Mathematics Extension 2

HSC Marking Feedback 2021

Question 11

Part (a)

Students should:

- apply index laws to multiply complex numbers using Euler form
- add the arguments when multiplying complex numbers.

In better responses, students were able to:

- immediately identify the complex form required and add the arguments when multiplying complex numbers.

Areas for students to improve include:

- using efficient techniques for questions with small mark values in an examination.

Part (b)

Students should:

- understand the use of sigma notation in the Mathematics Extension 2 course.

In better responses, students were able to:

- immediately write down the integer powers of i.

Areas for students to improve include:

- writing $i = \sqrt{-1}$ and manipulating it with powers.

Part (c)

Students should:

- know the syllabus formulae well, in particular the formulae around vectors
- use the Reference Sheet.

In better responses, students were able to:

- substitute the correct values into the required formula $\cos\theta = \frac{a \cdot b}{|a||b|}$ and move quickly to the correct solution.

Areas for students to improve include:

- working with simple arithmetic

- rounding off where instructed.

Part (d) (i)

Students should:

- apply a range of efficient strategies when finding square roots of complex numbers.

In better responses, students were able to:

- concisely find the two square roots of $-i$ using geometry or algebraic methods
- manipulate complex equations like $z^2 = -i$ in an efficient and effective manner.

Areas for students to improve include:

- using the geometry of complex numbers to quickly assess responses
- having versatility across different approaches instead of applying a rehearsed approach to all questions.

Part (d) (ii)

Students should:

- apply a range of efficient strategies when solving quadratic equations.

In better responses, students were able to:

- complete the square quickly and accurately to make z the subject before using part (i).

Areas for students to improve include:

- connecting parts (i) and (ii) of the questions.

Part (e)

Students should:

- perform division of complex numbers in Cartesian form.

In better responses, students were able to:

- rationalise the denominator correctly and collect like terms.

Areas for students to improve include:

- avoiding simple arithmetic errors
- using $i^2 = -1$.

Part (f)

Students should:

- manipulate partial fractions across a range of forms.

In better responses, students were able to:

- use a combination of techniques involving identities and in particular, substitute appropriate values to determine the unknown coefficients.

Areas for students to improve include:

- solving simultaneous equations with three variables.

Question 12

Part (a)

Students should:

- decompose an integral with quadratic denominator into components.

In better responses, students were able to:

- complete the square and separate the integrand into integrals that lead to arctan and logarithmic functions
- recognise the degree of the numerator was one less than the degree of the denominator.

Areas for students to improve include:

- recognising the integrals of the form $\int \frac{P(x)}{Q(x)}\,dx$ that can be decomposed
- completing the square accurately to facilitate integration.

Part (b) (i)

Students should:

- know the definition of a converse.

In better responses, students were able to:

- write the converse: 'If n is even, then n^2 is even'.

Areas for students to improve include:

- know the definitions of converse, contrapositive, counter-example and contradiction.

Part (b) (ii)

Students should:

- know how to prove simple numerical statements that involve odd and even numbers
- use algebraic approaches to prove number results.

In better responses, students were able to:

- succinctly set up and show that 'If n was even, then n^2 was even'
- establish an even number, $n = 2k$, and show $n^2 = 2(2k^2)$.

Areas for students to improve include:

- improving clarity of arguments when proving results
- avoiding careless algebraic errors, such as $4k^2 = 2(2k)$
- avoiding simple arithmetic and algebraic errors.

Part (c)

Students should:

- know the implications of the statement that vectors are perpendicular and intersect.

In better responses, students were able to:

- write the direction vectors and then find their scalar product, which provided $p = 2$

- equate the two vectors and solve simultaneously for μ and λ, which then allowed them to find q using p as well.

Areas for students to improve include:

- practising the skill of determining direction vectors
- using the scalar product effectively
- solving simultaneous equations without error.

Part (d)

Students should:

- use mathematical induction to prove inequalities.

In better responses, students were able to:

- clearly set up the induction process and express their logic
- clearly show where they used $P(k)$: $\sqrt{k!} > 2^k$
- clearly justify why $\sqrt{k+1} > 2$, for $k \geq 9$.

Areas for students to improve include:

- setting out a proof by mathematical induction with a simple logical process
- showing why inequalities are true for specified values of k.

