2018 HSC Mathematics Extension 2 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	C
3	D
4	C
5	A
6	A
7	D
8	B
9	B
10	C

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\begin{aligned} zw &= (2+3i)(1-i) \\ &= 2-2i+3i+3 \\ &= 5+i \end{aligned}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains $\overline{z}$, or equivalent merit	1

Sample answer:

$$\begin{aligned} \overline{z}-\frac{2}{w} &= 2-3i-\frac{2}{1-i} \\ &= 2-3i-\frac{2}{(1-i)}\frac{(1+i)}{(1+i)} \\ &= 2-3i-\frac{(2+2i)}{1+1} \\ &= 2-3i-1-i \\ &= 1-4i \end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains the value of a, or equivalent merit	2
• Obtains $64 + 16a + b = 0$, or equivalent merit	1

Sample answer:

$$p(x) = x^3 + ax^2 + b$$
$$p(r) = r^3 + ar^2 + b$$

$$p(r) = 0$$
$$r^3 + ar^2 + b = 0 \quad ①$$

$$p'(x) = 3x^2 + 2ax$$
$$p'(4) = 3(4)^2 + 2a(4)$$

$$p'(4) = 0$$
$$48 + 8a = 0$$
$$a = -6$$
$$p(4) = 0$$

$$(4)^3 + a(4)^2 + b = 0$$
$$64 - 6(16) + b = 0$$
$$b = 32$$

Substitute into ①

$$r^3 - 6r^2 + 32 = 0$$
$$p(-2) = 0$$
$$r = -2$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	4
• Obtains primitive of the form $p\ln\lvert x+1\rvert+q\ln\lvert x^2-3\rvert$, or equivalent merit	3
• Obtains the values of a, b and c, or equivalent merit	2
• Obtains the value of a, or equivalent merit	1

Sample answer:

$$\frac{x^2-x-6}{(x+1)\left(x^2-3\right)}=\frac{a\left(x^2-3\right)+(bx+c)(x+1)}{(x+1)\left(x^2-3\right)}$$

$$=\frac{ax^2-3a+bx^2+bx+cx+c}{(x+1)\left(x^2-3\right)}$$

$$=\frac{(a+b)x^2+(b+c)x-3a+c}{(x+1)\left(x^2-3\right)}$$

Equating coefficient

$a+b=1,\ b+c=-1,\ -3a+c=-6$

$$b+c=-1 \quad ①$$
$$-3a+c=-6 \quad ②$$
$$① - ②$$
$$b+3a=5 \quad ③$$
$$a+b=1 \quad ④$$
$$③ - ④$$
$$2a=4$$
$$a=2$$

Substitute into ④

$$2+b=1$$
$$b=-1$$

Substitute into ①

$$-1+c=-1$$
$$c=0$$

$$\int\frac{x^2-x-6}{(x+1)\left(x^2-3\right)}$$

$$=\int\frac{2}{x+1}-\frac{x}{x^2-3}dx$$

$$=2\ln\lvert x+1\rvert-\frac{1}{2}\ln\lvert x^2-3\rvert+c$$

Question 11 (d) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$w = i(5 + 2i)$

$w = 5i - 2$

$w = -2 + 5i$

Question 11 (d) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$v = u + w$

$v = 5 + 2i + -2 + 5i$

$v = 3 + 7i$

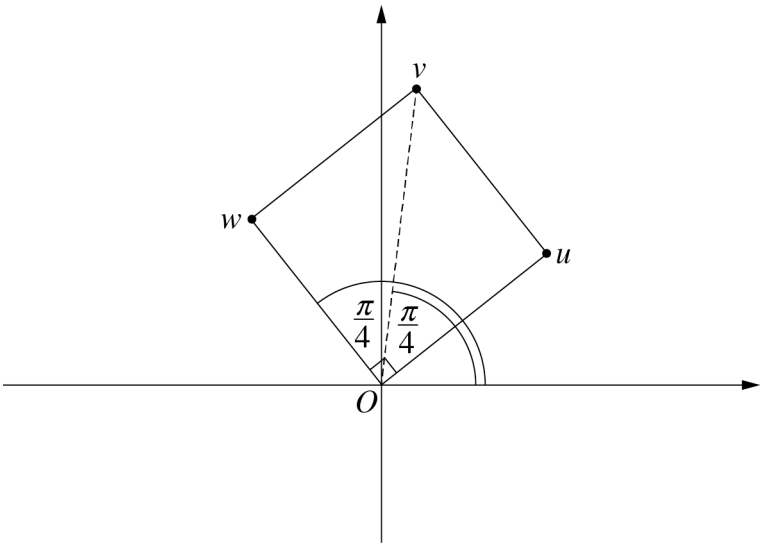
Question 11 (d) (iii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\arg\left(\frac{w}{v}\right) = \arg w - \arg v$$

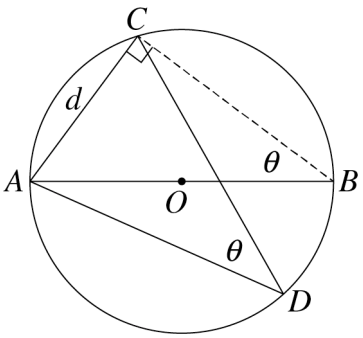
$$= \frac{\pi}{4}$$



Question 11 (e)

Criteria	Marks
• Provides correct explanation	2
• Explains why it is sufficient to consider $\triangle ABC$, or equivalent merit	1

Sample answer:



$\angle ABC = \angle ADC$ (angles standing on same arc are equal)

$= \theta$

$\angle BCA = 90°$ (angle in a semi-circle is a right angle)

$$\sin\theta = \frac{d}{2r}$$

$$\sin D = \frac{d}{2r}$$

$$d = 2r\sin D$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	3
• Obtains a correct integral for the volume in terms of y, or equivalent merit	2
• Obtains the area of a general slice, or equivalent merit	1

