Mathematics Extension 2

HSC Marking Feedback 2022

Question 11

Part (a)

Students should:

- know to multiply the numerator and denominator by the conjugate of the denominator
- show all calculations.

In better responses, students were able to:

- expand the product of the numerators correctly showing all four terms
- show the denominator as the sum of the squares of the moduli
- simplify the complex number.

Areas for students to improve include:

- multiplying two negative multiples of i
- showing relevant mathematical reasoning and/or calculations.

Part (b)

Students should:

- recognise the integrand as the product of a compound trigonometric function and a multiple of the derivative of its inner function
- rearrange in reverse chain rule form or let u equal the inner function of the compound function or use the double angle results to find the primitive function in terms of x.

In better responses, students were able to:

- rewrite the integrand in reverse chain rule form
- find the primitive of the integrand efficiently.

Areas for students to improve include:

- understanding and using the reverse chain rule
- choosing the simplest suitable u substitution
- using the correct constant factor with the double angle results
- looking for the most efficient method to find a primitive function.

Part (c) (i)

Students should:

- find the modulus and argument of $-\sqrt{3}+i$ showing all calculations and/or a sketch of its position on the complex plane
- write the complex number in exponential form.

In better responses, students were able to:

- sketch the position of the complex number on the complex plane with an exact ratio triangle in the correct position
- use the exact ratio triangle or formula, with suitable working, to find the modulus and argument
- check that the argument matched the quadrant in which the complex number lies.

Areas for students to improve include:

- identifying the correct quadrant in which a complex number lies in order to evaluate the correct argument
- showing relevant mathematical reasoning and/or calculations.

Part (c) (ii)

Students should:

- use their answer from part (i)
- use De Moivre's theorem to find the modulus and argument of a complex number
- convert the complex number from exponential form to Cartesian form showing all calculations.

In better responses, students were able to:

- use De Moivre's theorem to find the modulus and argument of the complex number
- find the principal argument
- convert $e^{i\theta}$ to Cartesian form using exact triangles
- simplify the complex number in Cartesian form.

Areas for students to improve include:

- showing relevant mathematical reasoning and/or calculations.

Part (d)

Students should:

- find the 2 vectors whose tails are at B, find their dot product and moduli, and use the angle between 2 vectors formula to find the angle to the nearest degree
- find the 3 vectors forming the triangle, find their lengths and use the cosine rule to find the correct angle to the nearest degree.

In better responses, students were able to:

- use vector formulae correctly.

Areas for students to improve include:

- recognising the correct vectors to use to find the required angle
- using vector formulae

- understanding that the angle between two vectors can be acute or obtuse.

Part (e)

Students should:

- find the gradient of the line then use the point-gradient formula or find the equation of the line in vector form then solve simultaneous equations, to find the equation of the line in gradient-intercept form.

In better responses, students were able to:

- recognise the gradient of the line from the direction vector, then use the point-gradient formula.

Areas for students to improve include:

- understanding the equation of a two-dimensional line in both vector and Cartesian form
- choosing the most efficient method when using lines in vector form
- writing their answer in the form required.

Part (f)

Students should:

- find the primitive function of an integrand involving trigonometric functions using t-results.

In better responses, students were able to:

- substitute the appropriate t-results and determine dx
- simplify the integrand correctly
- find the primitive function correctly using absolute value signs and rewrite their answer in terms of x.

Areas for students to improve include:

- using t-results and simplifying the resulting integrands
- writing the final answer using the same variable as the original integrand
- Understanding when absolute value signs are required.

Question 12

Part (a)

Students should:

- prove the arithmetic mean is greater than or equal to the geometric mean for non-negative values of a and b.

In better responses, students were able to:

- use the result $(\sqrt{a}-\sqrt{b})^2 \geq 0$ or $(a-b)^2 \geq 0$ and replace a with $\sqrt{a}$ and b with $\sqrt{b}$
- use the result $(a-b)^2 \geq 0$ and complete the square
- compare the difference between the LHS and RHS to show $\frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}-\sqrt{b})^2}{2} \geq 0.$

Areas for students to improve include:

- clarity of justifications when proving results

- learning a standard method to prove this result.