Part (e) (i)

Students should:

- use vectors to show geometric results.

In better responses, students were able to:

- use the information provided to show the result required
- explain their use of vectors to arrive at the desired result.

Areas for students to improve include:

- defining the vectors used if they are not defined in the question. For example, Let $\overrightarrow{AB} = \underset{\sim}{a}$
- using vector notation correctly.

Part (e) (ii)

Students should:

- show the result given using the suggested expressions.

In better responses, students were able to:

- define the vectors used in showing the result given
- show each step of the process and explain where the given results were used.

Areas for students to improve include:

- showing the result using vectors, not describing the situation and jumping to given conclusions.

Part (e) (iii)

Students should:

- manipulate vector expressions
- determine the scalar quantity that one vector is of another vector, knowing the difference between positive and negative vectors.

In better responses, students were able to:

- use the given results in the previous part to successfully determine the value of λ.

Areas for students to improve include:

- utilising the continuity between parts of a question
- understanding that $\overrightarrow{HS} = -\overrightarrow{SH}$ and other such relationships may be used to show given results.

Question 13

Part (a)

Students should:

- apply De Moivre's theorem to find the arguments and approximate moduli for the fourth roots of a complex number and correctly place these on an Argand diagram
- know the geometrical relationship between the n^{th} roots of a complex number. That is, the roots have the same length, and the roots are equally spaced about the Argand diagram.

In better responses, students were able to:

- generate the correct arguments for the four roots
- determine that the correct modulus of each root lay in the interval $1 \le \left|z^{\frac{1}{4}}\right| \le |a + ib|$
- calculate the argument of the first fourth root and use this to determine the argument of the following roots since the roots are equally spaced about the Argand diagram
- calculate the argument of each root using De Moivre's theorem.

Areas for students to improve include:

- understanding a complex number in polar form and applying De Moivre's theorem to find the arguments and moduli of the roots of complex numbers.
- using the Reference Sheet to assist with clarity and precision
- finding the n^{th} root of a complex number algebraically
- transferring their knowledge from solving equations with numerical values to this generalised form of $a + ib$
- representing the n^{th}roots of a complex number geometrically
- improving knowledge of De Moivre's theorem with the understanding of both transformations to the argument and modulus separately. That is, if $z = r\operatorname{cis}\theta$, $\arg\left(z^{\frac{1}{4}}\right) = \frac{1}{4}\theta$ and $\left|z^{\frac{1}{4}}\right| = r^{\frac{1}{4}}$.

Part (b)

Students should:

- provide an appropriate substitution, apply this substitution with correct limits and then integrate the new expression correctly to determine the definite integral
- select an appropriate substitution
- have thorough knowledge of all steps in integration by substitution, from changing the limits to calculating $\frac{dx}{du}$
- manipulate algebra and substitute correctly.

In better responses, students were able to:

- use $u = x^2 - 9$ or $u^2 = x^2 - 9$ as their substitution and correctly determined the definite integral
- use $x = 3\sec\theta$ successfully, although this method proved longer with some difficult limits
- use other approaches such as integration by parts, algebraic manipulation or use $u = x^2$ as the substitution, although this proved to be the most challenging.

Areas for students to improve include:

- avoiding simple errors when converting the integral in terms of x to an integral in term of u
- improving the setting out of solutions and showing all working. Splitting the x^3 term into x^2 and x would have avoided errors.
- thinking of a suitable substitution before persisting with inappropriate substitutions. For example, some students tried to use $u = 3\sin\theta$ or $u = x^3$
- recognising which trigonometric substitutions are suited to which algebraic expressions
- finding the correct limits for u. This was particularly an issue when using the substitution $x = 3\sec\theta$
- avoiding rounding their limits instead of providing exact values
- avoiding unnecessary steps such as converting the expression in terms of u back into an expression in terms of x
- avoiding simple errors such as dropping the $\frac{1}{2}$ in front of the integral sign
- avoiding simple errors when integrating, particularly when fractional powers are involved.

Part (c) (i)

Students should:

- recognise the need to use integration by parts to solve the recurrence relation
- explicitly state all four parts involved in integration by parts.

