rvert \leq \lvert w+z\rvert + \lvert z\rvert$		
$\therefore \lvert z+w\rvert \geq \lvert w\rvert - \lvert z\rvert \quad - (2)$		
combining (1) and (2)		
$\lvert z+w\rvert \geq \pm(\lvert z\rvert - \lvert w\rvert)$	1	
$\therefore \lvert z+w\rvert \geq \big\lvert \lvert z\rvert - \lvert w\rvert \big\rvert$		
ii) let $w = \frac{1}{z}$		
$\therefore \left\lvert \lvert z\rvert - \left\lvert \frac{1}{z}\right\rvert \right\rvert \leq \left\lvert z + \frac{1}{z}\right\rvert$		
$\left\lvert \lvert z\rvert - \left\lvert \frac{1}{z}\right\rvert \right\rvert \leq 4$		
$\therefore -4 \leq \lvert z\rvert - \left\lvert \frac{1}{z}\right\rvert \leq 4$	1	
$-4 \leq \lvert z\rvert - \frac{1}{\lvert z\rvert}$		
$-4\lvert z\rvert \leq \lvert z\rvert^2 - 1$		
$\lvert z\rvert^2 + 4\lvert z\rvert - 1 \geq 0$		
$\lvert z\rvert = \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2}$		
$= \frac{-4 \pm \sqrt{20}}{2}$		
$= -2 \pm \sqrt{5}$		
$\therefore \lvert z\rvert \geq \sqrt{5} - 2$,		
$\lvert z\rvert \leq -2 - \sqrt{5}$		
as $\lvert z\rvert \geq 0$		
$\therefore \lvert z\rvert \geq \sqrt{5} - 2$		
$\lvert z\rvert - \frac{1}{\lvert z\rvert} \leq 4$		1 mark for each case
$\lvert z\rvert^2 - 1 \leq 4\lvert z\rvert$		
$\lvert z\rvert^2 - 4\lvert z\rvert - 1 \leq 0$		
$\lvert z\rvert = \frac{-4 \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2}$		
$= \frac{4 \pm \sqrt{20}}{2}$		
$= 2 \pm \sqrt{5}$		
$\therefore 2 - \sqrt{5} \leq \lvert z\rvert \leq 2 + \sqrt{5}$		
as $\lvert z\rvert \geq 0$		
$\therefore 0 \leq \lvert z\rvert \leq 2 + \sqrt{5}$		
$\therefore \sqrt{5} - 2 \leq \lvert z\rvert \leq 2 + \sqrt{5}$ (intersection of the two inequalities)	1	minimum value with explanation.
$\therefore$ minimum is $\sqrt{5} - 2$		

MATHEMATICS Extension 2: Question 9		
Suggested Solutions	**Marks**	**Marker's Comments**
b) i) $\int \frac{\sin\theta}{1-\cos^2\theta}\,d\theta$ let $u=\cos\theta$ $\frac{du}{d\theta} = -\sin\theta$ $-du = \sin\theta\, d\theta$ $= \int \frac{1}{1-u^2}(-du)$ $= \int \frac{1}{u^2-1}\,du$ $= \int \frac{1}{(u-1)(u+1)}\,du$ $\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$ $1 = A(u+1) + B(u-1)$ let $u=1$ $\therefore A = \frac{1}{2}$ let $u=-1$ $\therefore B = -\frac{1}{2}$ $\therefore \int \frac{1}{(u-1)(u+1)}\,du = \frac{1}{2}\int\left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$ $= \frac{1}{2}\left(\ln\|u-1\| - \ln\|u+1\|\right) + C$ $= \frac{1}{2}\ln\left\|\frac{u-1}{u+1}\right\| + C$ $= \frac{1}{2}\ln\left\|\frac{\cos\theta-1}{\cos\theta+1}\right\| + C$	1 1 1	
ii) $e^x = \sin\theta$ $x = \ln\|\sin\theta\|$ $\frac{dx}{d\theta} = \frac{\cos\theta}{\sin\theta}$ $dx = \frac{\cos\theta}{\sin\theta}\,d\theta$ OR $\frac{d(e^x)}{dx} = \frac{d(\sin\theta)}{d\theta} \times \frac{d\theta}{dx}$ $e^x = \cos\theta\,\frac{d\theta}{dx}$ $\sin\theta = \cos\theta\,\frac{d\theta}{dx}$ $\quad \therefore dx = \frac{\cos\theta}{\sin\theta}\,d\theta$	1	