Part (b)

Students should:

- integrate the acceleration function twice with respect to time, to obtain functions for velocity and displacement respectively
- know that maximum velocity occurs when acceleration is zero.

In better responses, students were able to:

- obtain a correct equation for x in terms of t, recognise maximum velocity occurred when $t = 2$, and substitute $t = 2$ into their equation to find the value of x.

Areas for students to improve include:

- clarity of their solutions
- using the velocity equation to find the displacement equation
- calculating constants of integration using given information
- setting acceleration equal to zero to find when the maximum velocity occurs.

Part (c) (i)

Students should:

- use Newtons second law $F = ma$ and the resistive equation given in the question
- use the equation $a = v\frac{dv}{dx}$ to determine an expression for $\frac{dv}{dx}$.

In better responses, students were able to:

- recognise the particle was moving horizontally
- recognise the resistive force was negative and given in terms of velocity
- simplify $F = ma$ by substituting the mass as 1 kg and then use the equation $a = v\frac{dv}{dx}$.

Areas for students to improve include:

- identifying the initial conditions, particularly the direction of travel
- using the equation $a = v\frac{dv}{dx}$ when acceleration is given in terms of velocity.

Part (c) (ii)

Students should:

- integrate $\frac{dv}{dx}$ to find the displacement in terms of velocity, using the given function from part (i).

In better responses, students were able to:

- integrate the expression and find the constants of integration correctly, keeping the equation in factored form
- use the laws of logarithms to simplify the answer.

Areas for students to improve include:

- applying integration rules and procedures

- using initial conditions to find the constant of integration.

Part (d)

Students should:

- decompose an integral with a linear and quadratic denominator into components
- integrate an expression accurately, using logarithmic laws to simplify the expression into the required form.

In better responses, students were able to:

- decompose the expression into partial fractions
- use integration techniques correctly and simplify the expression using logarithmic laws to express the final answer in the form $\frac{1}{2} ln\left(\frac{4+n^2}{8(n-1)^2}\right)$.

Areas for students to improve include:

- recognising the need for a remainder of $Bx + C$ when given a denominator of $4 + x^2$
- clear setting out of partial fractions to accurately find the required integral
- using a variety of logarithmic laws.

Part (e)

Students should:

- apply a variety of complex number properties to prove an expression is purely imaginary.

In better responses, students were able to:

- work in exponential form and factorise using a form of $e^{i\theta}$ as a common factor
- factorise by taking out a common factor of z
- apply De Moivre's theorem and double angle formulae to simplify the expression
- multiply the numerator and denominator by the conjugate of the denominator.

Areas for students to improve include:

- understanding and working with complex numbers in exponential form
- considering both the real and imaginary parts, before proving that $Re(z) = 0$
- applying De Moivre's theorem and working accurately in polar form
- applying appropriate double angle results accurately.

Question 13

Part (a)

Students should:

- know how to work with even (and odd) numbers in proof
- not equate prime with odd
- state either the contrapositive or contradiction correctly.

In better responses, students were able to:

- use the contrapositive
- factorise $(2^k)^2 - 1$ correctly
- recognise that the expression had 2 factors other than 1 and itself
- show the expression was divisible by 3 when n was even, either by factorising using the binomial expansion or with an inductive proof.

Areas for students to improve include:

- writing a proof statement, in this case the contrapositive or contradiction
- recognising when an inductive proof is best
- setting out solutions to proofs
- writing a concluding statement using the language of proof.

Part (b)

Students should:

- know the process of proof by mathematical induction
- know how to use an assumption in an inductive proof
- test for the base case.

In better responses, students were able to:

- be clear and logical in their presentation of the inductive proof
- apply the double angle results or use a similar approach
- establish that only the positive result was valid when taking the square root.

Areas for students to improve include:

- understanding that when taking a square root there are two possible results, and recognising which of the 2 cases to include
- using the double angle or sum to product results efficiently
- setting out a proof by mathematical induction.

