Mathematics Extension 2

HSC Marking Feedback 2023

General feedback

Students should:

- show relevant mathematical reasoning and/or calculations
- read the question carefully to ensure that they do not miss important components of the question
- have a clear understanding of key words in the question and recognise the intent of the question and its requirements such as show, solve, evaluate, hence, calculate, derive
- use the Reference Sheet where appropriate
- ensure the solution is legible and follows a clear sequence
- engage with any stimulus material provided and refer to it in their response when required by the question
- check their solution answers the question
- round off numerical solutions only at the final step of the solution
- construct graphs neatly, with precision and display all relevant information as required by the question
- interpret information presented in graphs across a range of contexts
- understand when to use relevant calculator functions
- carefully note any information in the questions which supplies units of measurement.

Section II

Question 11 (a)

In better responses, students were able to:

- use the quadratic formula appropriately
- write the answer in Cartesian form.

Areas for students to improve include:

- simplifying the discriminant
- checking numbers/formula to avoid transcription errors.

Question 11 (b)

In better responses, students were able to:

- correctly calculate the scalar product
- use DEG mode on the calculator
- round to the nearest degree accurately.

Areas for students to improve include:

- taking care when calculating the scalar product.

Question 11 (c)

In better responses, students were able to:

- find the direction vector, showing working
- write the required vector in an acceptable form
- understand that there were several acceptable answers.

Areas for students to improve include:

- taking care to not transcribe incorrect numbers from their final answer
- determining the direction vector.

Question 11 (d)

In better responses, students were able to:

- draw an accurate diagram using the given information
- use vectors to answer the question succinctly
- argue the proof using both lengths of sides and parallel sides.

Areas for students to improve include:

- drawing and labelling diagrams using correct cyclic naming
- using vectors to make the proof easier
- using clear and logical statements throughout the proof.

Question 11 (e)

In better responses, students were able to:

- determine the value of n and the period T.

Areas for students to improve include:

- understanding when acceleration is zero, x is the centre of motion
- learning the links between $\ddot{x} = -n^2(x - c)$ and the centre and period of motion.

Question 11 (f)

In better responses, students were able to:

- correctly use partial fractions and determine constants
- correctly integrate partial fractions
- simplify the definite integral.

Areas for students to improve include:

- integrating $\frac{2}{x+1}$ to obtain $2\ln|x+1|$, and know that $\ln|-1| = \ln(1) = 0$
- integrating $\frac{3}{x-3}$ to obtain $3\ln|x-3|$, and know that $\ln|-3| = \ln(3)$.

Question 12 (a)

In better responses, students were able to:

- assume by contradiction that $\sqrt{23}$ is rational and let $\sqrt{23} = \frac{p}{q}$, where p and q are integers with a highest common factor of 1
- rearrange the equation to find $p^2 = 23q^2$
- state that 23 divides p^2 and hence 23 divides p, since 23 is prime
- let $p = 23m$ for $m \in \mathbb{Z}$ and show that $q^2 = 23m^2$, and so similarly 23 divides q
- state that this contradicts p and q having a highest common factor of 1 and so $\sqrt{23}$ must be irrational.

Areas for students to improve include:

- defining any pronumerals they use in the proofs
- stating all the successive implications used in reaching conclusions rather than skipping steps
- stating the conclusion of the proof.

Question 12 (b)

In better responses, students were able to:

- start with $(x-y)^2 \geq 0$
- expand the LHS and add $x^2 + y^2$ to both sides then rearrange successfully
- start with a result known to be true and proceed until the proof is completed
- choose an efficient method of proof.

Areas for students to improve include:

- understanding that when a proof starts with a result that is not yet known to be true a logical fallacy is created
- understanding the significance of x and y being real rather than positive, especially when dividing an inequality by $2xy$.

