The Scots College

2024

HSC Year 12 Assessment Task 2

Tuesday 2nd April 2024

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen.
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- A one-page A4 handwritten sheet can be used for this assessment.
- For questions in Section II, show all relevant mathematical reasonings and/or calculations.
- Answer each question in Section II at the start of a NEW writing booklet.

Total marks: 70

- **Section I – 10 marks** (pages 3–6)
  - Attempt Questions 1–10.
  - Allow about 15 minutes for this section.
  - Use the multiple-choice answer sheet provided.
- **Section II – 60 marks** (pages 8–13)
  - Attempt Questions 11-14.
  - Allow about 1 hour and 45 minutes for this section.

Areas of Assessment	Marks possible
Complex Numbers	/28
Proofs	/18
Vectors	/24
Total:	**/70**

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Section I

10 marks
Attempt Questions 1–10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet provided for Questions 1-10.

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1. The distance between the two points z and $-\overline{z}$ in the complex plane is given by?

A. $2\,Re(z)$

B. $2\,Im(z)$

C. $2\,|z|$

D. $2\,Re(z) + 2\,Im(z)$

2. What is the contrapositive of the conditional statement below?

"If you can't stand the heat, then stay out of the kitchen"

A. "If you stay out of the kitchen, then you can't stand the heat."

B. "If you can stand the heat, then stay in the kitchen."

C. "If you stay in the kitchen, then you can stand the heat."

D. "If you stay in the kitchen, then you can't stand the heat."

3. Given that $(1+i)^n = mi$, where m is a non-zero real constant, then $(1+i)^{2n+2}$ simplifies to?

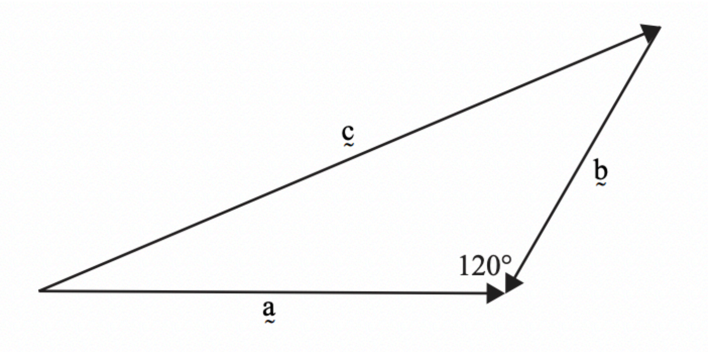
A. 0

B. 1+ $m^2 i$

C. $-2m^2 i$

D. m^4

4. Vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are indicated in the diagram below:



What statement must hold TRUE?

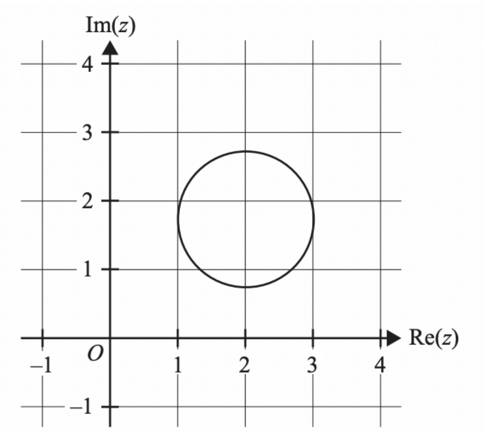
A. $\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 - \left|\underset{\sim}{a}\right| \left|\underset{\sim}{b}\right|$

B. $\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 + \left|\underset{\sim}{a}\right| \left|\underset{\sim}{b}\right|$

C. $\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 - \left|\underset{\sim}{a}.\underset{\sim}{b}\right|$

D. $\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 + \left|\underset{\sim}{a}.\underset{\sim}{b}\right|$

5. The graph of the circle given by $\left|z - 2 - \sqrt{3}i\right| = 1$, where $z \in \mathbb{C}$, is show below:



For points that lie on the circle, what is the maximum value of z?

A. 8

B. $\sqrt{3} + 1$

C. $\sqrt{13}$

D. $\sqrt{7} + 1$

6. What is the negation of the statement: $p > 3$ or $p < -3$?

A. $p \leq 3$ and $p \geq -3$

B. $p \leq 3$ or $p \geq -3$

C. $p < 3$ and $p > -3$

D. $p < 3$ or $p > -3$

7. Which of the following points lie on the line given by the vector equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$?