In better responses, students were able to:

- use the recurrence relation to clearly write u, du, v and dv and apply this correctly with $u = (\ln x)^n$ and $dv = 1$
- successfully use the substitution method of $u = \ln x$ and $x = e^u$.

Areas for students to improve include:

- using the correct choices for u and dv when performing integration by parts and clearly show the proof step by step
- clearly setting out solutions when using integration by parts
- remembering the exponent on $u = (\ln x)^n$ when manipulating algebra
- remembering to multiply by the derivative of $\ln x$ when differentiating this expression.

Part (c) (ii)

Students should:

- apply their knowledge of volumes of solids of revolution from Mathematics Extension 1 to recognise that a difference of volumes is required for this question
- use part (i) and the recurrence formula to help solve difficult integrals efficiently
- follow the prompt within the question
- recognise the different types of shapes that are generated when the specified regions are rotated around the x-axis and find their volumes using integration skills and the given volume formula where required.

In better responses, students were able to:

- write correct expressions for V_A and V_B
- clearly state the volumes required and express these as a difference of volumes
- use volumes of cylinders and spheres to efficiently answer the question
- notice the connection between this question and part (i) and use the recurrence relationship correctly
- generate an expression for I_0, or find I_1 and then use integration by parts
- use the 'or otherwise' approach by integrating $(\ln x)^2$ by parts.

Areas for students to improve include:

- setting out recurrence formula clearly
- setting work out clearly and in a logical order and looking to use part (i) when prompted
- generating an expression for I_0 , rather than finishing at I_1 which simplifies more easily
- gaining a deeper understanding of volumes of solids of revolution, by visualising the volumes of the cylinders and hemispheres that are generated and the limits that are used for these regions
- understanding that the formula $V = \pi \int_a^b {y_1}^2 - {y_2}^2 dy \neq \pi \int_a^b (y_1 - y_2)^2 dy$
- understanding that the formula $V = \pi \int_a^b {y_1}^2 - {y_2}^2 dy = \pi \int_a^b {y_1}^2 dy - \pi \int_a^b {y_2}^2 dy$
- identifying the correct limits of integration
- visualising and drawing solids of revolution and connecting this to the formulae above.

Question 14

Part (a)

Students should:

- identify the style of definite integral question that is appropriate for a t-substitution.

In better responses, students were able to:

- fully simplify fractions before proceeding to partial fractions
- clearly show the substitution of dx, change the limits of integration and develop and simplify the expression involving $\cos x$ in terms of t
- take algebraic steps to allow their path to the solution to involve integers.

Areas for students to improve include:

- simplifying algebraic fractions before proceeding, keeping their next steps as straightforward as possible
- factorising using the difference of two squares accurately
- knowing the expression for $\frac{dx}{dt}$ when using the t results and have a deeper understanding of the process involved with the t-substitution approach.

Part (b) (i)

Students should:

- decompose mg parallel to the slope and then add the resistive force to find resultant force
- find the components of vectors.

In better responses, students were able to:

- construct an accurate diagram representing the problem
- know where and how to construct the triangle to decompose mg into appropriate components.

Areas for students to improve include:

- translating a word problem into a force diagram
- knowing how to decompose mg parallel to the plane
- being familiar with mechanical models of systems involving gravity forces.

Part (b) (ii)

Students should:

- set acceleration to zero due to constant velocity
- be able to solve a quadratic equation
- identify which direction is positive in the question
- correctly round to one decimal place.

In better responses, students were able to:

- keep the quadratic equation as simple as possible
- use the quadratic formula correctly to solve a quadratic equation

- correctly apply the quadratic equation, identifying which of the two solutions were correct
- correctly answer the question to one decimal place as requested.

Areas for students to improve include:

- reading the question to respond to the appropriate number of decimal places
- solving a quadratic equation using an appropriate method
- understanding which direction is positive and which is negative and making an appropriate selection
- being familiar with mechanical models of systems involving gravity forces.

Part (c) (i)

Students should:

- use De Moivre's theorem correctly
- complete a binomial expansion
- equate real parts in an expansion and express their result in terms of $\cos\theta$ only.

In better responses, students were able to:

- state the binomial expansion and De Moivre's theorem separately
- write the most generic form of the binomial expansion and not work with it algebraically until equating real parts, thereby avoiding introducing algebraic errors into the expansion
- clearly write the binomial expansion with its numerous powers and operators correctly
- connect the real components of both expressions to successfully finalise the question.

