Mathematics Extension 2

HSC Marking Feedback 2020

Question 11

Part (a) (i)

Students should:

- convert complex numbers to modulus-argument form.

In better responses, students were able to:

- calculate the modulus of a complex number.

Areas for students to improve include:

- finding the modulus of a complex number.

Part (a) (ii)

Students should:

- carry out complex number arithmetic involving a conjugate.

In better responses, students were able to:

- find the conjugate of a complex number
- perform complex number multiplication.

Areas for students to improve include:

- being familiar with finding the conjugate of a complex number
- correctly multiplying two complex numbers.

Part (b)

Students should:

- know the techniques needed to apply integration by parts to a function involving $\ln x$.

In better responses, students were able to:

- choose the correct functions for integration by parts
- apply integration by parts and substitute the limits correctly.

Areas for students to improve include:

- using the correct choice for $u, \frac{du}{dx}, v$ and $\frac{dv}{dx}$ in integration by parts questions
- taking care when applying the integration by parts formula

- showing the substitution of limits clearly.

Part (c)

Students should:

- use $v\frac{dv}{dx}$ to find an expression for x and evaluate the constant using the given conditions.

In better responses, students were able to:

- convert acceleration to $v\frac{dv}{dx}$
- manipulate and simplify $v\frac{dv}{x} = v^2 + v$ to gain $\frac{dx}{dv} = \frac{1}{v+1}$
- integrate and gain an expression in $\ln(v+1)$
- interpret the initial conditions to evaluate the constant.

Areas for students to improve include:

- using the most appropriate expression for acceleration
- simplifying an algebraic expression before integration
- calculating the constant of integration using the given information.

Part (d)

Students should:

- manipulate vectors using vector arithmetic.

In better responses, students were able to:

- add and subtract vectors
- know the condition for two vectors to be perpendicular
- multiply two vectors involving pronumerals then solve the equation equal to zero.

Areas for students to improve include:

- taking care with basic number and algebra skills when using vector arithmetic, in particular, expanding brackets involving negatives
- reading questions carefully.

Part (e)

Students should:

- solve a complex quadratic equation by finding the square root of a complex number.

In better responses, students were able to:

- find the discriminant of quadratic equation
- find the square root of a complex number showing clear working of how it was obtained

  find the value of z in $a + ib$ form.

Areas for students to improve include

- knowing the most efficient technique to solve a quadratic equation
- knowing how to algebraically find the square root of a complex number

- using brackets correctly to avoid careless errors in obtaining z
- taking more care with basic arithmetic and algebraic calculations.

Question 12

Part (a) (i)

Students should:

- draw a force diagram and add their own annotations to their copy, for example, addition force components
- show working that actually gets the given result
- clearly show how they arrived at their calculation.

In better responses, students were able to:

- write an equation in which the sum of the vertical forces is equal to zero
- ensure that their calculation was the same as the given result
- resolve the vertical and horizontal forces separately.

Areas for students to improve include:

- not getting confused with the $50g$ (thinking 50 grams) when g was given in the question to be 10 m/s^2
- writing down relevant calculations.

Part (a) (ii)

Students should:

- clearly show how they arrived at their calculation
- understand the difference between vertical and horizontal forces and resolving a force.

In better responses, students were able to:

- write an equation in which the sum of the horizontal forces is equal to F_{net}.

Areas for students to improve include:

- writing with clarity.

Part (a) (iii)

Students should:

- know that $F_{net} = ma$ and that the mass was given and the acceleration was from part (ii)
- recognise that the motion is only in the horizontal direction.

In better responses, students were able to:

- write an integration statement
- understand that when $t = 3$, this implies to use $\frac{dv}{dt}$ for acceleration
- find the acceleration and integrate to find v.

Areas for students to improve include:

- understanding that when the box was moving there was no vertical velocity

- understanding there is no diagonal velocity
- knowing when to use $\ddot{x} = \frac{dv}{dt}$ and/or $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
- knowing $F_{net} = ma$ and so $a = \frac{F}{m}$
- knowing when to include resistive motion.

Part (b) (i)

Students should:

- derive the equations from the acceleration vector or using components
- write vectors clearly if using this technique
- realise that for three marks, significant working must be shown.

