Mathematics Extension 2

HSC marking feedback 2024

General feedback

Students should:

- show relevant mathematical reasoning and/or calculations
- read the question carefully to ensure that they do not miss important components of the question
- have a clear understanding of key words in the question and recognise the intent of the question and its requirements such as show, solve, evaluate, hence, calculate, derive
- use the Reference Sheet where appropriate
- ensure the solution is legible and follows a clear sequence
- engage with any stimulus material provided and refer to it in their response when required by the question
- check their solution answers the question
- round off numerical solutions only at the final step of the solution
- construct graphs neatly, with precision and display all relevant information as required by the question
- interpret information presented in graphs across a range of contexts
- understand when to use relevant calculator functions
- carefully note any information in the questions which supplies units of measurement.

Section II

Question 11 (a)

In better responses, students were able to:

- identity the need to use integration by parts
- establish a correct anti-derivative with a constant.

Areas for students to improve include:

- choosing the correct expressions of $u = x$ and $dv = e^x$.

Question 11 (b) (i)

In better responses, students were able to:

- use the definition of a conjugate to correctly add the 2 complex numbers.

Areas for students to improve include:

- checking their expressions to avoid transcription errors.

Question 11 (b) (ii)

In better responses, students were able to:

- efficiently square a complex number
- choose an appropriate form of a complex number before applying a power.

Areas for students to improve include:

- using the correct value of $i^2 = -1$.

Question 11 (c)

In better responses, students were able to:

- apply the formula for the angle between 2 vectors
- correctly calculate the dot product and modulus
- use radians mode when prompted.

Areas for students to improve include:

- differentiating between radians and degrees
- understanding the effect of the negative sign on a trigonometric ratio.

Question 11 (d)

In better responses, students were able to:

- use an appropriate substitution to simplify the integral
- apply t-result formulae to the integrand and related boundaries
- convert the integral into a simple algebraic form.

Areas for students to improve include:

- understanding the appropriate forms for t-result substitutions
- correctly applying boundary values with differentials.

Question 11 (e) (i)

In better responses, students were able to:

- efficiently determine the modulus and argument of a complex number.

Areas for students to improve include:

- applying square roots as part of the modulus formula.

Question 11 (e) (ii)

In better responses, students were able to:

- raise a complex number to a high power using polar form
- convert between polar and cartesian form.

Areas for students to improve include:

- demonstrating understanding of appropriate quadrants for differing arguments.

Question 11 (f)

In better responses, students were able to:

- graph the region described by a modulus inequality and an argument inequation
- carefully deal with boundary values in particular points where the argument is undefined.

Areas for students to improve include:

- using dotted lines and open circles to highlight points excluded from regions
- using argument inequations
- subtracting vectors related to complex numbers.

Question 12 (a) (i)

In better responses, students were able to:

- follow the dot product procedure properly
- multiply a vector by a scalar.

Areas for students to improve include:

- finding the dot product of a vector with itself.

Question 12 (a) (ii)

In better responses, students were able to:

- recognise the algebraic expression for the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$, and/or draw the associated vector diagram.

Areas for students to improve include:

- reviewing their calculations to avoid basic errors involving addition or multiplication by 0
- communicating simple calculations in working in response to the requirement to 'show that'
- subtracting negative numbers.

Question 12 (b)

In better responses, students were able to:

- break the expression into the required format for partial fractions
- use methodical progress to determine the unknown quantities
- check that the result of the partial fraction breakdown satisfies the original expression.

Areas for students to improve include:

- using absolute values for the natural logarithm anti-derivative
- checking the results of the simultaneous equations
- using the reference sheet to identify basic anti-derivatives.

Question 12 (c) (i)

In better responses, students were able to:

- clearly communicate understanding of the modulus of a complex number.

Areas for students to improve include:

- explaining why $b + 12 = 0$ satisfies the imaginary part for this example
- equating real and Imaginary parts of an equation involving complex numbers.

Question 12 (c) (ii)

In better responses, students were able to:

- clearly communicate understanding of the modulus of a complex number $z = a + ib$.

Areas for students to improve include:

- completing the response by supplying an answer for z after calculating a.

Question 12 (d)

In better responses, students were able to:

- recognise that considering n odd/even is the most appropriate approach
- clearly communicate solutions for both cases.