Question 12 (c) (i)

In better responses, students were able to:

- use a diagram
- show in algebra or in words why the resultant force is negative, has a magnitude of $mg\sin\theta$, and why there is no net force in the $\underset{\sim}{j}$ direction
- construct a diagram with θ at the appropriate angle, with mg on the hypotenuse and $-mg\sin\theta$ and $-mg\cos\theta$ on the other sides, and with an arrow showing the normal reactive force
- show that the net force is $\underset{\sim}{F} = \begin{pmatrix} -mg\sin\theta \\ \left|\underset{\sim}{R}\right| - mg\cos\theta \end{pmatrix}$ and state that $\left|\underset{\sim}{R}\right| = mg\cos\theta$.

Areas for students to improve include:

- identifying the important components of a result to be proved and stating how each can be derived.

Question 12 (c) (ii)

In better responses, students were able to:

- divide $\underset{\sim}{F}$ by m to find the acceleration vector and use integration to find the velocity vector
- use definite or indefinite integrals to show that the constant of integration is zero
- find $v(t)$ as a vector in the $\underset{\sim}{i}$ direction rather than a scalar.

Areas for students to improve include:

- reading the question carefully to find the answer using all pronumerals required in the question
- using a definite integral to integrate both sides, or show that the LHS is $\underset{\sim}{v} - 0$, to show that the constant of integration is zero
- using an indefinite integral clearly to find the value of the constant c rather than assuming it is zero.

Question 12 (d)

In better responses, students were able to:

- convert $2 - 2i$ into exponential form
- find the modulus of the cube root by finding the cube root of the modulus
- find three arguments for the cube roots by either dividing the argument by 3 then adding or subtracting multiples of $\frac{2\pi}{3}$, or adding multiples of 2π to the argument then dividing by 3
- show the three cube roots in exponential form.

Areas for students to improve include:

- finding the cube root of a surd

- finding three appropriate arguments for the cube roots
- ensuring that complex numbers in exponential form have i included in the exponent.

Question 12 (e) (i)

In better responses, students were able to:

- show that since all coefficients are real that the conjugate root theorem applies
- show that $2 - i$ is the conjugate of $2 + i$ so must also be a zero.

Areas for students to improve include:

- understanding that the conjugate root theorem only applies when the coefficients are real
- stating that $2 - i$ is a zero because it is the conjugate of the zero given in the question.

Question 12 (e) (ii)

In better responses, students were able to:

- represent the two unknown roots as α and β
- find the sum and product of the roots
- find the values of α and β by inspection.

Areas for students to improve include:

- understanding that the two unknown roots could either both be real, or could be a pair of complex conjugates
- writing the roots in the form α and β rather than $a + ib$ and $a - ib$.

Question 13 (a)

In better responses, students were able to:

- complete the square in the denominator
- split the linear numerator into correct algebraic parts
- use the reference sheet to apply the reverse chain rule and standard integrals.

Areas for students to improve include:

- completing the square when quadratics have a negative coefficient of x^2
- applying the reverse chain rule as stated in the reference sheet.

Question 13 (b) (i)

In better responses, students were able to:

- apply correct mathematical structures for proofs
- factorise or complete the square to prove a result.

Areas for students to improve include:

- identifying the difference between solving an inequation and proving a result
- completing a proof for all values of a variable, not specific values.

Question 13 (b) (ii)

In better responses, students were able to:

- state clearly the inductive step in the proof for $n = k + 1$
- use the assumption to complete the inequality proof.

Areas for students to improve include:

- including the use of required to prove statements at all levels of the proof
- connecting the result from part (i) in part (ii).

Question 13 (c) (i)

In better responses, students were able to:

- use appropriate mathematics to show the initial velocities.

Areas for students to improve include:

- reasoning of trigonometric connections between angle and velocity at any point on a trajectory.

Question 13 (c) (ii)

In better responses, students were able to:

- state the components of the force equations
- use the appropriate time equations and definite integrals.

Areas for students to improve include:

- understanding that air resistance applies to both components of the motion
- using initial conditions to find constants of integration.

Question 13 (c) (iii)

In better responses, students were able to:

- separate variables and apply a definite integral.

Areas for students to improve include:

- using constants of integration.