Part (c) (i)

Students should:

- understand that there are 5 evenly spaced solutions around the unit circle
- establish that -1 is a solution which is positioned at $\theta = \pi$ around the circle.

In better responses, students were able to:

- draw a diagram
- use the exponential form $\left(z = re^{i\theta}\right)$ or $z = r[\cos\theta + i\sin\theta]$ to find the roots
- clearly state their solutions in the domain $[-\pi, \pi][-\pi, \pi]$
- indicate that the solutions were $\frac{2\pi}{5}$ apart.

Areas for students to improve include:

- ensuring complete solutions are provided, not just the arguments of the complex numbers

- using different notations when working with roots of complex numbers.

Part (c) (ii)

Students should:

- substitute into, and expand, algebraic expressions involving quadratics.

In better responses, students were able to:

- factorise $z^5 + 1 = 0$ and use the result to simplify the expression
- manipulate the algebraic expression obtained to show that it equalled zero.

Areas for students to improve include:

- understanding the difference between the solution to a polynomial and the argument for complex equations.

Part (c) (iii)

Students should:

- know how to factorise a quadratic
- understand and be able to show that $z + \frac{1}{z} = 2\cos\theta$.

In better responses, students were able to:

- use the quadratic formula to factorise $u^2 - u - 1 = 0$
- determine that $z + \frac{1}{z} = 2\cos\frac{3\pi}{5}$ and that this is a second quadrant angle
- equate $2\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{2}$ and simplify to give the required result
- justify which of the quadratic solutions is correct.

Areas for students to improve include:

- connecting parts (i) and (ii) of the question
- determining which quadrant and consequently which solution is required.

Question 14

Part (a) (i)

Students should:

- understand the conditions for vectors to be parallel, and hence non-parallel
- use correct mathematical language, for example equating components (not coefficients).

In better responses, students were able to:

- use logic, via contradiction or otherwise, to explain why $\lambda = \mu = 0$
- assume the vectors were non-parallel, then use $\mu \neq \lambda \neq 0$ to show why this leads to a contradiction
- explain that if 2 vectors are not parallel, then it is not possible to express one as a scalar multiple of the other

- focus on the geometric definition of parallel vectors and use $\lambda\vec{u} = -\mu\vec{v}$ to explain that the only way the vectors lead to the origin is if $\lambda = \mu = 0$
- use mathematical language in their solutions, including *scalar multiple, linear combination, equating components.*

Areas for students to improve include:

- understanding that non-parallel lines have different direction vectors
- distinguishing between direct and indirect approaches of proof (like proof by contradiction) when applying it to 'if – then' statements.

Part (a) (ii)

Students should:

- connect the information in part (i) to part (ii)
- manipulate the equations to show the parameters are equal.

In better responses, students were able to:

- group the components so the equation resembled part (i)
- use the result of part (i) to explain why $\lambda_1 - \lambda_2 = 0$ and $\mu_1 - \mu_2 = 0$.

Areas for students to improve include:

- identifying connections between parts of a question
- not drawing incorrect assumptions from previous parts, for example the vector sum of zero in part (i) does not mean the vector sum is zero in part (ii).

Part (a) (iii)

Students should:

- understand vector notation and geometry
- know a tetrahedron can be a regular or irregular shape
- use their knowledge of vectors to show the result.

In better responses, students were able to:

- identify and use the connections between parts (i), (ii) and (iii)
- work algebraically to simplify expressions accurately and equate components
- solve simultaneous equations to find the parameter $\left(\lambda = \frac{4}{7}\right)$
- develop a relationship between linear combination of vectors and use part (ii) to show the required result
- use $\overrightarrow{SK} = \frac{1}{4}\overrightarrow{SB} + \frac{1}{3}\overrightarrow{SC}$ to create relationships involving $\overrightarrow{SL}$ or $\overrightarrow{BL}$ in the presence of $\overrightarrow{SC}$, $\overrightarrow{SB}$ and $\overrightarrow{BC}$ and apply part (ii) to derive $\mu = \frac{4}{7}$.