Sample answer:

$$\triangle V = \frac{1}{2} \times x \times x \times \sin 60° \triangle y$$

$$= \frac{\sqrt{3}}{4} x^2 \triangle y$$

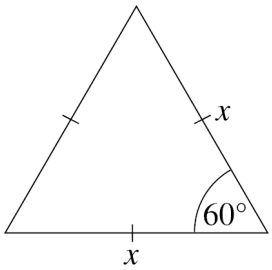
$$V = \int_{-1}^{1} \frac{\sqrt{3}}{4} x^2 \, dy$$

$$= \frac{\sqrt{3}}{2} \int_{0}^{1} \left(1 - 2y^2 + y^4\right) dy$$

$$= \frac{\sqrt{3}}{2} \left[y - \frac{2}{3} y^3 + \frac{1}{5} y^5 \right]_0^1$$

$$= \frac{\sqrt{3}}{2} \left(1 - \frac{2}{3} + \frac{1}{5} - 0 \right)$$

$$= \frac{4\sqrt{3}}{15}$$



$$x = 1 - y^2$$

$$x^2 = 1 - 2y^2 + y^4$$

Question 12 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly finds derivative of a product using implicit differentiation, or equivalent merit	1

Sample answer:

$$x^2 + xy + y^2 = 3$$

Find $\dfrac{dy}{dx}$

$$2x + x\frac{dy}{dx} + y + 2y.\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx}(x + 2y) = -(2x + y)$$

$$\frac{dy}{dx} = \frac{-(2x + y)}{x + 2y}$$

Question 12 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains a quadratic from which the solutions can be found, or equivalent merit	1

Sample answer:

$\dfrac{dy}{dx} = \dfrac{-(2x+y)}{x+2y}$ $\quad x^2 + xy + y^2 = 3$

$\dfrac{dy}{dx} = 0$ $\quad$ when $\quad -(2x+y) = 0$

$$2x = -y$$

$$\text{or} \quad y = -2x$$

Substituting into the equation for the curve

$$x^2 + x(-2x) + (-2x)^2 = 3$$

$$x^2 - 2x^2 + 4x^2 = 3$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1$$

when $x = 1 \quad y = -2$

when $x = -1 \quad y = 2$

Points on the curve where $\dfrac{dy}{dx} = 0$ are (1, –2) and (–1, 2).

Question 12 (c)

Criteria	Marks
• Provides correct solution	3
• Rewrites integrand as $1-\dfrac{5}{(x+1)^2+4}$, or equivalent merit	2
• Rewrites integrand as $1-\dfrac{5}{x^2+2x+5}$, or equivalent merit	1

Sample answer:

$$\int \frac{x^2+2x}{x^2+2x+5}\,dx$$

$$= \int \frac{x^2+2x+5}{x^2+2x+5} - \frac{5}{x^2+2x+5}\,dx$$

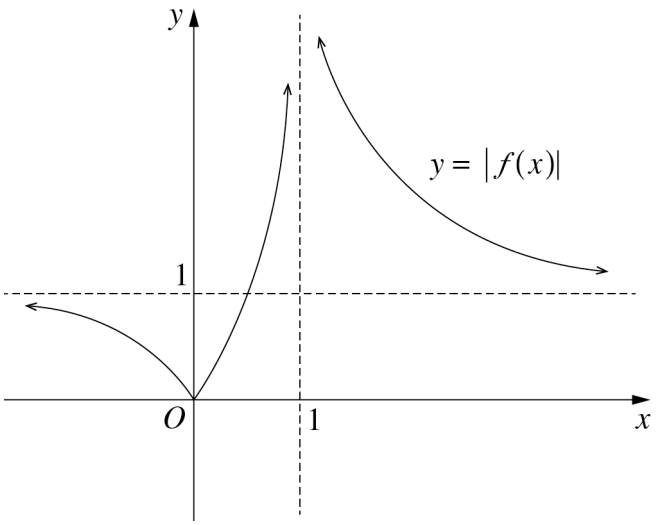
$$= \int 1 - \frac{5}{(x+1)^2+4}\,dx$$

$$= x - \frac{5}{2}\tan^{-1}\frac{(x+1)}{2} + c$$

Question 12 (d) (i)

Criteria	Marks
• Provides correct sketch	1

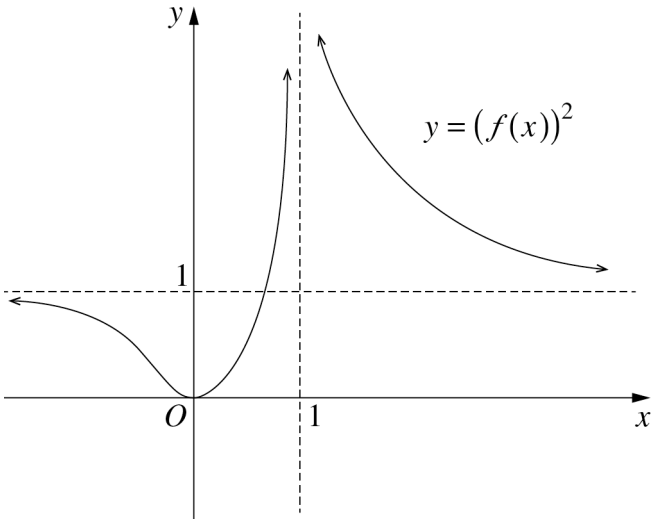
Sample answer:



Question 12 (d) (ii)

Criteria	Marks
• Provides correct sketch	2
• Shows minimum turning point at the origin or equivalent merit	1

