NORTHERN BEACHES SECONDARY COLLEGE

MANLY SELECTIVE CAMPUS

Year 12

Trial Examination

2020

Mathematics Extension II

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Write your name on the front of every booklet.
- In Questions 11 to 16 show relevant mathematical reasoning and/or calculations.
- NESA approved calculators and templates may be used.
- Weighting: 45%

Section I Multiple Choice

- 10 marks
- Attempt all questions.
- Answer Sheet provided
- Allow about 15 minutes for this section

Section II Free Response

- 90 marks
- Start a separate booklet for each question.
- Each question is of equal value.
- All necessary working should be shown in every question.
- Allow about 2 hour and 45 minutes for this section.

Section I

10 marks

Attempt Questions 1 – 10

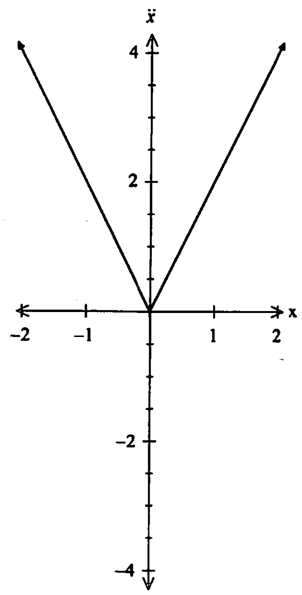
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

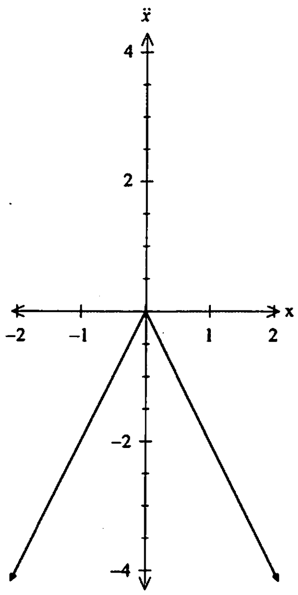
Q1. A particle is moving along a straight line. The displacement of the particle from a fixed point O is given by x. The graphs below show acceleration $\ddot{x}$ as a function of displacement x.

Which one of the graphs below best represents a particle moving in simple harmonic motion?

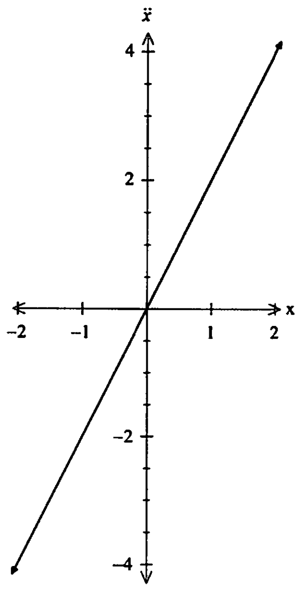
A



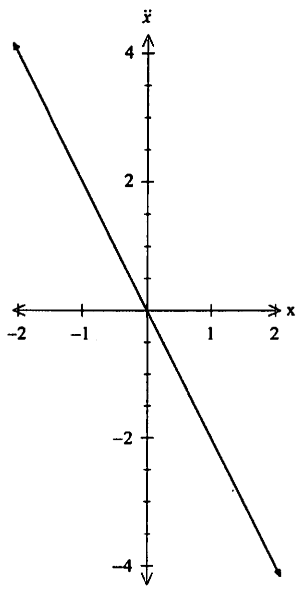
B



C



D



Q2. Which integral has the smallest value?

A. $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x \ dx$

B. $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x \ dx$

C. $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \cos x \ dx$

D. $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \tan x \ dx$

Q3. Which expression is equal to $\displaystyle\int \frac{1}{\sqrt{1 - 4x^2}} \ dx$?

A. $\dfrac{1}{2} \sin^{-1} \dfrac{x}{2} + C$

B. $\dfrac{1}{2} \sin^{-1} 2x + C$

C. $\sin^{-1} \dfrac{x}{2} + C$

D. $\sin^{-1} 2x + C$

Q4. Which symbol belongs in the box in the following statement?