In better responses, students were able to:

- justify all constants with reference to $t = 0$
- use a diagram to assist them find initial velocity components
- perform vector calculations so that both components were solved simultaneously
- use definite integrals as a means of defining constants.

Areas for students to improve include:

- using vector notation properly, if they choose to go that way
- improving vector notation and understanding, that if used correctly, was more succinct and clearer
- refraining from simply quoting $\dot{x} = u\ \cos\theta$ with no justification or making errors in notation such as $t = 0, \dot{x} = 0, \therefore \dot{x} = u\cos\theta$
- refraining from the use of rote learnt responses and leaving out critical details
- finding velocity and the various constants rather than writing acceleration and displacement components/vectors which were given in the question
- knowing what is important to include in a ‘show that’ question in order to score the allocated marks.

Part (b) (ii)

Students should:

- write clearly – often u and x, g and 5 looked the same
- keep their working logical, without the need for multiple arrows to indicate where to find working.

In better responses, students were able to:

- substitute t into the equation for y and simplify the resulting expression with clarity and precision
- factor out the common factor, $-\frac{gx^2}{2u^2}$ very clearly, without losing squares on the u or x.

Areas for students to improve include:

- taking care with details, ensuring that squares do not go missing, negative signs appear as required, and pronumerals are currently included or omitted
- improving handwriting for ease of reading

- showing sufficient steps to justify progression
- making changes to working clear.

Part (b) (iii)

Students should:

- use clear mathematical arguments to justify their answer
- find the discriminant ($\Delta > 0$) when looking for distinct solutions.

In better responses, students were able to:

- use the simplest quadratic in $\tan\theta$ to find the discriminant
- simplify the discriminant into a form which made it easier to apply the given condition $u^2 > gR$
- construct a logically ordered proof that showed 'If $u^2 > gR$ then $\Delta > 0$'.

Areas for students to improve include:

- taking care to be precise with their notation. It was common to see $\Delta \geq 0$ or Δ is real
- reducing algebraic errors
- realising that substituting $u^2 > gR$ into the quadratic does not result in the quadratic expression being positive
- writing larger to avoid error in powers and fractions
- reading the clues in the question: this part required only the discriminant and so time and effort spent on simplifying the quadratic created opportunities for more errors
- keeping powers in fractions legible resulting in less omissions or unintended sign changes.

Question 13

Part (a)

Students should:

- determine the equation for the displacement of a particle undergoing simple harmonic motion.

In better responses, students were able to:

- use the equation found on the Reference Sheet and replace each of the variables to suit this question
- interpret the information in the question correctly
- use a diagram to assist in finding all the missing parts to the equation.

Areas for students to improve include:

- knowing what is on the Reference Sheet
- knowing that simple harmonic motion questions can be asked in Mathematics Extension 2
- knowing the difference between when to use the $x = a\cos nt + x_0$ or $x = a\cos(nt + \alpha) + x_0$ forms and when to use the sine equations, what the phase change means, how to find the amplitude, and how the period gives the constant, n.

Part (b)

Students should:

- understand and use the vector equation $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\, \underset{\sim}{b}$
- find the point of intersection of two lines.

In better responses, students were able to:

- equate the components in order to then find the point of intersection.

Areas for students to improve include:

- remembering to find the point of intersection once the variables are found
- checking the solution to make sure the same point of intersection can be found using either one of the variables.

Part (c) (i)

Students should:

- be able to prove the arithmetic mean is greater than the geometric mean.

In better responses, students were able to:

- use the diagram and show the connection between the diagram and the inequality
- use another method to prove the result.

Areas for students to improve include:

- learning a standard method to prove this result.

Part (c) (ii)

Students should:

- use the result in part (i) to prove an additional result through substitution or use another method.

In better responses, students were able to:

- substitute into the result in part (i) to prove the additional inequality
- use another method to prove the result.

Areas for students to improve include:

- understanding a variety of methods that can be used to prove inequalities. These could be substitution into a known result, using $LHS - RHS$ approach and proving positive, using proof by contradiction, or starting with a true positive inequality and building an argument from there.

Part (d) (i)

Students should:

- be able to prove a known relationship in complex numbers.

In better responses, students were able to:

- show the relationship between the complex number and its conjugate.