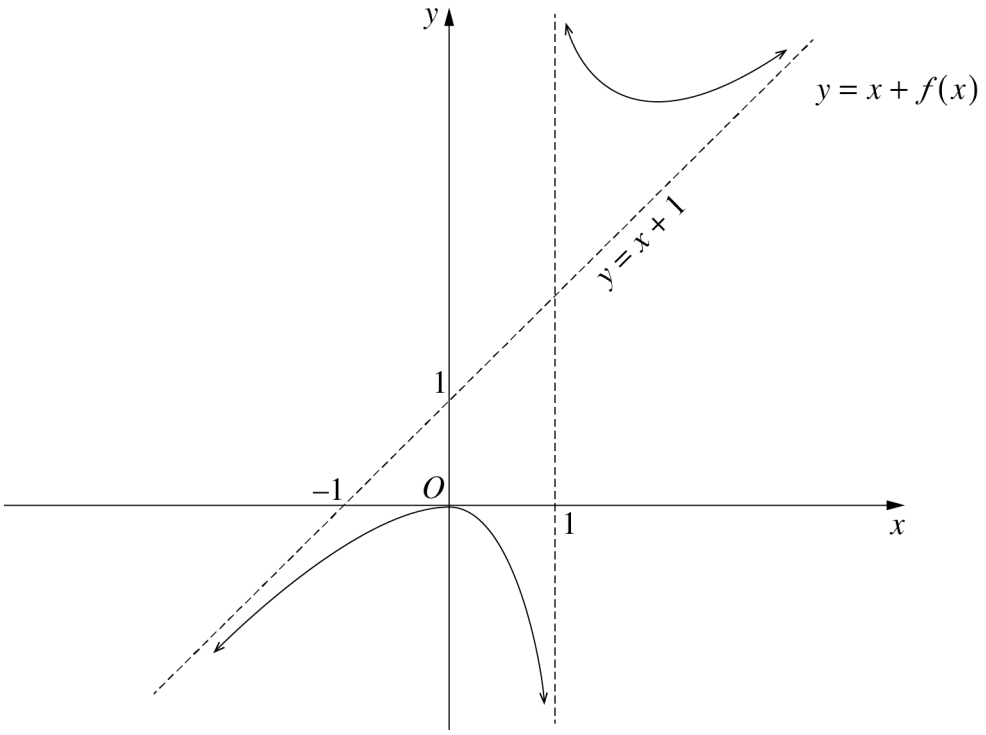
Sample answer:



Question 12 (d) (iii)

Criteria	Marks
• Provides correct sketch	2
• Provides sketch of curve with correct shape, or equivalent merit	1

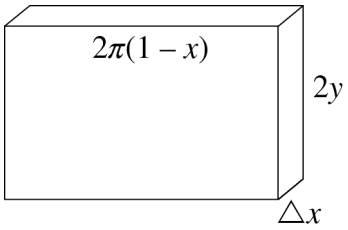
Sample answer:



Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Obtains a correct integral for the volume, or equivalent merit	2
• Correctly finds dV, or equivalent merit	1

Sample answer:



$$V = 4\pi \int_0^1 (1-x)\cdot\sqrt{x}\,(1-x)\,dx$$

$$V = 4\pi \int_0^1 \sqrt{x}\left(1-2x+x^2\right)dx$$

$$V = 4\pi \int_0^1 x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + x^{\frac{5}{2}}\,dx$$

$$V = 4\pi \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}}\right]_0^1$$

$$V = 4\pi \left[\left(\frac{2}{3}(1)^{\frac{3}{2}} - \frac{4}{5}(1)^{\frac{5}{2}} + \frac{2}{7}(1)^{\frac{7}{2}}\right) - 0\right]$$

$$\text{Volume} = \frac{64\pi}{105} \quad \text{u}^3$$

Question 13 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains $\lvert z\rvert^2$ in terms of 2θ, or equivalent merit	1

Sample answer:

$$\begin{aligned}|z| &= \sqrt{(1-\cos 2\theta)^2 + (\sin 2\theta)^2} \\ &= \sqrt{1 - 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta} \\ &= \sqrt{2 - 2\cos 2\theta} \\ &= \sqrt{2\left(1 - 1 + 2\sin^2 \theta\right)} \\ &= \sqrt{4\sin^2 \theta} \\ &= 2\sin\theta \quad \text{as required}\end{aligned}$$

Question 13 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds $\tan(\arg(z))$ in terms of $\sin\theta$ and $\cos\theta$, or equivalent merit	1

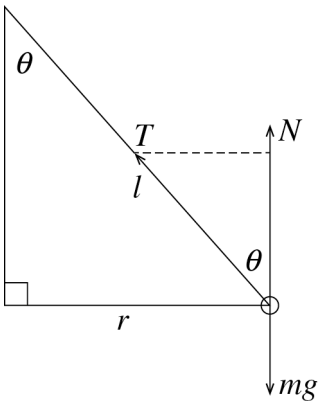
Sample answer:

$$\begin{aligned}\arg(z) &= \tan^{-1}\left(\frac{\sin 2\theta}{1-\cos 2\theta}\right) \\ &= \tan^{-1}\left(\frac{2\sin\theta\cos\theta}{2\sin^2\theta}\right) \\ &= \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right) \\ &= \tan^{-1}(\cot\theta) \\ &= \frac{\pi}{2} - \theta\end{aligned}$$

Question 13 (c)

Criteria	Marks
• Provides correct solution	3
• Obtains an expression for N independent of T, or equivalent merit	2
• Obtains an equation for the vertical components of the forces, or equivalent merit	1