$y = e^x \quad \boxed{} \quad x = \log_e y$

A. $\forall$

B. $\exists$

C. $\Rightarrow$

D. $\Leftrightarrow$

Q5. The following solution demonstrates that $\sqrt{2}$ is an irrational number

Let $\sqrt{2} = \frac{p}{q}$, where p and q are positive integers and have no common factors

$2 = \frac{p^2}{q^2}$ on squaring both sides

$p^2 = 2q^2$

$\therefore p^2$ is even, $\therefore$ p is even

Let $p = 2k$, where k is an integer

$4k^2 = 2q^2$

$\therefore$ $q^2 = 2k^2$

$\therefore q^2$ is even, $\therefore$ q is even.

Which method was used in this proof?

A. Direct proof

B. Proof by contradiction

C. Proof by contrapositive

D. Proof by counter-examples.

Q6. Given that $w^5 = 1$ and w is a complex number what is the value of

$1 + w + w^2 + w^3 + w^4 + w^5$?

A. 1

B. 0

C. w

D. $-w$

Q7. Let $\arg(z) = \frac{\pi}{5}$ for a certain complex number z. What is $\arg(z^7)$?

A. $-\frac{7\pi}{5}$

B. $-\frac{3\pi}{5}$

C. $\frac{2\pi}{5}$

D. $\frac{3\pi}{5}$

Q8. If $\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -m\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ where m is a real constant, the vector $\underset{\sim}{a} - \underset{\sim}{b}$ will be perpendicular to the vector $\underset{\sim}{b}$ when m equals

A. 0 only

B. 2 only

C. 0 or 2

D. 0 or -2

Q9. Points A, B and C have position vectors
$\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = -3\underset{\sim}{j} + \underset{\sim}{k}$ respectively.

The cosine of the angle ABC is equal to ?

A. $\frac{7}{\sqrt{6}\sqrt{13}}$

B. $-\frac{1}{\sqrt{6}\sqrt{13}}$

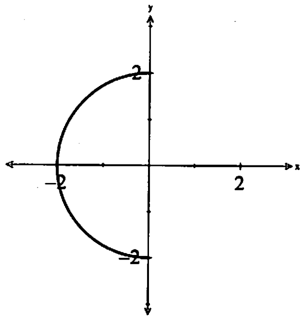
C. $-\frac{7}{\sqrt{21}\sqrt{6}}$

D. $-\frac{2}{\sqrt{6}\sqrt{13}}$

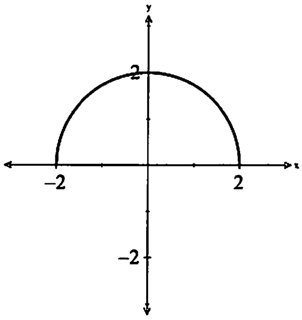
Q10. Which of the following graphs is correct for the following parametric function?

$x = -|2\cos t| \qquad y = |2\sin t|$

A



B



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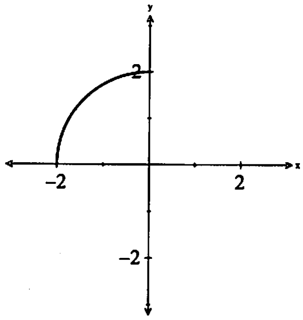

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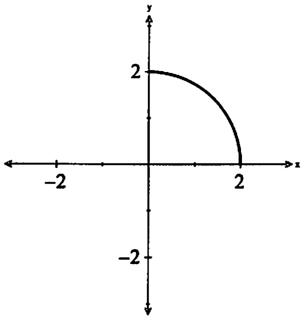


End of Multiple Choice

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 **15 marks**

a) Consider the following statement.

$$\forall\, m \in \mathbb{Z}^+ \;\; \exists\, n \in \mathbb{Z}^+ \text{ such that } \frac{1}{n} - \frac{1}{n+1} < \frac{1}{m}$$

i. Write in symbolic notation the negation of the above statement (without the negation symbol present). **1**

ii. Prove that the original statement is true. **1**

b) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find $\displaystyle\int \frac{dx}{1 + \cos x - \sin x}$ **3**

c) Show by Mathematical Induction that $8^n + 2 \times 7^n - 1$ is divisible by 7 for $n \geq 1$ **3**

Question 11 continues on the next page.