A. $\begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$

B. $\begin{pmatrix} 5 \\ -8 \\ 4 \end{pmatrix}$

C. $\begin{pmatrix} 11 \\ -17 \\ -2 \end{pmatrix}$

D. $\begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$

8. A unit vector that is perpendicular to $5\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ is given by?

A. $2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$

B. $\frac{1}{29}(2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k})$

C. $\frac{1}{\sqrt{29}}(2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k})$

D. $\frac{1}{\sqrt{30}}(2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k})$

9. One of the complex solutions to $z^5 = -a$ where a is a positive real constant, is $a^{\frac{1}{5}}\, cis\left(\frac{\pi}{5}\right)$.

One of the other solutions is a real number and is equal to?

A. $a^{\frac{1}{5}}\, cis\left(\frac{7\pi}{5}\right)$

B. $-a^{\frac{1}{5}}$

C. $a^{\frac{1}{5}}$

D. $a^{\frac{1}{5}}\, cis\left(\frac{9\pi}{5}\right)$

10. *A, B* and *C* are three collinear points with position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively.

B lies between *A* and *C*.

If $\left|\overrightarrow{BC}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|$ then $\underset{\sim}{c}$ is equal to?

A. $\frac{3}{2}(\underset{\sim}{b}) - \frac{1}{2}(\underset{\sim}{a})$

B. $\frac{3}{2}(\underset{\sim}{a}) - \frac{1}{2}(\underset{\sim}{b})$

C. $\frac{3}{2}(\underset{\sim}{b}) - \frac{3}{2}(\underset{\sim}{a})$

D. $\frac{1}{2}(\underset{\sim}{a}) - \frac{3}{2}(\underset{\sim}{b})$

END OF SECTION I

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Section II
60 marks
Attempt Questions 11–14.
Allow about 1 hour and 50 minutes for this section.

Answer each question on a NEW writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW writing booklet.

(a) (i) Write $z = \sqrt{2} - \sqrt{2}i$ in the form $re^{i\theta}$. **2**

(ii) Hence, write z^{22} in the form $x + iy$. **2**

(b) Prove by contrapositive that for all positive integers *n,* if the sum of all positive factors is $1 + n$, then *n* is a prime number. **3**

(c) The complex numbers u and ω are defined by $u = -1 + 7i$ and $\omega = 3 + 4i$.

(i) Find in the form $x + iy$, where x and y are real numbers, the complex numbers: **2**

(α) $u - 2\omega$

(β) $\frac{u}{\omega}$

(ii) In an Argand diagram with the origin O, the points P, Q and R represent the complex numbers u, ω and $u - 2\omega$ respectively. **2**

Prove that angle $POQ = \frac{\pi}{4}$.

(iii) State the geometrical relation between the line segments OQ and RP. **2**

(d) Find correct to the nearest whole degree, the angle between the line vectors: **2**

$$\underset{\sim}{r}_1 = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} \text{ and } \underset{\sim}{r}_2 = 3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$$

Question 12 (15 marks) Start a NEW writing booklet.

(a) (i) On the one Argand diagram sketch the two complex equations given by: **2**

$z\,\overline{z} = 4$ and $|z + \overline{z}| = |z - \overline{z}|$, where $z \in \mathbb{C}$.

(ii) State all points of intersection of the two relations above. **2**

Express your answer(s) in the form $a + ib$.

(b) Prove by Mathematical Induction that: **3**

$f(n) = 2^{3n+1} + 3(5^{2n+1})$ is divisible by 17, for all positive integers n.

(c) A line ℓ passes through the points $\mathrm{A}(2, -1, 3)$ and $\mathrm{B}(1, 3, -5)$.

(i) Show that the vector equation of the line ℓ is given by: **2**

$$\underset{\sim}{r} = \left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) + \lambda\left(-\underset{\sim}{i} + 4\underset{\sim}{j} - 8\underset{\sim}{k}\right) \text{ where } \lambda \in \mathbb{R}.$$

(ii) Determine if the point $\mathrm{C}(4, -9, 13)$ lies on $\underset{\sim}{r}$. **1**

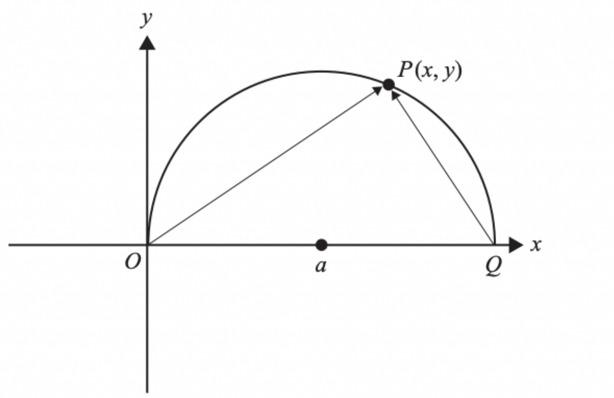
(iii) A second line has equation $\frac{x-2}{1} = \frac{y+2}{-2} = \frac{z-1}{12}$. **2**

Determine where this line and $\underset{\sim}{r}$ intersect.