Areas for students to improve include:

- being accurate with writing down the powers correctly
- taking care with basic number and algebraic skills when expanding a complex binomial expansion when it involves powers and negatives and powers of i.

Part (c) (ii)

Students should:

- correctly identify the link form part (i) to establish the connection between $\mathrm{Re}\left(e^{\frac{i\pi}{10}}\right)$ and $\cos\left(\frac{\pi}{10}\right)$.
- solve the resulting polynomial equation
- make decisions about the values of $\sqrt{\frac{5\pm\sqrt{5}}{8}}$ and choose the appropriate one, using trigonometric theory.

In better responses, students were able to:

- apply part (i) to form a quadratic equation that could be solved via factorising and the quadratic equation after using a substitution
- form a polynomial by solving $\cos 5\theta = 0$ and determining its distinct roots
- identify that $\cos\left(\frac{\pi}{10}\right)$ is not equal to zero and therefore a root of the quartic

- decide that the result is either $\cos\left(\frac{\pi}{10}\right)$ or $\cos\left(\frac{3\pi}{10}\right)$
- compare the value of $\cos\left(\frac{\pi}{10}\right)$ to $\cos\left(\frac{3\pi}{10}\right)$ using either a graph and/or stating it is a decreasing function in the appropriate domain and decide that it is the greater of the two.

Areas for students to improve include:

- finding the roots of the polynomial from part (i) that results in 0 using $\cos 5\theta = 0$
- comparing $\cos\left(\frac{3\pi}{10}\right)$ to a root and not an arbitrary value
- correctly justifying that $\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$ through the use of a correct comparison or demonstrating the cosine function is decreasing over the domain of $\left(0, \frac{\pi}{2}\right)$.

Question 15

Part (a) (i)

Students should:

- use the relationship between the arithmetic mean and geometric mean for two terms more than once to achieve a desired result
- prove the statement given for all non-negative values of a, b and c.

In better responses, students were able to:

- know what substitutions for x and y would be most beneficial, for example $x = \frac{a^2+b^2}{2}$ and $y = c$ or similar.

Areas for students to improve include:

- showing all steps in the proof, using the given inequality as required
- using the relationship between the arithmetic mean and geometric mean in its many forms.

Part (a) (ii)

Students should:

- use the suggested result, or otherwise
- know the extended versions of the relationship between the arithmetic mean and geometric mean.

In better responses, students were able to:

- use the extended version; $\sqrt[6]{abcdef} \leq \frac{a+b+c+d+e+f}{6}$
- use the result from part (i) recognising that $\sqrt{abc} \leq \frac{a^2+c^2+2b}{4}$ and $\sqrt{abc} \leq \frac{b^2+c^2+2a}{4}$ are also true.

Areas for students to improve include:

- using part (i) to assist in understanding and solving this part
- knowing the different forms of the relationship between the arithmetic mean and geometric mean.

Part (b) (i)

Students should:

- use a general expression for an odd number, such as $n = 2k - 1$ in this case
- prove the statement for all odd values of n.

In better responses, students were able to:

- substitute $n = 2k - 1$ or $n = 2k + 1$ into t_n, students using $n = 2k - 1$ were more successful as it simplified easily. Those substituting $n = 2k + 1$ had an extra step to show that $t_{2k+1} = 2k^2 + 3k + 1 = 2(k + 1)^2 - (k + 1) = h_{k+1}$.

Areas *for students to improve include:*

- understanding that proving for all odd values of n does not mean simply trying a few odd numbers, like the ones given in the question
- reducing simple algebraic errors by writing every step of the proof and not skipping steps.

Part (b) (ii)

Students should:

- prove that the expression for t_n where n is even cannot equal the expression for a hexagonal number
- prove that there cannot be any value of t_n for even n which is a hexagonal number.

In better responses, students were able to:

- prove that $t_{2k} = h_n$ was only possible if one solution was negative and the other was a positive fraction
- show from part (i) that the odd indexed triangular numbers covered all hexagonal numbers and so there are no hexagonal numbers that map to the even indexed triangular numbers.