Areas for students to improve include:

- supporting statements and ideas with mathematical calculations or proof
- ensuring that the response adds information to what is supplied in the question.

Question 12 (e) (i)

In better responses, students were able to:

- identify the direction of a vector
- complete the subtraction of the relevant vectors given in the question.

Areas for students to improve include:

- carefully subtracting vector components
- understanding the format for the vector equation of a line.

Question 12 (e) (ii)

In better responses, students were able to:

- maintain consistent methods when calculating the parameters for different vector components
- carefully test and compare results.

Areas for students to improve include:

- calculating multiple parameters to reduce the possibility of an error in calculation.

Question 13 (a) (i)

In better responses, students were able to:

- find the vector $\overrightarrow{AB}$ in component form, then find $\left|\overrightarrow{AB}\right|^2$ showing all working.

Areas for students to improve include:

- finding the difference of two vectors or expanding perfect squares.

Question 13 (a) (ii)

In better responses, students were able to:

- use the formula for the axis of symmetry and the concavity of the parabola to show that $\left|\overrightarrow{AB}\right|^2$ has a minimum value when $p = 2$, then substitute into the expression in (a) to find the minimum value of $\left|\overrightarrow{AB}\right|^2$, then take its square root to find the minimum distance
- use calculus to find the minimum turning point of $\left|\overrightarrow{AB}\right|^2$, or use the dot product of AB and ℓ, the vector projection of $\overrightarrow{AB}$ onto ℓ or $\overrightarrow{OA}$ onto ℓ, the scalar projection of $\overrightarrow{AB}$ onto ℓ or $\overrightarrow{OA}$ onto ℓ, or complete the square on $\left|\overrightarrow{AB}\right|^2$.

Areas for students to improve include:

- explaining why a value found using the axis of symmetry or by letting the first derivative equal zero or by completing the square gives a minimum value.

Question 13 (b)

In better responses, students were able to:

- find $n = 2$ and $c = -1$ from the equation of motion, then find $a = \sqrt{5}$ by either:
  - (1) finding an expression for v^2 using definite integration using $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4(x+1)$
  - (2) finding an expression for v^2 using indefinite integration using $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4(x+1)$ then rearranging into the form $v^2 = n^2(a^2 - (x-c)^2)$
  - (3) using their equation for v^2 to find the extremities of motion
  - (4) finding expressions for displacement and velocity as functions of sine and cosine
- find that the distance travelled in a full period of motion is $4a = 4\sqrt{5}$, or twice the distance between the extremities of motion.

Areas for students to improve include:

- taking more care when solving simple equations within a longer solution
- understanding when and how to correctly use the formula $v^2 = n^2(a^2 - (x-c)^2)$
- knowing that the distance travelled in one full period of motion is four times the amplitude or twice the distance between the extremities of motion, and that no further calculus is required.

Question 13 (c) (i)

In better responses, students were able to:

- integrate $v\frac{dv}{dx} = -kv^2$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -kv^2$ to find an equation linking x and v, then correctly rearrange to make v the subject, clearly explaining why only the positive solution was required.

Areas for students to improve include:

- recognising that $\ddot{x} = v\frac{dv}{dx}$ gives a simpler solution than $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ in this case since we are trying to find a function of v rather than v^2
- removing constant factors from integrals before integration to reduce the complexity of the algebra and reduce simple errors
- manipulating equations including logarithms or exponentials with a minus sign correctly, showing all steps clearly especially when using indefinite integrals
- using definite integrals to obtain the solution more efficiently.

Question 13 (c) (ii)

In better responses, students were able to:

- substitute correctly into $v = 40e^{-kx}$ and rearrange the equation one step at a time to reach the given result.

Areas for students to improve include:

- showing all steps when proving a given result, especially the last steps to reach the result.

Question 13 (c) (iii)

In better responses, students were able to:

- integrate $\frac{dv}{dt} = -kv^2$ or $\frac{dx}{dt} = 40e^{-kx}$ to find an expression involving v and t or x and t respectively, then correctly substitute to find t.

Areas for students to improve include:

- recognising that $\ddot{x} = \frac{dv}{dt}$ gives a simpler solution than using the expression for $\frac{dx}{dt}$ since we need to link v and t in the question
- leaving all expressions in terms of k then substituting $k = \frac{\ln 4}{15}$ at the end to find a numerical expression for t
- using definite integrals to obtain the solution more efficiently.