Sample answer:



$$T\cos\theta + N = mg \qquad ①$$

$$T\sin\theta = mr\omega^2 \qquad ②$$

$$T = \frac{mr\omega^2}{\sin\theta}$$

Substitute into ①

$$\frac{mr\omega^2}{\sin\theta}\cdot\cos\theta + N = mg$$

$$\frac{mr\omega^2\cdot l}{r}\cdot\cos\theta + N = mg$$

$$N = mg - m\omega^2 l\cos\theta$$

$N \geq 0$ to remain in contact

$$mg - m\omega^2 l\cos\theta \geq 0$$

$$m\omega^2 l\cos\theta \leq mg$$

$$\omega^2 \leq \frac{mg}{ml\cos\theta}$$

$$\omega^2 \leq \frac{g}{l\cos\theta}$$

Question 13 (d) (i)

Criteria	Marks
• Provides correct solution	3
• Correctly finds the distance PS or equivalent merit	2
• Correctly finds the coordinates of R and S, or equivalent merit	1

Sample answer:

$$P\left(cp, \frac{c}{p}\right), \quad S\left(0, \frac{c(p+q)}{pq}\right)$$

$$PS = \sqrt{(0-cp)^2 + \left(\frac{c(p+q)}{pq} - \frac{c}{p}\right)^2}$$

$$PS = \sqrt{c^2p^2 + \left(\frac{cp+cq-cq}{pq}\right)^2}$$

$$PS = \sqrt{c^2p^2 + \frac{c^2p^2}{p^2q^2}}$$

$$PS = \sqrt{c^2\left(p^2 + \frac{1}{q^2}\right)}$$

$$PS = c\sqrt{p^2 + \frac{1}{q^2}}$$

$$Q\left(cq, \frac{c}{q}\right), \quad R\left(c(p+q), 0\right)$$

$$QR = \sqrt{(cp+cq-cq)^2 + \left(0 - \frac{c}{q}\right)^2}$$

$$QR = \sqrt{c^2p^2 + \frac{c^2}{q^2}}$$

$$QR = \sqrt{c^2\left(p^2 + \frac{1}{q^2}\right)}$$

$$= c\sqrt{p^2 + \frac{1}{q^2}}$$

$$= PS$$

Question 13 (d) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds M, or equivalent merit	1

Sample answer:

$y = tx - at^2$

At $x = 0$, $y = -at^2$ $\quad\therefore$ y-intercept is $\left(0, -at^2\right)$

At $y = 0$, $0 = tx - at^2$

$x = at$ $\quad\therefore$ x-intercept is $(at, 0)$

Using part (i), the midpoint of AB will also be the midpoint of the x and y-intercepts.

$$\therefore M \text{ is } \left(\frac{0+at}{2}, \frac{-at^2+0}{2}\right)$$

$$\text{ie } \left(\frac{at}{2}, \frac{-at^2}{2}\right)$$

substitute M into $2x^2 = -ay$

$$\text{LHS} = 2\times\left(\frac{at}{2}\right)^2$$

$$= \frac{2a^2t^2}{4}$$

$$= \frac{a^2t^2}{2}$$

$$= -a\left(-\frac{at^2}{2}\right)$$

$$= -ay$$

$$= \text{RHS}$$

$\therefore$ M lies on the parabola $2x^2 = -ay$

Question 13 (d) (ii)

Alternative solution

$$y = tx - at^2 \qquad ①$$

$$xy = c^2 \qquad ②$$

$$y = \frac{c^2}{x}$$

Substitute into ①

$$\frac{c^2}{x} = tx - at^2$$

$$c^2 = tx^2 - at^2 x$$

$$tx^2 - at^2 x - c^2 = 0$$

$$x = \frac{at^2 \pm \sqrt{\left(-at^2\right)^2 - 4t\left(-c^2\right)}}{2t}$$

$$x = \frac{at^2 \pm \sqrt{a^2t^4 + 4c^2t}}{2t}$$

$$M = \left(\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2} \right)$$

$$M = \left(\frac{2at^2}{4t}, \quad ? \right)$$

$$M = \left(\frac{at}{2}, \quad ? \right)$$

$x = \frac{at}{2}$ lies on $y = tx - at^2$

$$y = t \cdot \frac{at}{2} - at^2$$

$$y = -\frac{at^2}{2}$$

$$\therefore M = \left(\frac{at}{2}, \quad -\frac{at^2}{2} \right)$$

$$x = \frac{at}{2}, \quad y = -\frac{at^2}{2}$$

$$t = \frac{2x}{a}, \quad y = -\frac{a}{2}\left(\frac{2x}{a} \right)^2$$

$$y = -\frac{a}{2}\left(\frac{4x^2}{a^2} \right)$$

$$y = -\frac{2x^2}{a}$$

$$2x^2 = -ay$$

Question 14 (a)

Criteria	Marks
• Provides correct solution	3
• Obtains correct integral, or equivalent merit	2
• Correctly obtains $d\theta$, or equivalent merit	1

Sample answer:

$$I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$$

$$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{2-\frac{1-t^2}{1+t^2}} \times \frac{1+t^2}{1+t^2}$$

$$= \int_0^1 \frac{2 \cdot dt}{2\left(1+t^2\right)-\left(1-t^2\right)}$$

$$= \int_0^1 \frac{2dt}{1+3t^2}$$

$$= \frac{2}{3}\int_0^1 \frac{dt}{\frac{1}{3}+t^2}$$

$$= \frac{2}{3} \cdot \frac{1}{\frac{1}{\sqrt{3}}}\left[\tan^{-1}\frac{t}{\frac{1}{\sqrt{3}}}\right]_0^1$$