Areas for students to improve include:

- drawing or copying diagrams and marking with the information given in the question
- identifying key vector relationships in a geometric setting and develop the necessary relationships
- understanding the necessity for a parameter to establish the required relationship

- representing and distinguishing between vectors in different forms such as vector form and equation form
- using vector notation, particularly writing the direction of vectors without error.

Part (a) (iv)

Students should:

- show $\overrightarrow{AP}$ lies on $\overrightarrow{AL}$ by using the information in the question and the result from part (iii).

In better responses, students were able to:

- use the result in part (iii) to make a connection with the point required
- recognise the relevance of the negative in the obtained parameter as meaning point P was not located between A and L but was still on the line
- manipulate the vector $\overrightarrow{BC}$ into $\overrightarrow{AP} = -6\overrightarrow{AB} - 8\overrightarrow{AC}$ by using $\overrightarrow{BL} = \frac{4}{7}\overrightarrow{BC}$ and realising $\overrightarrow{AL} = \overrightarrow{AB} + \overrightarrow{BL}$ and $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ by using substitution and the use of factorising.

Areas for students to improve include:

- drawing diagrams if referring to their own vectors or variables
- understanding the difference between an interval and a line/vector
- applying the information found in previous parts
- understanding the necessity for the use of a parameter to map out all the points on the line AL
- being mindful of negative signs when simplifying expressions.

Part (b) (i)

Students should:

- show all steps in their solution, especially for a 'show' question where the result is given.

In better responses, students were able to:

- identify that when $n = 0$, $J_0 = \int_0^1 e^{-x} dx$
- substitute and integrate accurately.

Areas for students to improve include:

- interpreting a general formula
- showing detail in solutions including substitution of limits
- taking care when substituting limits to avoid errors
- understanding and using index notation.

Part (b) (ii)

Students should:

- consider the effect on the integral of increasing the value of n
- recognise how to create an inequality, in this case consider what relationships can be compared
- integrate to show that the inequality holds.

In better responses, students were able to:

- recognise that $e^{-x} \leq 1$ and so $e^{-x}x^n \leq x^n$ in the interval [0,1]
- draw a graph to show their understanding of the question
- see the connection between the definite integral and area.

Areas for students to improve include:

- practising the use of arguments such as $\int_0^1 e^{-x}x^n dx \leq \int_0^1 x^n dx$ rather than relying on specific cases such as when $n = 0$ or when $n = 1$
- approaching the problem geometrically by drawing an appropriate graph
- understanding how inequalities in integration can be verified using area graphs.

Part (b) (iii)

Students should:

- recognise the need to use integration by parts
- write the formula for integration by parts before substituting
- show full working including substitution of limits.

In better responses, students were able to:

- identify their use of integration by parts, integrate correctly and show the substitution of limits leading to the result.

Areas for students to improve include:

- setting out solutions when using integration by parts
- operating with negative signs and simplifying with care
- converting expressions back into recursive form $\left(J_n = nJ_{n-1} - \frac{1}{e}\right)$ from integral form
- using subscripts.

Part (b) (iv)

Students should:

- understand the process of mathematical induction
- work efficiently and effectively with sigma notation and subscripts
- apply mathematical induction or use repeated substitution.

In better responses, students were able to:

- use sigma notation and factorise in order to arrive at the result
- develop a pattern
- write down accurate expressions for J_k and J_{k+1}
- understand how to manipulate the sigma notation to obtain the required expression
- correctly manipulate the summation operator to show the statement holds for $n = k + 1$ by using the condition in part (iii).

Areas for students to improve include:

- taking care when dealing with series expressions and ensuring that all terms and signs are accounted for when simplifying expressions
- taking care with subscripts and the use of brackets with sigma notation
- understanding the changes to the value on the top of the sigma notation, in this case k changing to $k+1$ when a new term is added.

Part (b) (v)

Students should:

- understand how limits work
- know that sigma notation limits are important
- use part (ii) to place a bound on J_n and evaluate the limit after suitable algebraic manipulations.