Areas for students to improve include:

- showing the result by writing the complex number in modulus-argument form.

Part (d) (ii)

Students should:

- use binomial expansion and the result in part (i) to show the desired relationship.

In better responses, students were able to:

- expand carefully and then use the result in the part (i) to show the desired result.

Areas for students to improve include:

- being careful with the use of the result from part (i) to ensure that the final answer was correct
- being careful not to make simple errors such as stating that $2^4 = 8$.

Part (d) (iii)

Students should:

- integrate correctly.

In better responses, students were able to:

- use the result in part (ii) and then integrate correctly.

Areas for students to improve include:

- making use of formulae given in examinations, such as the Reference Sheet results
- being aware that this question could have been attempted even if the previous part was not attempted. Students should look out for the types of questions where a previous result can be used.

Question 14

Part (a) (i)

Students should:

- be able to interpret the expression $z_2 = e^{\frac{i\pi}{3}} z_1$ geometrically and link it to relevant properties of an equilateral triangles.

In better responses, students were able to:

- clearly state that z_2 is the rotation of z_1 by $\frac{\pi}{3}$ with no change of length, so that the included angle is $\frac{\pi}{3}$ and the moduli are equal
- state that the triangle is isosceles with an apex angle of $\frac{\pi}{3}$ so the triangle is equilateral.

Areas for students to improve include:

- understanding the geometrical significance of a complex number in polar form without converting to other form
- giving a clear explanation of why moduli are equal and angle is $\frac{\pi}{3}$
- practising questions where they need to explain using a mixture of words, diagrams and mathematical language
- knowing the most appropriate form of the complex number to use in particular situations
- knowing the properties of an equilateral triangle.

Part (a) (ii)

Students should:

- be able to prove a relationship involving z_1 using result from part (i).

In better responses, students were able to:

- cube both sides of $z_2 = e^{\frac{i\pi}{3}} z_1$, rearrange and factorise to gain the result
- take out a factor of z_1^2 from LHS, convert $1 + e^{\frac{i2\pi}{3}}$ into mod-arg form then convert back to $e^{\frac{i\pi}{3}}$, and rearrange to form the RHS
- take out a factor of $z_1^2 e^{\frac{i\pi}{3}}$ from the LHS then use conjugate results.

Areas for students to improve include:

- choosing which complex number form is most appropriate depending on the type of calculation to be performed
- making sure all necessary steps are included in a proof
- understanding that z_1 and z_2 are complex not real numbers so using the cosine rule is invalid.

Part (b)

Students should:

- take the reciprocal of $\frac{dv}{dt}$ and integrate, either finding c or with limits then substitute to solve.

In better responses, students were able to:

- substitute $k = 0.01$ at the start and rearrange the equation of motion into a simpler form such as $\frac{dv}{dt} = \frac{10000 - v^2}{1000}$
- invert the fraction and form the partial fractions
- integrate the partial fractions including evaluating c then rearrange to make v the subject and substitute given conditions

Areas for students to improve include:

- recognising appropriate and most efficient integration techniques to use for $\int \frac{dx}{1 - a^2x^2}$
- integrating expressions involving a variable in the denominator with an unknown constant coefficient

- recognising when to use partial fractions in applied questions, rather than standalone integration questions
- avoiding the use of memorised formulas for integrating partial fractions
- taking care with the primitive of $\int \frac{1}{1-kv} dv$ to not forget the $-\frac{1}{k}$ factor in front of $\ln|1-kv|$
- thinking about whether the answer they gain is reasonable.

Part (c)

Students should:

- understand the process of mathematical induction and manipulate the LHS to match RHS involving an inequality.

In better responses, students were able to:

- prove the base case for $n = 2$
- clearly show the inductive step using the correct assumption for $n = k + 1$ case
- work from the true result (LHS) to gain the result to be proved (RHS) with clear working and appropriate use of $<$ and $=$ signs.

Areas for students to improve include:

- practising induction where $n = 1$ is not the base case
- taking care with algebra manipulation involving algebraic fractions
- being able to write clear and logical reasoning for an inequality proof
- understanding when an inequality rather than an equality sign is required

Part (d)

Students should:

- be able to complete a proof by contradiction that involves irrational numbers.