Areas for students to improve include:

- understanding how to disprove statements and that providing counter-examples does not disprove a statement for all even n
- showing every step to avoid careless and costly arithmetic and algebraic errors.

Part (c) (i)

Students should:

- create a correct equation of motion making $\frac{dv}{dt}$ the subject and integrating until the time when $v = 0$.

In better responses, students were able to:

- use a definite integral approach, such as $\int_0^T dt = -\int_u^0 \frac{dv}{g+kv^2} = \int_0^u \frac{dv}{1+\frac{k}{g}v^2}$, which was generally quicker and more successful
- reverse the limits of integration to eliminate the negative from the equation
- use the limit method or substituted $v = 0$ to find the given equation for t.

Areas for students to improve include:

- drawing a force diagram and determining a correct equation of motion
- knowing what is required as a result, t or x, which then determines the use of $\frac{dv}{dt}$ or $v\frac{dv}{dx}$.

Part (c) (ii)

Students should:

- create a correct equation of motion making $v\frac{dv}{dx}$ the subject and integrating until the maximum height when $v = 0$, is reached.

In better responses, students were able to:

- integrate using $v\frac{dv}{dx}$ using definite or indefinite integrals with definite integrals providing the solution quicker and easier
- draw a force diagram and indicate the positive direction to be used.

Areas for students to improve include:

- accurately applying the arithmetic of logarithms
- taking care with the algebra and not losing constants
- ensuring that the solution is logical, clear and legible.

Part (d)

Students should:

- prove that there is no value of $n \geq 2$ for which $2^n + 3^n = 5^n$.

In better responses, students were able to:

- state that $5^n = (2+3)^n$ then use a binomial expansion and noting the middle terms in the expansion are all positive, so $2^n + 3^n \neq 5^n$
- successfully show that $2^n + 3^n < 5^n$ using mathematical induction, and so $2^n + 3^n \neq 5^n$
- use contradiction, showing that $5^n - 3^n = (5-3)(5^{n-1} + 5^{n-2}3 + 5^{n-3}3^2 + \cdots + 3^{n-1})$ which gives a statement like $2^{n-1} = 5^{n-1} + 5^{n-2}3^1 + 5^{n-3}3^2 + \cdots + 3^{n-1}$, for $n \geq 2$.

Areas for students to improve include:

- understanding that mathematical induction cannot be used to disprove a statement, but only prove that a statement is true for all values
- using binomial expansions and remembering that there are coefficients $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}$ within the expansion.

Question 16

Part (a) (i)

Students should:

- operate with the magnitude of a vector, using appropriate notation
- utilise appropriate identities from the Vectors topic.

In better responses, students were able to:

- state and apply the triangle inequality as an identity and use it show the lower bound to the expression $|x| + |y| + |z|$ is 1.

Areas for students to improve include:

- understanding the difference between $\left|x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}\right|$ and $|x + y + z|$.

Part (a) (ii)

Students should:

- be clear with their intent in show questions that involve inequalities.

In better responses, students were able to:

- identify a correct relationship initially, typically $\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|} = \cos\theta$, leading to a correct inequality involving $|\underset{\sim}{a} \cdot \underset{\sim}{b}|$ and $|\underset{\sim}{a}|\,|\underset{\sim}{b}|$.

Areas for students to improve include:

- using absolute value or modulus symbols across vectors and scalars.

Part (a) (iii)

Students should:

- seek to find the intent behind a multi-part question.

In better responses, students were able to:

- carefully choose vectors from part (ii) to apply in this part, leading to the required upper bound for $|x| + |y| + |z|$ of $\sqrt{3}$.

Areas for students to improve include:

- understanding inequalities using vectors.

Part (b)

Students should:

- be able to answer questions involving links between multiple topics.

In better responses, students were able to:

- link the ideas of projectile motion, dot products, discriminant theory and trigonometry to find a range of angles
- justify their answers.

Areas for students to improve include:

- manipulating algebraic expansions
- applying discriminant theory to quadratic equations.

Part (c)

Students should:

- make geometric and algebraic connections across complex numbers.

In better responses, students were able to:

- realise a functional relationship, $y = x \tan x$, and use it to guide them to a correct region satisfying the required relationship in z.

Areas for students to improve include:

- using the Cartesian form when dealing with graphs involving complex numbers.