In better responses, students were able to:

- use the limit and sigma notations clearly and efficiently
- show full working
- manipulate the expression, then take the limit as $n \to \infty$ on both sides to arrive at the result
- rearrange the given expression to make the required subject e
- apply the information from part (ii)
- recognise $n! - \frac{n!}{e} \sum_{r=0}^{n} \frac{1}{r!} = 0$.

Areas for students to improve include:

- identifying connections between parts of a question
- understanding limits and using limit notation correctly
- manipulating algebraic expressions involving factorial notation
- operating with inequalities.

Question 15

Part (a) (i)

Students should:

- decompose forces into horizontal and vertical components
- eliminate tension forces using simultaneous equations
- obtain the expression for $\tan\theta$ by solving the vertical forces and horizontal forces simultaneously.

In better responses, students were able to:

- draw diagrams which clearly showed the horizontal and vertical components of the forces
- identify the horizontal component as $T_1 \cos\theta = T_2 \cos\emptyset$ and the vertical component as $T_1 \sin\theta = Mg + T_2 \sin\emptyset$
- use a division strategy to solve the simultaneous equations

- clearly show all steps in solving the simultaneous equations to obtain $\tan\theta = \tan\emptyset + \frac{Mg}{T_2\cos\emptyset}$.

Areas for students to improve include:

- understanding that equilibrium means that opposite forces are equal
- understanding that constant velocity means acceleration is zero, so the force is zero
- equating horizontal and vertical forces
- using correct symbols and being careful with similar symbols such as θ and ϕ
- applying trigonometry correctly to the described forces
- using the correct notation and variables for the forces.

Part (a) (ii)

Students should:

- use the result given in part (i)
- equate $\frac{Mg}{T_2\cos\phi}$ as always greater than zero
- represent $\tan\theta$ and $\tan\phi$ as ratios of distances
- identify that $\tan\theta$ and $\tan\phi$ are equal at the given value
- use $M > 0$ and $g > 0$ to construct an inequality or to identify a contradiction.

In better responses, students were able to:

- equate $\tan\theta$ and $\tan\phi$ clearly from part (i)
- demonstrate clear and logical steps to show $P < \frac{2h}{3}$ by using contradiction or establishing an inequality.

Areas for students to improve include:

- identifying when a problem can be solved using a contradiction.

Part (b)

Students should:

- draw a diagram to describe the simple harmonic motion
- determine the amplitude, the period and the centre of motion
- find the maximum force using $F = ma$.

In better responses, students were able to:

- find the amplitude, period, centre of motion and $n = \frac{2\pi}{T} = 80\pi$
- find $\ddot{x} = -(80\pi)^2(x - 0.11)$, substitute an extremity to determine the maximum acceleration and then multiply by the mass 0.8 kg to obtain the maximum force.

Areas for students to improve include:

- accuracy in calculations
- deriving the second derivative of a trigonometric function with a coefficient being a multiple of π
- understanding what the unit of measurement, Newtons, represents
- understanding the difference between frequency and period.

Part (c)

Students should:

- recognise the need to use integration by parts
- explicitly state all four parts involved in integration by parts.

In better responses, students were able to:

- correctly change the limits of the integration
- simplify the integral before applying by parts
- separate the question into clear logical steps
- select optimal integration by parts components.

Areas for students to improve include:

- simplifying inverse trigonometric function expressions
- expanding brackets correctly
- integration by parts, particularly using the correct choices for u and dv when performing integration by parts
- showing all steps in working.

Part (d)

Students should:

- use the triangle inequality
- interpret the inequality $|z| > 0$
- use the hint to find $|z| \leq \left|z - \frac{4}{z}\right| + \left|\frac{4}{z}\right|$.

In better responses, students were able to:

- draw a clear diagram to display given information
- demonstrate their knowledge of the triangle inequality and select the appropriate one to use
- use the $|z|$ definition to restrict the solution to the quadratic inequality
- equate $|z| \leq \left|z - \frac{4}{z}\right| + \left|\frac{4}{z}\right|$ to $|z|^2 - 2|z| - 4 \leq 0$ and correctly solve the quadratic inequation.