In better responses, students were able to:

- let $\log_n(n+1) = \frac{p}{q}$ as the contradiction
- remove $\log_n$ to gain $n^p = (n+1)^q$
- clearly state that if n was even then n^p was even, but that $n+1$ must be odd and so $(n+1)^q$ must be odd or vice versa which is a contradiction
- take out a factor of n on one side and on the other do the same by expanding $(n+1)^q$ and factorising to leave a remainder of 1 to prove the contradiction.

Areas for students to improve include:

- taking care when choosing variables to avoid using the same pronumeral for two different variables eg $\log_n(n+1) = \frac{m}{n}$
- working with logarithms being equal to fractions
- working with logarithms with bases other than e or 10
- creating clear and succinct explanations of why a statement is a contradiction.

Question 15

Part (a) (i)

Students should:

- know the correct expansions for $(A+B)^3$
- know how to set out a basic proof.

In better responses, students were able to:

- utilise the factorisation of $k^3+1=(k+1)(k^2-k+1)$ requiring a simple substitution.

Areas for students to improve include:

- being careful with factorisation, particularly towards the end of an algebraic expression – quite often the result $3(9M^3-9M^2+M)$ was obtained.

Part (a) (ii)

Students should:

- understand a broad range of techniques for proving results, including direct methods and the indirect methods which use contrapositive, converse, contradiction and counter-example.

In better responses, students were able to:

- write the contrapositive clearly and concisely
- understand truth tables and implications.

Areas for students to improve include:

- knowing the different methods of proof.

Part (a) (iii)

Students should:

- understand a broad range of techniques for proving results, including direct methods and the indirect methods which use contrapositive, converse, contradiction and counter-example
- learn how to use the language and write the notation of proof correctly.

In better responses, students were able to:

- use algebraic techniques effectively: $k^3+1=3M \rightarrow 3M-k^3=1$ or $k+1=k-k^3+3M \rightarrow k(1-k)(1+k)+3M$ and used the fact that one of three consecutive integers is a multiple of three
- use the contrapositive and identify cases to test where $k+1 \neq 3M$.

Areas for students to improve include:

- understand a broad range of techniques for proving results, including direct methods and the indirect methods which use contrapositive, converse, contradiction and counter-example
- writing concisely and knowing when to finish their solution.

Part (b) (i)

Students should:

- understand vector notation, knowing the difference between a scalar and a vector
- know how to 'show' and 'prove' without starting and finishing with the same result.

In better responses, students were able to:

- use similar triangles theory to provide reasons for the proof and show parts
- use strong vector theory associated with the point of intersection of two vectors
- use internal division theory effectively.

Areas for students to improve include:

- learning how to use the information given in a question to work towards a mathematical solution
- using proper vector notations, including the difference between $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$
- understanding vector equations of lines to solve problems.

Part (b) (ii)

Students should:

- use the vector content taught in the course
- know how to 'show' and 'prove' without starting and finishing with the same result
- understand vector notation, knowing the difference between a scalar and a vector.

In better responses, students were able to:

- write solutions very clearly and logically, with explanations where required
- make connections between the parts of the question.

Areas for students to improve include:

- realising that a question of one mark should not take a page of vector notation
- understanding that $\frac{m}{m+n}$ is a ratio which indicates the proportion of the vector, eg $\frac{m}{m+n}\overrightarrow{AB}$.

Part (b) (iii)

Students should:

- draw enlarged diagrams to help facilitate and visualise their working of the problem
- annotate diagrams with labels for all vectors that might be useful
- extract triangles that may be proven to be similar and label correctly.

In better responses, students were able to:

- use equations of lines using vector notation and solve simultaneously
- use similar triangles to establish ratios and deduce the relative lengths of other vectors.

Areas for students to improve include:

- avoiding substitution into previous answers without further explanation
- making sure that vector notations are correct.

Part (b) (iv)

Students should:

- use previous parts, or otherwise, to prove the result
- make the proof clear and the connections obvious.

In better responses, students were able to:

- provide clear reasoning for the result to be proven
- use the previous results in parts (ii) and (iii) with clear explanation and connections.

Areas for students to improve include:

- ensuring that previous results are applied accurately, especially with respect to the order of ratios
- understanding that 'or otherwise' may mean there are other easier ways, they just have not been thought of at this time
- using the answers provided can guide your direction in solving a problem.