Question 11 continued.

d) A complex number z is defined by the equation $\arg(z - 2 - i) = \dfrac{3\pi}{4}$.

i. Plot on an argand diagram all the points which satisfy this relationship. **2**

ii. What is the minimum value that $|z|$ can take? **1**

e) Using the substitution $u^2 = 4 - x^2$, or otherwise, evaluate **4**

$$\int_0^2 x^3\sqrt{4 - x^2}\ dx$$

End of Question 11

Question 12 **15 marks**

a) i. Use the method of partial fractions to show that **3**

$$\int \frac{2}{x^3 + x^2 + x + 1}\,dx = \log_e \left| \frac{x+1}{\sqrt{x^2+1}} \right| + \tan^{-1} x + C$$

ii. Hence show that **3**

$$\int_{\frac{1}{2}}^{2} \frac{2}{x^3 + x^2 + x + 1}\,dx = \tan^{-1}\left(\frac{3}{4}\right)$$

b) The displacement x of a particle P moving along a straight line with respect to a fixed point O is given by

$$x = 6\sin\left(2t + \frac{\pi}{4}\right) + \sin 2t$$

i. Show that P is moving in simple harmonic motion about O. **2**

ii. State the period of the motion. **1**

iii. Find the amplitude of the motion. Leave your answer in exact form. **2**

c) Let $z = 2e^{\frac{2\pi i}{3}}$ and $w = \sqrt{2}\,e^{\frac{i\pi}{4}}$

i. Convert z and w to Cartesian form **1**

ii. Find $\frac{z}{w}$ in Cartesian form and polar form. **2**

iii. Hence, find the exact values of $\cos\left(\frac{5\pi}{12}\right)$ and $\sin\left(\frac{5\pi}{12}\right)$ **1**

End of Question 12

Question 13 **15 marks**

a) The point $B\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is on the interval AC and is twice as far from $A\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ as it is from $C\begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$. Use vectors to find the coordinates of B. **2**

b) A particle is moving in such a way that its speed v ms^{-1} is given by

$$v^2 = 2 - x - x^2$$

where x is the displacement of the particle from a fixed point O.

i. Show that the particle is moving in simple harmonic motion. **2**

ii. What is the maximum distance of the particle from the origin? **2**

iii. The particle is initially at the point $x = -\dfrac{1}{2}$. **1**

At what time does the particle first return to this point?

c) Given that $a_1 = 2$ and $a_n = \dfrac{a_{n-1}}{n}$ for $n \geq 2$, prove that $a_n = \dfrac{2}{n!}$ for $n \geq 1$ **3**

d) Let n be a natural number.

i. Show that if n is composite, then there exists a factor of n not equal to 1, and at most equal to $\sqrt{n}$ **2**

ii. State the converse of the proposition in part i. **1**

iii. Write down the contrapositive of the proposition in part i. **2**

Question 13 completed.

Question 14 **15 Marks**

a) Evaluate $\int_0^1 \tan^{-1} x \, dx$. Express your answer in simplest exact form. **3**