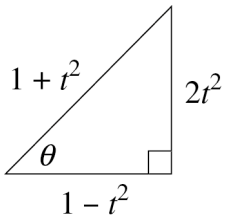
$$= \frac{2\sqrt{3}}{3}\left[\tan^{-1}\sqrt{3t}\right]_0^1$$

$$= \frac{2\sqrt{3}}{3}\left(\tan^{-1}\sqrt{3}-\tan^{-1}0\right)$$

$$= \frac{2\sqrt{3}}{3} \cdot \frac{\pi}{3}$$

$$= \frac{2\sqrt{3}}{9}\pi$$

Let $t = \tan\frac{\theta}{2}$



$$dt = \frac{1}{2}\sec^2\frac{\theta}{2} \cdot d\theta$$

$$2dt = \left(1+t^2\right)d\theta$$

$$d\theta = \frac{2dt}{1+t^2}$$

$\theta = 0, \quad t = 0$

$\theta = \frac{\pi}{2}, \quad t = 1$

Question 14 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains x, or equivalent merit	2
• Obtains $\frac{dx}{dv}=\frac{v}{g-kv^2}$, or equivalent merit	1

Sample answer:

$$\ddot{x}=g-kv^2$$

$$v\cdot\frac{dv}{dx}=g-kv^2$$

$$\frac{dx}{dv}=\frac{v}{g-kv^2}$$

$$x=\int\frac{v}{g-kv^2}dv$$

$$x=-\frac{1}{2k}\ln\left(g-kv^2\right)+c$$

$$x=0,\ v=0 \quad \Rightarrow 0=-\frac{1}{2k}\ln\left(g-k(0)^2\right)+C$$

$$\Rightarrow c=\frac{1}{2k}\ln(g)$$

$$\therefore x=-\frac{1}{2k}\ln\left(g-kv^2\right)+\frac{1}{2k}\ln(g)$$

$$=-\frac{1}{2k}\ln\left(\frac{g-kv^2}{g}\right)$$

$$=-\frac{1}{2k}\ln\left(1-\frac{k}{g}v^2\right)$$

$$\text{Let } x=h\Rightarrow \quad -2kh=\ln\left(1-\frac{k}{g}v^2\right)$$

$$1-\frac{k}{g}v^2=e^{-2kh}$$

$$1-e^{-2kh}=\frac{k}{g}v^2$$

$$v^2=\frac{g}{k}\left(1-e^{-2kh}\right)$$

$$v=\sqrt{\frac{g}{k}\left(1-e^{-2kh}\right)}$$

$$v>0$$

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Correctly applies integration by parts, or equivalent merit	2
• Attempts to apply integration by parts or equivalent merit	1

Sample answer:

$$I_n = \int_{-3}^{0} x^n \sqrt{x+3}\, dx \qquad n = 0, 1, 2\ldots$$

Let $u = x^n \qquad v^1 = (x+3)^{\frac{1}{2}}$

$$u^1 = nx^{n-1} \qquad v = \frac{2(x+3)^{\frac{3}{2}}}{3}$$

$$I_n = \left[\frac{2x^n(x+3)^{\frac{3}{2}}}{3}\right]_{-3}^{0} - \int_{-3}^{0} \frac{2nx^{n-1}(x+3)^{\frac{3}{2}}}{3} dx$$

$$I_n = (0-0) - \frac{-2n}{3}\int_{-3}^{0} x^{n-1}(x+3)\sqrt{x+3}\, dx$$

$$I_n = -\frac{2n}{3}\int_{-3}^{0} \left(x^n\sqrt{x+3} + 3x^{n-1}\sqrt{x+3}\right) dx$$

$$I_n = -\frac{2n}{3}\int_{-3}^{0} x^n\sqrt{x+3}\, dx - 2n\int_{-3}^{0} x^{n-1}\sqrt{x+3}\, dx$$

$$I_n = -\frac{2n}{3}I_n - 2nI_{n-1}$$

$$I_n\left(1+\frac{2n}{3}\right) = -2nI_{n-1}$$

$$I_n\left(\frac{3+2n}{3}\right) = -2nI_{n-1}$$

$$I_n = \frac{-6n}{3+2n}I_{n-1}$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds I_2 in terms of I_0, or equivalent merit	1

Sample answer:

$$I_2 = \frac{-12}{7} I_1$$

$$= \frac{-12}{7} \times \frac{-6}{5} I_0$$

$$I_0 = \int_{-3}^{0} \sqrt{x+3}\, dx$$

$$= \left[\frac{2(x+3)^{\frac{3}{2}}}{3} \right]_{-3}^{0}$$

$$= \frac{2}{3}(3)^{\frac{3}{2}} - 0$$

$$= \frac{2 \times 3\sqrt{3}}{3}$$

$$= 2\sqrt{3}$$

$$\therefore I_2 = \frac{72}{35} \times 2\sqrt{3}$$

$$= \frac{144\sqrt{3}}{35}$$

Question 14 (d) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$P(A \text{ wins}) = \left(\frac{1}{3}\right)^n$$

Question 14 (d) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$P(C \text{ never wins}) = \left(\frac{2}{3}\right)^n$$

This includes when A wins all and B wins all.

$$P(A \text{ wins all games or } B \text{ wins all games}) = 2 \times \left(\frac{1}{3}\right)^n$$

$\therefore$ Required probability where A and B both win a game is: $\left(\frac{2}{3}\right)^n - 2\left(\frac{1}{3}\right)^n$

Question 14 (d) (iii)

Criteria	Marks
• Provides correct solution	2
• Correctly deals with the case where one player wins all the games, or equivalent merit	1

Sample answer:

If A, B or C wins all the games

Probability is: $3 \times \left(\frac{1}{3}\right)^n = \frac{1}{3^{n-1}}$

If two people win at least one game but not the third person

Probability is: $3\left[\left(\frac{2}{3}\right)^n - 2\left(\frac{1}{3}\right)^n\right]$

$$= \frac{2^n - 2}{3^{n-1}}$$

P(each player wins at least one game)

$$= 1 - \left[\frac{1}{3^{n-1}} + \frac{2^n - 2}{3^{n-1}}\right]$$

$$= \frac{3^{n-1} - 1 - 2^n + 2}{3^{n-1}}$$

$$= \frac{3^{n-1} - 2^n + 1}{3^{n-1}}$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Finds coordinates of B, or equivalent merit	1