Question 12 continues on the next page.

Question 12 (continued)

(d) OPQ is a semi-circle of radius a with equation $y = \sqrt{a^2 - (x-a)^2}$.
$P(x, y)$ is a point on the semicircle OPQ as shown in the diagram below.



(i) Express the vectors $\overrightarrow{OP}$ and $\overrightarrow{QP}$ in terms of $a, x, y, \underset{\sim}{i}$ and $\underset{\sim}{j}$ where $\underset{\sim}{i}$ is a unit vector in the direction of the positive x-axis and $\underset{\sim}{j}$ is a unit vector in the direction of the positive y-axis. **2**

(ii) Hence show that $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{QP}$. **1**

Question 13 (15 marks) Start a NEW writing booklet.

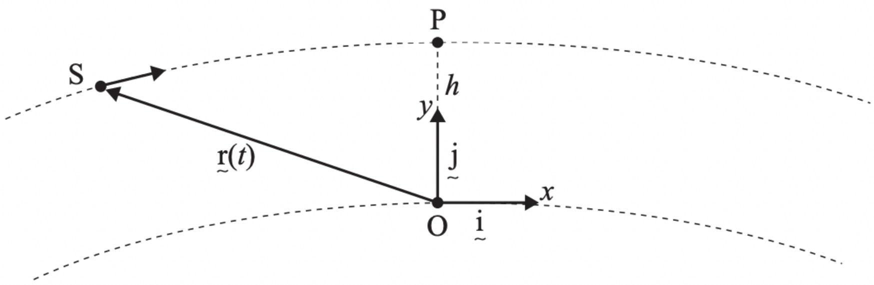
(a) Solve the equation $iz^2 + 2z - 3i = 0$, giving your answers in the form $x + iy$, where x and y are real and exact values. **3**

(b) The position vector of the International Space Station (S), when visible above the horizon from a radar tracking location (O) on the surface of the Earth, is modelled by:

$$\underset{\sim}{r}\,(t) = 6800\,sin[\pi(1.3t - 0.1)]\,\underset{\sim}{i} + [6800\,cos[\pi(1.3t - 0.1)] - 6400]\,\underset{\sim}{j}$$

for $t \in [0, 0.154]$,

Where $\underset{\sim}{i}$ is a unit vector relative to O and $\underset{\sim}{j}$ is a unit vector vertically up from the point O. The time *t* is measured in hours and displacement components are measured in kilometres.



(i) Find the height, *h* km, of the space station above the surface of the Earth when it is at point P, directly above point O. (Answer to the nearest km). **1**

(ii) Find the speed of the space station in km/h. (Answer to nearest whole km/h) **2**

(iii) Find the equation of the path of the space station in cartesian form. **2**

(iv) Find the times when the space station is at a distance of 1000km from the radar tracking location at O. (Answer in hours correct to 2 decimal places) **2**

Question 13 continues on the next page.

Question 13 (continued)

(c) Consider the complex number $z = (a - i)^3$ where $a \in \mathbb{R}^+$ **2**

Find a given that $\arg(z) = -\frac{\pi}{2}$.

(d) Prove by contradiction that, for all $x \in \left[\frac{\pi}{2}, \pi\right]$, $\sin x - \cos x \geq 1$ **3**

Question 14 (15 marks) Start a NEW writing booklet.

(a) If x, y and z are positive real numbers, prove that:

(i) $\frac{x}{y} + \frac{y}{x} \geq 2$ **2**

(ii) Hence, prove that $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9$ **2**

(b) Prove by Mathematical Induction that for all positive integers n, where $n \geq 1$ **3**

$$\sin x + \sin(x + y) + \sin(x + 2y) + \cdots + \sin[x + (n-1)y] = \frac{\sin\left[x + \left(\frac{n-1}{2}\right)y\right]\sin\left(\frac{ny}{2}\right)}{\sin\left(\frac{y}{2}\right)}$$

(c) (i) Find the equation of the sphere that passes through the point $(6, -2, 4)$ and has a centre of $(-1, 2, 1)$. **1**

(ii) Hence find the equation of the cross-sectional area of where the sphere intersects with the yz-plane. **2**

(d) (i) Show that $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ **2**

(ii) Find all solutions of the complex number z, such that $\sin z = 2$. **3**

END OF SECTION II

END OF EXAMINATION

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