MATHEMATICS Extension 2: Question ....9...		
Suggested Solutions	**Marks**	**Marker's Comments**
iii) $\int \frac{\sqrt{1-e^x}}{\sqrt{1+e^x}}\,dx$ let $e^x = \sin\theta$, $dx = \frac{\cos\theta}{\sin\theta}\,d\theta$ $= \int \frac{\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}}\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$ $= \int \frac{\sqrt{1-\sin^2\theta}\ \cos\theta}{(1+\sin\theta)\sin\theta}\,d\theta$	1	changing x to θ and rationalising either the numerator or denominator.
$= \int \frac{\cos^2\theta}{(1+\sin\theta)\sin\theta}\,d\theta$ $= \int \frac{1-\sin^2\theta}{(1+\sin\theta)\sin\theta}\,d\theta$ $= \int \frac{1-\sin\theta}{\sin\theta}\,d\theta$	1	
$= \int\left(\frac{1}{\sin\theta} - 1\right)d\theta$ $= \int \frac{\sin\theta}{\sin^2\theta}\,d\theta - \theta$ $= \int \frac{\sin\theta}{1-\cos^2\theta}\,d\theta - \theta$ $= \frac{1}{2}\ln\left\|\frac{\cos\theta-1}{\cos\theta+1}\right\| - \theta + C$ from (i)	1	
OR $\int(\operatorname{cosec}\theta - 1)\,d\theta$ $= \int \frac{\operatorname{cosec}\theta(\operatorname{cosec}\theta+\cot\theta)}{\operatorname{cosec}\theta+\cot\theta}\,d\theta - \theta$ $= \int \frac{\operatorname{cosec}^2\theta+\operatorname{cosec}\theta\cot\theta}{\operatorname{cosec}\theta+\cot\theta}\,d\theta - \theta$		

MATHEMATICS Extension 2: Question…9…		
Suggested Solutions	**Marks**	**Marker's Comments**
$$\frac{d}{d\theta}(\operatorname{cosec}\theta+\cot\theta)=\frac{d}{d\theta}\left((\sin\theta)^{-1}+(\tan\theta)^{-1}\right)$$ $$=-(\sin\theta)^{-2}\cos\theta-(\tan\theta)^{-2}\sec^2\theta$$ $$=-\frac{\cos\theta}{\sin^3\theta}-\frac{\cos^2\theta}{\sin^2\theta\cos^2\theta}$$ $$=-\frac{\cos\theta}{\sin\theta}\times\frac{1}{\sin\theta}-\frac{1}{\sin^2\theta}$$ $$=-\cot\theta\operatorname{cosec}\theta-\operatorname{cosec}^2\theta$$ $$\therefore\int\frac{\operatorname{cosec}^2\theta+\operatorname{cosec}\theta\cot\theta}{\operatorname{cosec}\theta+\cot\theta}\,d\theta-\theta$$ $$=-\ln\lvert\operatorname{cosec}\theta+\cot\theta\rvert-\theta+C$$ OR $\ln\lvert\operatorname{cosec}\theta-\cot\theta\rvert-\theta+C$ OR $\ln\lvert\tan\frac{\theta}{2}\rvert-\theta+C$		must show working as these are not on the formula sheet.

Extension 2: Question 10		
Suggested Solutions	**Marks**	**Marker's Comments**
(a) The premise is that $p \in \mathbb{N}$ and $p \geq 2$. The statement to be proved is: $$\text{prime}(p) \rightarrow (p = 2 \vee \text{odd}(p))$$ The contrapositive is: $$\neg(p = 2 \vee \text{odd}(p)) \rightarrow \neg\text{prime}(p)$$ $$\equiv p \neq 2 \wedge \neg\text{odd}(p) \rightarrow \neg\text{prime}(p)$$ $$\equiv p \neq 2 \wedge \text{even}(p) \rightarrow \text{composite}(p)$$	1	The great majority handled the contrapositive statement well (first mark). To secure the second mark, candidates had to establish the conditions for the proof correctly. This means also identifying that $k > 1$ in $2k$, say, since the premise (a statement that's not to be contested) has that $p \geq 2$.
Assume $p > 2$ and p even. Then $\exists k \in \mathbb{Z}^+$ such that $$p = 2k$$	1	Third mark was achieved for correctly showing that this setup led to a non-trivial factorisation of p (i.e. a factorisation other than 1 and p).
Since $p > 2$, we must have $k > 1$, hence p has a non-trivial factorisation (i.e. a factorisation beyond 1 and itself). Hence p is composite. Since the contraposition is true, the original position is true; that is, given any natural number greater than or equal to 2, if p is prime, then p is odd.	1	A large proportion of students did not establish the conditions for the proof properly, or had trouble with concluding composite factorisation (they made it so much harder than necessary, such as establishing even and odd cases, or some other circuitous route).
(b) (i) Two polynomials are identically equal iff the coefficients on respective powers are equal. Hence expand RHS and equate with LHS to see: $$z^4 + z^3 + z^2 + z + 1 = z^4 + (a+b)z^3 + (2+ab)z^2 + (a+b)z + 1$$ from which we see immediately: $$a + b = 1, \qquad ab = -1$$	1	First mark was awarded for sufficient progress in establishing real numbers for a and b.
Solving simultaneously leads to $$a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2}$$	1	Second mark was obtained for finding a and b. Many students did not recognise that the spirit of the question was that of expanding the brackets and identifying LHS, RHS coefficients. Students jumped immediately into looking for the roots to $1 + z + \cdots +$