Sample answer:

Since $\angle AOB = \dfrac{\pi}{2}$

B has coordinates $\left(a\cos\left(\theta+\dfrac{\pi}{2}\right),\ a\sin\left(\theta+\dfrac{\pi}{2}\right)\right)$

ie $\left(-a\sin\theta, a\cos\theta\right)$

Q has same x-coordinate

$$\begin{aligned}\frac{y^2}{b^2} &= 1-\frac{x^2}{a^2}\\ &= 1-\frac{a^2\sin^2\theta}{a^2}\\ &= \cos^2\theta\\ \therefore\ y^2 &= b^2\cos^2\theta\\ y &= b\cos\theta \qquad (\text{since } y>0)\end{aligned}$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds the minimum value of $\tan(\angle POQ')$, or equivalent merit	2
• Finds an expression for $\tan(\angle POQ')$, or equivalent merit	1

Sample answer:

$$M_{OP} = \frac{b\sin\theta}{a\cos\theta} \quad (\cos\theta \neq 0)$$

$$M_{OQ'} = -\frac{b\cos\theta}{a\sin\theta} \ (\sin\theta \neq 0)$$

$$\therefore \tan\angle POQ' = \left| \frac{\frac{b\sin\theta}{a\cos\theta} - \left(-\frac{b\cos\theta}{a\sin\theta}\right)}{1 + \frac{b\sin\theta}{a\cos\theta}\left(-\frac{b\cos\theta}{a\sin\theta}\right)} \right|$$

$$= \left| \frac{ab\sin^2\theta + ab\cos^2\theta}{a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta} \right|$$

(Multiplying numerator and denominator by $a^2\sin\theta\cos\theta$)

$$= \frac{2ab}{\left(a^2 - b^2\right)\sin 2\theta} \quad \text{(all terms are positive so absolute value is not necessary)}$$

This expression is least when $\sin 2\theta$ is greatest
ie when $\sin 2\theta = 1$

Result follows

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly uses binomial theorem, or equivalent merit	1

Sample answer:

$$(\cos\theta + i\sin\theta)^8 = \cos 8\theta + i\sin 8\theta \quad \text{(De Moivre)}$$

$$\text{Also} \quad (\cos\theta + i\sin\theta)^8 = \cos^8\theta + \binom{8}{1}\cos^7\theta\, i\sin\theta + \binom{8}{2}\cos^6\theta\, i^2\sin^2\theta + \binom{8}{3}\cos^5\theta\, i^3\sin^3\theta +$$

$$\binom{8}{4}\cos^4\theta\, i^4\sin^4\theta + \cdots + i^8\sin^8\theta$$

$$= \cos^8\theta\binom{8}{1}\cos^7\theta\sin\theta\, i - \binom{8}{2}\cos^6\theta\sin^2\theta - \binom{8}{3}\cos^5\theta\sin^3\theta\, i +$$

$$\binom{8}{4}\cos^4\theta\sin^4\theta + \cdots + \sin^8\theta$$

Equating imaginary parts

$$\sin 8\theta = \binom{8}{1}\cos^7\theta\sin\theta - \binom{8}{3}\cos^5\theta\sin^3\theta + \binom{8}{5}\cos^3\theta\sin^5\theta - \binom{8}{7}\cos\theta\sin^7\theta$$

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains an expression for $\dfrac{\sin 8\theta}{\sin 2\theta}$ in terms of $\sin\theta$, or equivalent merit	2
• Correctly divides by $\sin 2\theta$, or equivalent merit	1

Sample answer:

$$\frac{\sin 8\theta}{\sin 2\theta} = \frac{\binom{8}{1}\cos^7\theta\sin\theta - \binom{8}{3}\cos^5\theta\sin^3\theta + \binom{8}{5}\cos^3\theta\sin^5\theta - \binom{8}{7}\cos\theta\sin^7\theta}{2\sin\theta\cos\theta}$$

$$= \frac{1}{2}\left[\binom{8}{1}\cos^6\theta - \binom{8}{3}\cos^4\theta\sin^2\theta + \binom{8}{5}\cos^2\theta\sin^4\theta - \binom{8}{7}\sin^6\theta\right]$$

$$= \frac{1}{2}\left[\binom{8}{1}\left(1-\sin^2\theta\right)^3 - \binom{8}{3}\left(1-\sin^2\theta\right)^2\sin^2\theta + \binom{8}{5}\left(1-\sin^2\theta\right)^2\sin^4\theta - \binom{8}{7}\sin^6\theta\right]$$

$$= \frac{1}{2}\left[8\left(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta\right) - 56\left(\sin^2\theta-2\sin^4\theta+\sin^6\theta\right)\right.$$

$$\left.+56\left(\sin^4\theta-\sin^6\theta\right) - 8\sin^6\theta\right]$$

$$= 4\left[1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta-7\sin^2\theta+14\sin^4\theta-7\sin^6\theta\right.$$

$$\left.+7\sin^4\theta-7\sin^6\theta-\sin^6\theta\right]$$

$$= 4\left[1-10\sin^2\theta+24\sin^4\theta-16\sin^6\theta\right]$$

Question 15 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Show that $x^n - 1 - n(x-1) = (x-1)\left(1 + x + \cdots + x^{n-1} - n\right)$