Question 13 (d)

In better responses, students were able to:

- start with an expression equal to either the left-hand side, the right-hand side or the difference between the sides, of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ or $a\sqrt{bc}+b\sqrt{ac}+c\sqrt{ab}=abc$, then use the arithmetic mean – geometric mean (AM-GM) three times to prove the result
- start with three AM-GM inequalities using appropriate variables and add them, then rearrange to prove the result
- use three separate inequalities and add them appropriately.

Areas for students to improve include:

- using more than one application of the AM-GM and using the variables required in pairs
- using the most complicated of the expression to be proved, in this case $a\sqrt{bc}+b\sqrt{ac}+c\sqrt{ab}$, as a hint to guide the choice of variables to be used in the AM-GM.

Question 14 (a)

In better responses, students were able to:

- use a direct proof by letting $a=2k+1$ for $k\in\mathbb{Z}$ to show that $a^2-1=4k(k+1)$, then noting that k and $k+1$ are consecutive numbers where one is even and the other odd and so their product must be even, then lastly let $k(k+1)=2p$ for $p\in\mathbb{Z}$, to conclude that a^2-1 is divisible by 8.
- use cases to prove that $k(k+1)$ is even
- use $a=4k+1$ and $a=4k+3$ (or $a=4k-1$) and prove that in both cases a^2-1 is divisible by 8
- use different methods of proof, for example, mathematical induction, proof by contrapositive and proof by contradiction can all be used to prove the result.

Areas for students to improve include:

- understanding that proving the result is true for only one of $a=4k+1$ or $a=4k+3$ or $a=4k-1$ only proves the result is true for some odd values of a, not all odd values of a as required
- using mathematical induction appropriately. The base case must be the lowest number, in this case $a=1$, and the step value must be 2
- setting up the proof properly to be able to prove the required result when using proof by contrapositive or proof by contradiction, rather than trying to prove the converse or inverse.

Question 14 (b)

In better responses, students were able to:

- prove $P(5)$ by showing that $252 < 256$, then correctly state $P(k)$ as ${}^{2k}C_k < 2^{2k-2}$ and $P(k+1)$ as ${}^{2k+2}C_{k+1} < 2^{2k}$, then start with the LHS of $P(k+1)$, convert from C notation to factorial notation, rearrange the expression to be $\frac{(2k+2)(2k+1)}{(k+1)(k+1)} \times \frac{(2k)!}{k!k!}$, replace $\frac{(2k)!}{k!k!}$ with ${}^{2k}C_k$, use $P(k)$ to replace ${}^{2k}C_k$ with 2^{2k-2} and introduce the correct inequality sign, then show that $\frac{(2k+2)(2k+1)}{(k+1)(k+1)} < 2^2$ in a clear and logical manner to complete the proof
- start with the RHS of $P(k+1)$, LHS - RHS of $P(k+1)$ or both sides of $P(k)$.

Areas for students to improve include:

- substituting $n = 5$ and $n = k+1$ correctly into both sides of ${}^{2n}C_n < 2^{2n-2}$ to prove the base case and to find $P(k+1)$
- converting from C notation to factorial notation, and operating with factorial notation
- manipulating simple algebraic terms correctly within a longer solution
- using factorisation, and replacing smaller terms with larger terms, to prove that $\frac{(2k+2)(2k+1)}{(k+1)(k+1)} < 2^2$ for all values of $k \geq 5$.

Question 14 (c)

In better responses, students were able to:

- use a diagram to show that $z - w$ and $z + w$ are the diagonals of a rectangle, and since they are equal $\left|\frac{z-w}{z+w}\right| = \frac{|z-w|}{|z+w|} = 1$
- let $\frac{z}{w} = ki$ and by dividing the numerator and denominator of $\left|\frac{z-w}{z+w}\right|$ by w show that $\left|\frac{\frac{z}{w}-1}{\frac{z}{w}+1}\right| = \frac{|ki-1|}{|ki+1|} = \frac{\sqrt{k^2+1}}{\sqrt{k^2+1}} = 1$
- use Pythagoras theorem with $|w|$ and $|z|$.