where WLOG (since the quadratic factors are symmetrical), a has been assigned the positive solution. (ii) Now, since $1 + z + z^2 + z^3 + z^4$ is factorisable into a pair of quadratics, and since $\alpha = e^{2\pi i/5}$ is a zero of the polynomial $P(z) = 1 + z + z^2 + z^3 + z^4$ (i.e. a polynomial with all real coefficients), we must have that $\overline{\alpha}$ is also a zero (conjugate root theorem). Hence $(z - \alpha)$ and $(z - \overline{\alpha})$ are factors of the polynomial, and so their product is, too: $$z^2 - (\alpha + \overline{\alpha})z + \|\alpha\|^2$$ Since the quadratic factors are irreducible ($\Delta = \left(\frac{1\pm\sqrt{5}}{2}\right)^2 - 4 < 0$), their factorisation into linear complex factors occurs in conjugate pairs. And since factors here are unique, and there are only two irreducible quadratic factors, we must have either: $$z^2 - (\alpha + \overline{\alpha})z + \|\alpha\|^2 \equiv z^2 + az + 1 \quad \dots(1)$$ or $$z^2 - (\alpha + \overline{\alpha})z + \|\alpha\|^2 \equiv z^2 + bz + 1 \quad \dots(2)$$ Now, if we take the first equation, given $\alpha + \overline{\alpha} = 2\Re(\alpha) = 2\cos\left(\frac{2\pi}{5}\right)$, we have: $$-(\alpha + \overline{\alpha}) = -2\Re(\alpha) = a \rightarrow \Re(\alpha) = -\frac{1+\sqrt{5}}{4} < 0$$ which would mean $\cos\frac{2\pi}{5} < 0$, which is false. Hence, we must have the second factor being that associated with α: $$-2\Re(\alpha) = b \rightarrow \Re(\alpha) = \frac{\sqrt{5}-1}{4}$$ i.e. $$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$$	 1 1	$z^4 = 0$, delivering $\cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}$, and using these are the 'real numbers' to be equated here with a and b. If done properly, the mark was awarded, since the cosines are real numbers, but this made the question less straightforward to answer going forward. The first mark was awarded for recognition of the connection between the zeros of the polynomial + conjugate pairings. We are presented with two possibilities for the assignment of a value to $\alpha = \cos\frac{2\pi}{5}$ given the construction, so students were required to reason explicitly in order to accept vs reject each possible solution in order to obtain the second mark. Why? Because (i) the target is known, and (ii) it may be assumed that the student, by '50:50 chance', selected the right solution with which to work, obtaining the answer (again, which is known), and ignoring the second solution as a non-contender. The third mark was obtained by extracting the radical expression for cosine of $2\pi/5$.