$x > 0, \quad x \in IR$

$$\left(x^n - 1\right) - n(x-1) = (x-1)\left(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1\right) - n(x-1)$$

$$= (x-1)\left(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1 - n\right)$$

$$= (x-1)\left(1 + x + x^2 + \cdots + x^{n-1} - n\right)$$

Question 15 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Recognises the two cases $x \geq 1,\ x < 1$, or equivalent merit	1

Sample answer:

Hence show that $x^n \geq 1 + n(x-1)$

When $x = 1, \quad x^n = 1, \quad \text{and} \quad 1 + n(x-1) = 1$
so $x^n \geq 1 + n(x-1)$

Suppose $0 < x < 1$ so $0 < x^k < 1$ for all integers $k > 0$

$(x-1) < 0 \quad \text{and} \quad 1 + x + x^2 + \cdots + x^{n-1} < 1 + 1 + \cdots + 1 \ (n \text{ terms})$

so $\qquad 1 + x + x^2 + \cdots + x^{n-1} < n$
so $\qquad 1 + x + x^2 + \cdots + x^{n-1} - n < 0$

when $x < 1 \qquad (x-1)\left(1 + x + x^2 + \cdots + x^{n-1} - n\right) > 0$

suppose $x > 1 \quad x^k > 1$ for all integers $k > 0$

$(x-1) > 0 \qquad 1 + x + x^2 + \cdots + x^{n-1} > n$

so $\qquad 1 + x + x^2 + \cdots + x^{n-1} - n > 0$

so when $x > 1 \quad (x-1)\left(1 + x + x^2 + \cdots + x^{n-1} - n\right) > 0$

Thus

$x^n - 1 - n(x-1) \geq 0$

so $\qquad x^n \geq 1 + n(x-1)$

Question 15 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Considers $x = \frac{a}{b}$, or equivalent merit	1

Sample answer:

Deduce that for positive real numbers

a and b

$a^n b^{1-n} \geq na + (1-n)b$

Let $x = \frac{a}{b} > 0$ and use part (ii)

$$\frac{a^n}{b^n} \geq 1 + n\left(\frac{a}{b} - 1\right)$$

$$a^n b^{-n} \geq \frac{b}{b} + \frac{na}{b} - \frac{bn}{b}$$

$$a^n b^{-n} \times b \geq na + b - bn$$

$$a^n b^{1-n} \geq na + b(1-n)$$

Question 16 (a)

Criteria	Marks
• Provides correct proof	3
• Establishes the inductive step, or equivalent merit	2
• Shows true for $n = 1$, or equivalent merit	1

Sample answer:

Prove for $n \geq 1$

$$x^{3^n} - 1 = (x-1)\left(x^2 + x + 1\right)\left(x^6 + x^3 + 1\right)\ldots\left(x^{2\times 3^{n-1}} + x^{3^{n-1}} + 1\right)$$

When $n = 1$

$$\text{LHS} = x^3 - 1 = (x-1)\left(x^2 + x + 1\right) \text{ as required} = \text{RHS}$$

Assume that the proposition is true when $n = k$

so $x^{3^k} - 1 = (x-1)\left(x^2 + x + 1\right)\ldots\left(x^{2\times 3^{k-1}} + x^{3^{k-1}} + 1\right)$

Show the proposition is true when $n = k + 1$

$$\begin{aligned} x^{3^{k+1}} - 1 &= x^{3^k.3} - 1 \\ &= \left(x^{3^k}\right)^3 - 1 \\ &= \left(x^{3^k} - 1\right)\left(\left(x^{3^k}\right)^2 + \left(x^{3^k}\right) + 1\right) \\ &= (x-1)\left(x^2 + x + 1\right)\ldots\left(x^{2\times 3^{k-1}} + x^{3^{k-1}} + 1\right)\left(x^{2\times 3^k} + x^{3^k} + 1\right) \end{aligned}$$

Since proposition is true for $n = k + 1$

By POMI $x^{3^n} - 1 = (x-1)\left(x^2 + x + 1\right)\ldots\left(x^{2\times 3^{n-1}} + x^{3^{n-1}} + 1\right)$

Question 16 (b) (i)

Criteria	Marks
• Provides correct proof	3
• Proves that $BF = HC$, or equivalent merit	2
• Observes that $BH = DY$, or equivalent merit	1

Sample answer:

In triangles BGF and BAC $\angle B$ is common

As $GF \parallel AC$

$\angle GFB = \angle ACB$ (corres)

so $\triangle BGF$ is similar to $\triangle BAC$

$$\frac{AC}{GF} = \sqrt{2}$$

so $$\frac{BC}{BF} = \sqrt{2}$$

similarly $$\frac{BC}{HC} = \sqrt{2}$$

hence $$BF = HC$$

$AB \parallel IH$ and $BC \parallel DE$

so $DYHB$ is a parallelogram

similarly $ZECF$ is a parallelogram

hence

$$\begin{aligned} DY &= BH && \text{(opposite sides in parallelogram)} \\ &= BC - HC \\ &= BC - FB \\ &= FC \\ &= ZE && \text{(opposite sides in parallelogram)} \end{aligned}$$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains an expression for YZ in terms of BC, or equivalent merit	1

Sample answer:

$$YZ = DE - DY - ZE$$

$$= DE - 2DY \qquad \text{(from (i))}$$

$$= \frac{BC}{\sqrt{2}} - 2(BC - HC)$$

$$= \frac{BC}{\sqrt{2}} - 2BC + 2HC$$

$$= \frac{BC}{\sqrt{2}} - 2BC + 2\frac{BC}{\sqrt{2}}$$

$$= 3\frac{BC}{\sqrt{2}} - 2BC$$

$$= BC\left(\frac{3 - 2\sqrt{2}}{\sqrt{2}}\right)$$

$$\therefore \frac{YZ}{BC} = \frac{3 - 2\sqrt{2}}{\sqrt{2}}$$

Question 16 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use $\alpha+\beta+\gamma=0$, or equivalent merit	1