Question 13 (c) (iv)

In better responses, students were able to:

- solve for $y = 0$ to obtain the time for finding the range

- use the graphs provided in order to obtain the correct time value.

Areas for students to improve include:

- connecting horizontal and vertical components of displacement in order to find the range.

Question 14 (a) (i)

In better responses, students were able to:

- find the cartesian form for the complex numbers as the arguments gave exact values
- apply the formula for $|z + w|^2$
- use algebraic techniques involving surds.

Areas for students to improve include:

- understanding of the modulus of complex numbers.

Question 14 (a) (ii)

In better responses, students were able to:

- apply geometry results involving the properties of a rhombus
- realise that the addition of complex numbers yields a diagonal of a parallelogram.

Areas for students to improve include:

- considering geometric approaches for complex numbers
- working mathematically with vectors.

Question 14 (a) (iii)

In better responses, students were able to:

- apply the cosine rule and part (i) to find the exact value of $\cos\frac{7\pi}{24}$.

Areas for students to improve include:

- using a diagram to convey a message.

Question 14 (b)

In better responses, students were able to:

- draw a diagram showing the related positions of both particles
- realise that the time of colliding $t = 5\pi$ could be obtained through symmetry.
- use trigonometric equations correctly to solve to find $t = 5\pi$.

Areas for students to improve include:

- generating the time equations for simple harmonic motion
- applying the sine or cosine graph for certain starting conditions.

Question 14 (c) (i)

In better responses, students were able to:

- calculate vertical resisted motion
- apply separation of variable techniques and definite integrals to generate related equations of motion.

Areas for students to improve include:

- choosing the resultant force vector for upwards motion involving gravity and resistance.

Question 14 (c) (ii)

In better responses, students were able to:

- establish two equivalent expressions for the height *H* in terms of the original velocity and final velocity
- use the absolute values involved in log expressions.

Areas for students to improve include:

- setting the downwards direction as positive for downwards motion
- generating the correct direction for gravity and resistance for downwards motion.

Question 15 (a) (i)

In better responses, students were able to:

- use integration by parts with $\sin^{n-1}\theta \times \sin\theta$, rewrite the result in terms of J_n and J_{n-2} and then rearrange to prove the result.

Areas for students to improve include:

- converting from an integral to J_n or J_{n-2}
- using correct notation when using integrals
- writing the powers of trigonometric functions in integrands.

Question 15 (a) (ii)

In better responses, students were able to:

- use the given substitution and trigonometric identities to find the integrand in terms of 2θ
- use the substitution $u = 2\theta$ to find the integral from 0 to π in terms of u
- use the symmetry of $\sin^n u$ around $u = \frac{\pi}{2}$ to show that it is twice the integral from 0 to $\frac{\pi}{2}$, then show that since the integral is independent of the variable used the result is proved.

Areas for students to improve include:

- differentiating $\sin^2\theta$ when integrating by parts
- converting integrands from $\sin\theta\cos\theta$ to $\sin 2\theta$
- operating with exponents in the form $an + b$

- considering second substitutions and symmetry when working with harder integrals.

Question 15 (a) (iii)

In better responses, students were able to:

- convert I_n to a function of J_{2n+1}
- use the result of (i) to convert to a function of J_{2n-1}
- use the result of (ii) to convert to a function of I_{n-1}.

Areas for students to improve include:

- converting integrals correctly into the given forms J_n or I_n
- understanding that I_n and J_n are not equivalent
- considering the use of results from earlier parts of a question as a more efficient method of solving problems.

Question 15 (b) (i)

In better responses, students were able to:

- use the vector form $\overrightarrow{XY}$ to clearly show $\overrightarrow{LP}$ as the sum or difference of vectors
- convert their expression into vectors in terms of $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$
- simplify their expression in logical steps to achieve the result to be proved.