(c) Let $$I_n = \int_0^a x^n\sqrt{a^2 - x^2}\,dx$$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{N}_0$. **There are two primary methods of solution presented, though some other methods were successfully managed.** **Method 1** (i) Let $u = x^{n-1} \rightarrow du = (n-1)x^{n-2}dx$ and $dv = x(a^2 - x^2)^{\frac{1}{2}}\,dx \rightarrow v = -\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}}$	1	The first mark was obtained for correctly identifying $'u'$ and $'dv'$, and deriving $'du'$ and $'v'$.
So $$I_n = \left[-\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}}x^{n-1}\right]_0^a + \frac{n-1}{3}\int_0^a x^{n-2}(a^2 - x^2)^{\frac{3}{2}}dx$$ $$= 0 + \frac{n-1}{3}\int_0^a x^{n-2}(a^2 - x^2)\sqrt{a^2 - x^2}\,dx$$ $$= \frac{n-1}{3}\left[a^2\int_0^a x^{n-2}\sqrt{a^2 - x^2}dx - \int_0^a x^n\sqrt{a^2 - x^2}\,dx\right]$$ $$= \frac{n-1}{3}a^2 I_{n-2} - \frac{n-1}{3}I_n$$	1	Many students went 'the wrong way' (although, it can still be done), in that they selected $dv = x^n dx \rightarrow v = \frac{x^{n+1}}{n+1}, u = \sqrt{a^2 - x^2} \rightarrow du = -\frac{x}{\sqrt{a^2-x^2}}dx$. This can be used, but it requires 'slight of hand', such as remapping the parameter n (one student did this successfully).
so $$\left(1 + \frac{n-1}{3}\right)I_n = \frac{n-1}{3}a^2 I_{n-2}$$ $$\frac{2+n}{3}I_n = \frac{n-1}{3}a^2 I_{n-2}$$		The second mark was obtained for correctly arriving at an expression involving I_{n-2}.
$$I_n = \frac{n-1}{n+2}a^2 I_{n-2}$$ as required.	1	Third mark for delivering the required expression.
Method 2 (ii) We have $$I_n = \int_0^a x^{n-2}x^2\sqrt{a^2 - x^2}\,dx$$ $$= \int_0^a x^{n-2}(a^2 - (a^2 - x^2))\sqrt{a^2 - x^2}\,dx$$ $$= a^2\int_0^a x^{n-2}\sqrt{a^2 - x^2}dx - \int_0^a x^{n-2}(a^2 - x^2)^{\frac{3}{2}}\,dx$$ $$= a^2 I_{n-2} - \int_0^a x^{n-2}(a^2 - x^2)^{\frac{3}{2}}\,dx \quad \dots(1)$$		The second method attracted marks upon completing the same milestones as above.

In $$\int_0^a x^{n-2}(a^2-x^2)^{\frac{3}{2}}\,dx$$ let $u=(a^2-x^2)^{\frac{3}{2}},\ v'=x^{n-2}$. Then $u'=\frac{3}{2}\sqrt{a^2-x^2}(-2x)=-3x\sqrt{a^2-x^2}$ and $v=\frac{x^{n-1}}{n-1}$. So $$\int_0^a x^{n-2}(a^2-x^2)^{\frac{3}{2}}\,dx = \left[(a^2-x^2)^{\frac{3}{2}}\cdot\frac{x^{n-1}}{n-1}\right]_0^a - \int_0^a \frac{x^{n-1}}{n-1}\cdot\left(-3x\sqrt{a^2-x^2}\right)dx = 0+\frac{3}{n-1}\int_0^a x^n\sqrt{a^2-x^2}\,dx = \frac{3}{n-1}I_n$$ Putting this result into (1): $$I_n=a^2I_{n-2}-\frac{3}{n-1}I_n$$ so $$\left(1+\frac{3}{n-1}\right)I_n=a^2I_{n-2}$$ giving $$I_n=\frac{n-1}{n+2}a^2I_{n-2}$$ as required.		
(iii) To find I_6: $$I_6=\frac{5}{8}a^2I_4 = \frac{5}{8}a^2\left(\frac{3}{6}a^2I_2\right) = \frac{5}{8}a^2\left(\frac{3}{6}a^2\left(\frac{1}{4}a^2I_0\right)\right) = \frac{5a^6}{64}I_0$$	1	The first mark was obtained for **sufficient** progress/demonstration of understanding of the recursion process (note: ‘sufficient’ was determined on a relative basis).

Now $$I_0 = \int_0^a \sqrt{a^2 - x^2}\,dx$$ which is equivalent to a quarter of the area of a circle of radius a, so $$I_0 = \frac{1}{4}\pi a^2$$ hence $$I_6 = \frac{5}{256}\pi a^8$$	1	The second mark was obtained for the correct answer. Many students made unfortunate errors at this point. Some students did not recognise that I_0 was an integral involving a geometrical situation (area of a quarter circle), and so engaged on a mission of integration. **A general comment:** much of the work was illegible, so it was difficult to understand what was being communicated. This lack of clarity included extremely poor 'handwriting', stopping arguments 'mid-stream' and venturing into unchartered 'paperial territory' to start anew, with an expectation that the marker would know what to do, and in some instances, piece together **their** argument (e.g. filling in missing steps/taking 'leaps of faith'). This may sound strange, but you are not looking for a final answer to present to the marker, **you are looking to make a sound, cogent argument**. The entirety of your work **IS** the answer. Your argument **IS** your answer. If a marker in the HSC cannot read your writing, you could very well lose out to someone of equal ability.