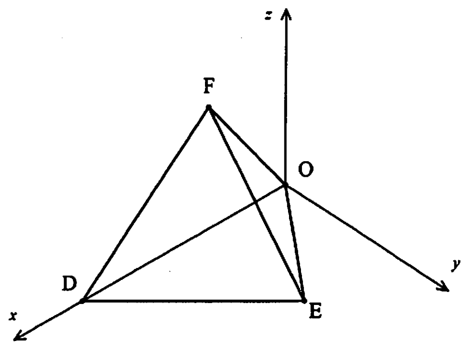
b) i. Prove that $\dfrac{x}{y}+\dfrac{y}{x} \geq 2$ where $x, y > 0$ **2**

ii. Hence, or otherwise, show that $(x+y)\left(\dfrac{1}{x}+\dfrac{1}{y}\right) \geq 4$ **1**

iii. Deduce that $(x+y+z)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right) \geq 9$ **2**

c) Find all the solutions for θ given $|e^{2i\theta} - 1| = \sqrt{3}$ satsfying $-\pi < \theta < \pi$ **3**

d) The faces of the tetrahedron *ODEF* are equilateral triangles of side length 1 unit. Its base *ODE* lies flat on the *xy* plane with two vertices at *O* and *D* (1,0,0) with *F* above the *xy* plane.



i. Show that the coordinates of *E* are $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}, 0\right)$ **1**

ii. Using vectors, prove the coordinates of the vertex *F* are $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{6}, \dfrac{\sqrt{6}}{3}\right)$. **3**

Question 14 completed.

Question 15 **15 marks**

a) The line l_1 has the equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

The line l_2 has the equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

Show that l_1 and l_2 do not meet. **3**

b) It is given that a and b are positive real numbers.

Consider the statement $\forall\ a\ (\forall\ b,\ a^{\ln b} = b^{\ln a})$.

Either prove the statement is true or provide a counter example. **2**

c) Let $P(x) = x^4 + 16x^3 + 108x^2 + 400x + 800$. $P(x)$ has roots $a + 2ib$ and $3a + ib$, where $a, b \in \mathbb{R}$.

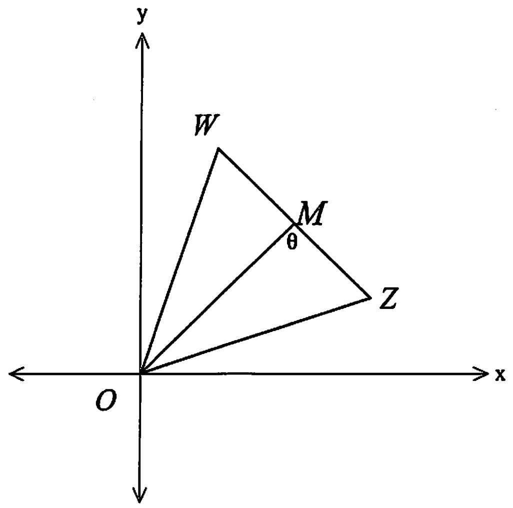
i. Find all the roots of $P(x)$ with integer values for a and b. **3**

ii. Factorise $P(x)$ into its quadratic factors with real coefficients. **2**

Question 15 continues on the next page.

Question 15 continued.

d) Let the complex numbers $W(w)$, $Z(z)$ and the origin form a triangle of area $1u^2$ on the Argand diagram. M is the midpoint of WZ. Let $\angle ZMO = \theta$.



i. Show that $|z+w|\,|z-w|\sin\theta = 4$ **2**

ii. Prove that $|z|^2 + |w|^2 = \dfrac{|z+w|^2 + |z-w|^2}{2}$ **2**

iii. If $\theta = \dfrac{\pi}{2}$, prove that $(|z| - |w|)(|z| + |w|) = 0$ **1**

Question 15 completed

Question 16 **15 marks**

a) Write the Cartesian equation $3x - 4y = 11$ as a vector equation. **2**

b) The point $\vec{v}$ is a point on the surface of a sphere with centre $P(1,3,1)$ and radius 14 units.

i. Write down the vector equation of the sphere. **1**

ii. The point $Q(2,1,4)$ lies on the surface of the sphere.
Find the Cartesian equation of the tangent to the sphere at Q. **3**

(You may assume that the radius drawn to the point of contact of the tangent is perpendicular to the tangent.)