Areas for students to improve include:

- recognising the relationship between the moduli of z and w is not given in the question then we cannot assume that they are equal
- drawing a clear diagram to help identify a more efficient approach.

Question 14 (d)

In better responses, students were able to:

- clearly state that the two instances of $\int \frac{1}{x}dx$ in the question each involve a constant of integration which may be different, or clearly state that $\frac{1}{x} \times x$ should be $\frac{1}{x} \times x + c$ as there is a constant of integration that is not normally shown
- identify that in solving $\int \frac{1}{x}dx = 1 + \int \frac{1}{x}dx$ we arrive at $\ln|x| + C_1 = 1 + \ln|x| + C_2$, where the two constants differ by 1, is a successful solution.

Areas for students to improve include:

- understanding that indefinite integrals refer to a function whose gradient function is equal to the integrand, not to areas underneath curves
- understanding that an integral is only defined where the integrand is defined, so in this case the fact that x could not be equal to zero was not relevant
- understanding that integration by parts works for all integrals, even those that can be solved more easily using standard integrals.

Question 14 (e) (i)

In better responses, students were able to:

- use arrow vector notation to show that $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$, then use given information to show that $\overrightarrow{OP} = \frac{3}{5}\overrightarrow{OA}$ and $\overrightarrow{PR} = k\overrightarrow{PQ} = k\left(\overrightarrow{OQ} - \overrightarrow{OP}\right)$ then substitute $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$ and rearrange to prove the result
- use $\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR}$ to prove the result.

Areas for students to improve include:

- recognising that if the question states an expression for $\overrightarrow{PR}$, then it is more efficient to use $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$ than break $\overrightarrow{OR}$ into the sum of any other vectors
- using operations with vector algebra correctly.

Question 14 (e) (ii)

In better responses, students were able to:

- create two simultaneous equations involving h and k to equate the coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then solve to show $k = \frac{1}{6}$
- subtract vectors to reach the zero vector.

Areas for students to improve include:

- understanding that when a vector can be represented by two expressions using scalar multiples of the sum of two non-parallel vectors, then the scalar multiples of each vector must be equal
- solving simultaneous equations within a longer solution.

Question 14 (e) (iii)

In better responses, students were able to:

- find an expression for $\overrightarrow{OT}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ by introducing a scalar, and using the result from part (ii) find $\overrightarrow{OR}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, find the value of the scalar, and find $\overrightarrow{OT}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Areas for students to improve include:

- solving simultaneous equations with letters from the Greek alphabet.

Question 15 (a) (i)

In better responses, students were able to:

- set up the equation of the vector from *A* to *B*
- compare the equation to the position vector for *M*
- draw a diagram illustrating the parallelogram formed by the vectors in the question
- express OM in terms of OA and AB
- show AM and MB are equal
- show AM is a multiple of AB.

Areas for students to improve include:

- identifying the difference between $\underset{\sim}{b} - \underset{\sim}{a}$ and $\underset{\sim}{a} + \underset{\sim}{b}$
- providing clear and logical working
- avoiding proof of AM + MB = AB to prove M is a point on the line AB
- attempting to equate vector AB with OM to show point of intersection
- using proper vector notation.

Question 15 (a) (ii)

In better responses, students were able to:

- understand the difference between a point and a vector
- understand the difference between vector $\overrightarrow{MC}$ and 'the line passing through *M* and *C'*.
- understand that the parameter of an equation is a quantity that determines if a point is in between two points.

Areas for students to improve include:

- identifying that the point of difference between parts (i) and (ii) implies an extra component to this response
- addressing the expression, 'lies between *M* and *C*' with evidence
- drawing a diagram so the response can focus on the relevant vectors instead of using guesswork
- understanding the difference between a vector between two points and a position vector.

Question 15 (a) (iii)

In better responses, students were able to:

- relate part (iii) to the previous parts of question 15
- relate complex numbers as vectors to those on an Argand diagram
- refer to the modulus information given
- refer to the cube root of xwz as well as the value xwz
- relate the geometrical aspect of roots of complex numbers and the geometry of the point G in relation to the unit circle.

Areas for students to improve include:

- communicating all aspects of the problem-solving process.