Areas for students to improve include:

- understanding the triangle inequality
- knowing how to interact algebraically with moduli.

Question 16

Part (a)

Students should:

- determine the relationships between the three vertices using complex vectors or two-dimensional geometry
- solve vector equations or multiple equations arising from two-dimensional geometry
- find the exact value of z_B.

In better responses, students were able to:

- define z_B in Cartesian form as $b + 5i$
- find vectors representing two of the sides of the equilateral triangle and create an equation linking them using exponential form
- equate the imaginary parts of the equation and solve to find the real part of z_B.

Areas for students to improve include:

- determining efficient techniques to approach questions involving complex numbers which are not scaffolded
- manipulating vector equations
- understanding complex vectors and using proper vector notation.

Part (b)

Students should:

- find a single force equation representing upward and downward flight and integrate with respect to time to find v, then integrate with respect to time to find the initial velocity
- find separate force equations for the upward and downward flights, find expressions for the time and maximum height on the upward flight, and time and maximum height on the downward flight, and solve for the initial velocity.

In better responses, students were able to:

- use a single force equation for the whole flight and efficiently choose the integration methods needed to find the initial velocity.

Areas for students to improve include:

- recognising that a single force equation can be used for upward and downward flight with resisted motion
- choosing integration methods efficiently in questions which are not scaffolded, in this case definite integrals were generally more efficient than indefinite integrals
- working with the algebra to simplify integrands or work with integrals.

Part (c) (i)

Students should:

- start with a result known to be true and proceed to the result to be proved
- use the given result, or the arithmetic mean-geometric mean (AM-GM) result, together with an expression for the surface area of a rectangular prism to proceed to the result to be proved.

In better responses, students were able to:

- select ab, bc and ca as the terms that needed to be substituted into either the given result or the AM-GM result with three terms
- efficiently manipulate indices and roots to achieve the result to be proved.

Areas for students to improve include:

- knowing how to find the surface area and volume of a rectangular prism
- understanding what the arithmetic mean of a set of numbers represents
- recognising which terms to substitute into the AM-GM result or the given result

- manipulating inequalities
- starting all proofs with a result known to be true rather than the result to be proved.

Part (c) (ii)

Students should:

- clearly link the proof to the result from part (i), recognising that the left-hand side of the inequality represents the volume of the rectangular prism
- state the sides of a cube are equal, and provide its volume and surface area
- substitute the expressions for volume and surface area into the inequality for part (i) to show that the left- and right-hand sides are equal
- clearly explain why this proves that the volume of a cube is the maximum possible volume for any rectangular prism with a fixed surface area.

In better responses, students were able to:

- set out their justification in a clear and concise manner
- clearly state each step of logic that was needed to prove the result
- clearly show that their proof answered the question.

Areas for students to improve include:

- knowing how to find the volume and surface area of a cube
- understanding the logical steps that are needed to prove a result
- setting out working in a clear and logical order
- explaining why their solution has proved the result.

Part (d)

Students should:

- recognise that the given information can refer to the complex roots of a polynomial, or to complex vectors
- show that the moduli of each complex number equals 1, then use conjugate rules to find the sum of the roots two at a time, then factorise the cubic equation to find that the only three solutions are $1, \pm i$
- show that the moduli of each complex number equals 1, draw a clear diagram showing that three unit vectors starting at 0 and ending at 1 form a rhombus, show that one of the vectors is 1 and the other two are negatives of each other, then use the given results to show that the other two roots are $\pm i$.

In better responses, students were able to:

- recognise that the question involved complex numbers as the roots of a polynomial
- manipulate equations involving complex numbers leaving the variable as z, rather than converting to exponential or Cartesian form
- recognise the need for and use conjugate rules
- use the sum and products of roots to find the cubic, then factorise to find the solutions
- prove that $1, \pm i$ are the only three possible solutions.

Areas for students to improve include:

- understanding and manipulating equations involving complex numbers
- applying conjugate rules in different contexts.