Areas for students to improve include:

- showing the path they are following in the form $\overrightarrow{XY}$ before converting into vectors in terms of $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$
- stating when they are using the fact that the vector to the midpoint of a side is the average of the two vectors leading from the same point to the ends
- stating when they are using the fact that the vector between two midpoints is half the vector representing the parallel side
- showing sufficient explanations when proving a result that has been given in the question
- using vector notation correctly
- using vectors that are defined in the question unless they define them clearly.

Question 15 (b) (ii)

In better responses, students were able to:

- use scalar products to correctly expand and simplify either the LHS or RHS
- correctly expand and simplify the other side and show they are equal or rearrange the expression until it is equal to the other side.

Areas for students to improve include:

- understanding the difference between scalar products and multiplication, and the difference between squaring the modulus of a vector and squaring the vector itself
- expanding the square of an expression with three terms.

Question 15 (c)

In better responses, students were able to:

- realise that z increases in a linear manner and let $z = t$
- clearly draw the graph of $y = f(x)g(x)$ or consider $\sqrt{9 - t^2}\cos(\pi t)$
- find the relevance to the parametric equations of x and y, taking careful note of the starting and end points of the spiral, plus the direction in which it travels
- find possible parametric equations.

Areas for students to improve include:

- understanding the parametric form of three-dimensional curves.

Question 16 (a) (i)

In better responses, students were able to:

- use the factorisation of the difference of two cubes
- substitute $e^{\frac{2i\pi}{3}}$ and simplify accurately.

Areas for students to improve include:

- during simplification not dropping 'i' when simplifying
- knowing that $e^{\frac{4i\pi}{3}} = e^{\frac{-2i\pi}{3}} = \overline{w}$.

Question 16 (a) (ii)

In better responses, students were able to:

- associate AC as a rotation of AB, that is $\overrightarrow{AC} = e^{\frac{i\pi}{3}} \times \overrightarrow{AB}$
- use the result from part (i), as $1 + w = -w^2$
- use vector notation correctly, $(c - a) = e^{\frac{i\pi}{3}} \times (b - a)$.

Areas for students to improve include:

- rotating by $e^{i\theta}$ in an anticlockwise direction or $e^{-i\theta}$ in a clockwise direction
- using the connections between parts (i), (ii) and (iii) are to guide possible methods of solution.

Question 16 (a) (iii)

In better responses, students were able to:

- identify that the product of the two expressions given in parts (i) an (ii) could possibly provide a solution if expanded and simplified using the result from part (i).

Areas for students to improve include:

- expanding and collection terms
- knowing when to connect vectors and complex numbers.

Question 16 (b) (i)

In better responses, students were able to:

- prove the given result using meticulous detail and not omit any cases
- use a variety of methods to prove that $x > \ln(x), \text{for } x > 0$
- use well-constructed statements about the possible values that x.

Areas for students to improve include:

- checking for errors when using $x \geq 0$ when using $f(x) = x - \ln(x)$
- knowing that $x > \ln(x)$ is equivalent to $e^x > x$.

Question 16 (b) (ii)

In better responses, students were able to:

- use Mathematical Induction with precision; $LHS - RHS > 0$ or Showing $LHS = RHS$ or showing $RHS = LHS$, and using part (i) to simplify
- recognise that adding logarithms gives the product leading to $n!$
- recognising that multiplying exponentials leads to a sum of 1 to n.

Areas for students to improve include:

- using patterns to identify parts of the result required
- using part (i), or otherwise; $1 > ln1, 2 > ln2, \ldots, n > ln(n)$
- realising that multiple solutions are possible, Mathematical Induction is not the only way to prove this result.

Question 16 (c)

In better responses, students were able to:

- determine that if $\frac{\pi}{2} < Arg\left(\frac{z}{w}\right) < \pi$ then $-\pi < Arg\left(\frac{w}{z}\right) < -\frac{\pi}{2}$
- use $\frac{\overline{w}}{z} = \frac{z}{w}$ and connected this with $\frac{xz+yw}{z}$
- determine that the solution must lie in the third quadrant.

Areas for students to improve include:

- using the information provided about the complex numbers w and z
- using the connection to the Argand diagram and the real numbers x and y.