Question 15 (b)

In better responses, students were able to:

- separate appropriate factors and follow the procedure of integration by parts
- include all relevant steps to communicate the procedure to satisfy the requirement to ‘show that’
- manipulate the integrand into the required form.

Areas for students to improve include:

- choosing the correct factors to differentiate and integrate
- setting out responses so they are easily followed and minimise errors in processes
- using the index laws correctly
- understanding that questions involving this form of recurrence relation most likely involve integration by parts.

Question 15 (c) (i)

In better responses, students were able to:

- respond to the requirement to ‘show that’ by finding an anti-derivative expression as well as the calculation for the desired constant of integration
- illustrate separate steps in the process to clarify the correct method used.

Areas for students to improve include:

- setting out responses so they are easily followed and minimise errors in processes
- differentiating algebraic terms with negative powers.

Question 15 (c) (ii)

In better responses, students were able to:

- create a cubic equation and solve it
- notice that the object was ‘at rest at $x = 6$’ gives one solution to the cubic, and hence find the required quadratic factor by using an appropriate method
- select the appropriate solution that relates to the movement of the object.

Areas for students to improve include:

- applying methods to solve a cubic equation
- reading the question carefully to extract information about the movement of the object.

Question 15 (d)

In better responses, students were able to:

- ‘complete the square’ in the integrand’s denominator
- use an appropriate trigonometric substitution
- apply relevant trigonometric identities in their responses
- use a right-angle triangle diagram to convert the substituted variable back to x.

Areas for students to improve include:

- recognising the forms of integrand that are best solved with trigonometric substitution
- using a substitution that will simplify the original expression
- carefully organising substitutions and derivatives on the page for clarity of communication
- identifying their most appropriate response when there have been multiple responses made.

Question 16 (a)

In better responses, students were able to:

- use derivatives to establish a relationship between orthogonal vectors
- find a set of equations relating the 3 variables a, b and k
- simplify the trigonometric relationship between the variables to establish that $k > 1$.

Areas for students to improve include:

- understanding that two tangential curves share a common point and a common tangent

  This would have given $b = \cos ka$ and $a = kb \sin ka$ from which the relationship

  $2a = k \sin 2ka$ would have been potentially achieved
- using contradiction methods as part of show and prove questions
- using functional connections between variables when trigonometric and linear functions are related.

Question 16 (b) (i)

In better responses, students were able to:

- use the binomial expansion involving complex numbers
- show that the sum of conjugate complex numbers $\gamma + \bar{\gamma} = 2Re(\gamma)$
- show, without a calculator, the root of a polynomial equation
- use the triple angle formula for cos to simplify their trigonometric expression
- draw a connection between the cube roots of the cube roots of unity.

Areas for students to improve include:

- understanding conjugate properties of complex numbers
- understanding that a cubic polynomial will have at most 3 complex roots
- understanding that De Moivre's theorem applies to complex numbers, not real numbers, that is $(\cos x)^3 \neq \cos 3x$.

Question 16 (b) (ii)

In better responses, students were able to:

- understand that the roots to the polynomial $z^3 - 3z + 1 = 0$ were $2\cos\frac{2\pi}{9}$, $2\cos\frac{4\pi}{9}$, $2\cos\frac{8\pi}{9}$
- apply the product of the roots relationship to show that $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\ \cos\frac{8\pi}{9} = \frac{-1}{8}$
- deduce with appropriate reasoning, that the product given by $\cos\frac{2^n\pi}{9}\cos\frac{2^{n+1}\pi}{9}\ \cos\frac{2^{n+2}\pi}{9}$ also takes the value $\frac{-1}{8}$.

Areas for students to improve include:

- ensuring they respond to references to other question parts, for example, 'by using part (i) to find a value …'
- attempting to establish a general relationship between variables in a relationship
- showing case values for $n = 1,2,3$ ... to establish a pattern.

Question 16 (c)

In better responses, students were able to:

- correctly use force equations for both variables with the appropriate directions
- make use of the starting conditions for each projectile
- work effectively with definite integrals to establish velocity and displacement equations in terms of time
- work through the algebra to seek a correct expression for time of impact.

Areas for students to improve include:

- differentiating between correct expressions for acceleration to meet the requirements of a question
- establishing an appropriate direction for positive motion and then altering variables to suit these conditions.