Question 16

Part (a) (i)

Students should:

- draw free body diagrams
- realise that the resultant force is not an actual force acting on an object but the 'result' of all of the forces acting upon an object
- choose the positive direction to be the direction the object is travelling
- write an expression for the net force on the whole system
- write the forces (including T) acting on both masses separately, then add them together.

In better responses, students were able to:

- handle the resolution of forces when more than one object (mass) is involved
- write the net force on the system in one expression
- calculate the overall mass as being $6m$.

Areas for students to improve include:

- using the linking force, in this case tension, to solve problems involving more than one object
- calculating the overall mass of the system to be $6m$ and not 'averaging' the masses.

Part (a) (ii)

Students should:

- use the techniques of differential equations as seen in Mathematics Extension 1 to solve dynamics problems
- remember the constant when using indefinite integrals
- write the expression for $\frac{dv}{dt}$ from part (i).

In better responses, students were able to:

- use definite integrals and separation of variables to efficiently solve the problem with minimal working
- obtain an expression for t in terms of v, or v in terms of t and then substitute $t = \frac{3m}{k} \ln 2$ to the derived expression for v
- execute the appropriate integration technique successfully and simplify the expression using logarithm laws.

Areas for students to improve include:

- using general integration techniques
- demonstrating knowledge of standard integrals
- using the log laws correctly
- remembering to add an arbitrary constant $(+\, c)$ in the primitive.

Part (b) (i)

Students should:

- recognise the need to use integration by parts to solve this recurrence relation
- know techniques to split an expression into the parts that are required
- explicitly state all four parts involved in an integration by parts
- avoid doing too many calculations in one step of a proof: whilst it may seem obvious to the writer it may not be to the reader of the proof.

In better responses, students were able to:

- explain the logic behind their proof when it may not be immediately obvious what they had done moving from one line to another in their proof
- indicate clearly the values for u, u', v and v'.

Areas for students to improve include:

- focusing on the setting out of solutions using integration by parts
- using the correct choices for u and $\frac{dv}{dx}$ when performing integration by parts
- crossing out and rewriting a correct solution, rather than trying to change a pronumeral or symbol by overwriting the incorrect part
- writing the given integral as $\int \sin^{2n}(2\theta)\sin(2\theta)d\theta$ or $\int \sin^{2n-1}(2\theta)\sin^2(2\theta)d\theta$
- using the chain rule to differentiate $\sin^{2n}(2\theta)$ correctly.

Part (b) (ii)

Students should:

- telescope their reduction formula including at least three terms at the beginning of the sequence, the second last term and find the last iterant of the formula I_0
- use the recurrence relation to write a full expansion of I_n

In better responses, students were able to:

- find I_0 correctly
- show clearly the final step to get the denominator $(2n + 1)$.

Areas for students to improve include:

- understanding that deduction and induction are two different types of proof
- explaining each step
- calculating the value for I_0.

Part (b) (iii)

Students should:

- look at the previous parts of the question and attempt to find a way of using them, particularly when the question instructs them to do so, eg compare the expression to be shown with expressions previously found and look for what is different
- use the substitution $x = \sin^2\theta$ or $x = \cos^2\theta$.

In better responses, students were able to:

- notice the connection between this question and previous parts of the question
- recognise the need for an appropriate substitution and take out a factor of $\frac{1}{2^{2n}}$.

Areas for students to improve include:

- clearly showing any substitutions being used, including any limits of integration
- looking for a suitable trigonometric substitution.

Part (b) (iv)

Students should:

- understand that over a specified domain the maximum value of a function is different to the maximum value of the area under the curve of the function
- justify the statements made in a proof
- realise that there are other forms of proof than 'by induction'
- choose one of the many possible arguments to prove the result
- prove or justify their steps, not just state them.

In better responses, students were able to:

- explain why $I_n \leq 1$, and not simply state it
- prove each important step
- clearly explain that a product of fractions and 1 is less than 1, or that powers of numbers between 0 and 1 get smaller.
- use diagrams and/or words to explain each step
- state I_0 from part (ii)

Areas for students to improve include:

- solving inequalities using graphs
- considering the values of the integrals, not just the function.