Sample answer:

$$(\beta-\gamma)^2=\beta^2-2\beta\gamma+\gamma^2$$

$$=\beta^2+2\beta\gamma+\gamma^2-4\beta\gamma$$

$$=(\beta+\gamma)^2-4\beta\gamma$$

$$=(\alpha+\beta+\gamma-\alpha)^2-\frac{4\alpha\beta\gamma}{\alpha}$$

$$=(0-\alpha)^2-\frac{4(-q)}{\alpha}$$

$$(\text{since } \alpha+\beta+\gamma=0 \text{ and } \alpha\beta\gamma=-q)$$

$$=\alpha^2+\frac{4q}{\alpha}$$

Question 16 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains cubic equation in y having noted $y+p\neq 0,$ or equivalent merit	2
• Lets $y=x^2+\dfrac{4q}{x}$, or equivalent merit	1

Sample answer:

We want to form an equation with roots

$$\alpha^2+\frac{4q}{\alpha},\quad \beta^2+\frac{4q}{\beta}\quad \text{and}\quad \gamma^2+\frac{4q}{\gamma}$$

Suppose x is a solution of the original equation

$$\text{Let}\qquad y=x^2+\frac{4q}{x}$$

$$\text{then}\quad xy=x^3+4q$$

$$=-px-q+4q\qquad \text{since } x^3+px+q=0$$

$$x(y+p)=3q$$

$$\text{Since } q\neq 0,\ \text{it follows } y+p\neq 0$$

$$\therefore\ x=\frac{3q}{y+p}$$

Since x satisfies original equation

$$\left(\frac{3q}{y+p}\right)^3+p\left(\frac{3q}{y+p}\right)+q=0$$

After multiplying by $(y+p)^3$

$$27q^2+3pq(y+p)^2+q(y+p)^3=0$$

This is an equation with roots $(\alpha-\beta)^2,\ (\beta-\gamma)^2$ and $(\gamma-\alpha)^2$

$$\text{Product of these roots}=\frac{-\text{ constant term}}{\text{loading coefficient}}$$

$$= -\frac{27q^3 + 3pq.p^2 + qp^3}{q} \quad (q \neq 0)$$

$$= -\left(27q^2 + 3p^3 + p^3\right)$$

$$= -\left(27q^2 + 4p^3\right)$$

Question 16 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Observes that $p(x)$ has distinct roots which are all real or 1 real and 2 complex, or equivalent merit	1

Sample answer:

Suppose $27q^2 + 4p^3 < 0$

Then $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 > 0$

The roots α, β and γ are distinct otherwise at least one factor would be zero.

If not all the roots are real then there must be one real root and two roots which are complex conjugates (since coefficients of the cubic are real).

Let the roots be α, $a + ib$, $a - ib$ where α, a, b are real

$$(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2$$

let α be the real root and $\beta = a + ib$ and $\gamma = a - ib$ where a, b are real

$$(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = (\alpha - \beta)^2(\beta - \gamma)^2(\alpha - \gamma)^2$$

$$= \Big[(\alpha - a) - ib\Big]^2\Big[2ib\Big]^2\Big[(\alpha - a) + ib\Big]^2$$

$$= \left(\Big[(\alpha - a) - ib\Big]\Big[(\alpha - a) + ib\Big]\right)^2 \times -4b^2$$

$$= -4b^2\Big[(\alpha - a)^2 + b^2\Big]^2$$

LHS > 0 RHS < 0 contradictions

Hence all roots are real.

2018 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	4.1	E8
2	1	3.2	E4
3	1	7.5	E4
4	1	1.5, 1.7	E6
5	1	5.1	E7
6	1	2.4	E3
7	1	2.5	E3
8	1	1.8	E6
9	1	2.1, 8.3	E3
10	1	1.4	E6

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	2.1	E3
11 (a) (ii)	2	2.1	E3
11 (b)	3	7.2	E4
11 (c)	4	4.1, 7.6	E8
11 (d) (i)	1	2.2	E3
11 (d) (ii)	1	2.2	E3
11 (d) (iii)	1	2.2	E3
11 (e)	2	8.1	E2
12 (a)	3	5.1	E7
12 (b) (i)	2	1.8	E6
12 (b) (ii)	2	1.8	E6
12 (c)	3	4.1	E8
12 (d) (i)	1	1.3	E6
12 (d) (ii)	2	1.6	E6
12 (d) (iii)	2	1.2	E6
13 (a)	3	5.1	E7
13 (b) (i)	2	2.2	E3
13 (b) (ii)	2	2.2	E3
13 (c)	3	6.3.1, 6.3.2	E5
13 (d) (i)	3	3.3	E3, E4
13 (d) (ii)	2	3.3	E3, E4
14 (a)	3	4.1	E8
14 (b)	3	6.2.3	E5

Question	Marks	Content	Syllabus outcomes
14 (c) (i)	3	4.1	E8
14 (c) (ii)	2	4.1	E8
14 (d) (i)	1	8	E2
14 (d) (ii)	1	8	E2
14 (d) (iii)	2	8	E2
15 (a) (i)	2	3.1	E4
15 (a) (ii)	3	3.1	E4
15 (b) (i)	2	2.4, 8	E3
15 (b) (ii)	3	8	E4
15 (c) (i)	1	8	E2, E9
15 (c) (ii)	2	8.3	E2, E9
15 (c) (iii)	2	8.3	E2, E9
16 (a)	3	8.2	E9
16 (b) (i)	3	8	E9
16 (b) (ii)	2	8	E9
16 (c) (i)	2	7.5	E4
16 (c) (ii)	3	7.5	E4
16 (c) (iii)	2	2.1, 7.5	E4