c) By considering $f'(x)$ where $f(x) = e^x - x$

i. Show that $e^x > x$ for $x \geq 0$ **2**

ii. Hence, use a calculus method and Mathematical Induction to show that for

$x \geq 0$, $e^x \geq \dfrac{x^n}{n!}$ for all $n \in \mathbb{Z}^+$ **3**

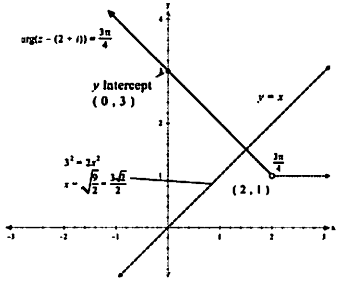
d) Find $\displaystyle\int \frac{\log_e x}{(1 + \log_e x)^2}\, dx$ **4**

End of paper

Q1	$\ddot{x} = -n^2 x$ Ie. Acceleration is a linear function opposite in sign to x	D
Q2	$0 < x < \frac{\pi}{4},\ 0 < \sin x < \cos x < 1$ $\therefore \sin^2 x < \cos^2 x$ $\sin x \times \sin x < \sin x \times \cos x$ $\therefore \sin^2 x < \sin x \cos x$ *because* $0 < \cos x < 1$ $\sin x < \frac{\sin x}{\cos x}$ $\sin x < \tan x$ $\sin^2 x < \sin x \tan x$ $\therefore \int_0^{\frac{\pi}{4}} \sin^2 x\,dx$ has the smallest value	A
Q3	$\int \frac{1}{\sqrt{1-4x^2}}\,dx$ $= \int \frac{1}{\sqrt{4\left(\left(\frac{1}{2}\right)^2 - x^2\right)}}\,dx$ $= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}}\,dx$ $= \frac{1}{2}\sin^{-1}(2x) + C$	B
Q4	$\Leftrightarrow$ Implies there is an x value such that $x = \log_e y$	C
Q5	Proof by contradiction	B
Q6	$w^5 - 1 = 0$ $\Sigma\,\alpha = -\frac{b}{a}$ $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ $\therefore$ $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 1$	A
Q7	$\arg(z) = \frac{\pi}{5}$ $\therefore$ $\arg(z^7) = \frac{7\pi}{5} = -\frac{3\pi}{5}$	B
Q8	**Perpendicular** $\therefore$ $(a-b).b = 0$ $a = -2i - j + 3k$ $b = -mi + j + 2k$ $a - b = (-2i - j + 3k) - (-mi + j + 2k)$ $= (m-2) - 2j + k$ $(a-b).b = -m(m-2) - 2 \times 1 + 2 \times 1$ $= -m(m-2)$ $-m(m-2) = 0$ $m = 0$ or $m = 2$	C
Q9	Application of the scalar product to $\overline{BA}$ and $\overline{BC}$ gives the required result. $\cos\theta = \frac{(\underset{\sim}{b} - \underset{\sim}{a}).(\underset{\sim}{b} - \underset{\sim}{c})}{\lvert\underset{\sim}{b} - \underset{\sim}{a}\rvert\,\lvert\underset{\sim}{b} - \underset{\sim}{c}\rvert}$ $\underset{\sim}{b} - \underset{\sim}{a} = \left(3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) - \left(2\underset{\sim}{i} + \underset{\sim}{j}\right) = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ $\underset{\sim}{b} - \underset{\sim}{c} = \left(3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) - \left(-3\underset{\sim}{j} + \underset{\sim}{k}\right) = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ $\lvert\underset{\sim}{b} - \underset{\sim}{a}\rvert = \sqrt{6}$ $\lvert\underset{\sim}{b} - \underset{\sim}{c}\rvert = \sqrt{13}$ $(\underset{\sim}{b} - \underset{\sim}{a}).(\underset{\sim}{b} - \underset{\sim}{c}) = 3 - 4 + 0 = -1$ $\cos\theta = -\frac{1}{\sqrt{6}\sqrt{13}}$	B
Q10	$x^2 + y^2 = 4 \quad x < 0,\ y > 0$	C

Q11		
a-i	$\forall\, m \in \mathbb{Z}^+ \;\; \exists\, n \in \mathbb{Z}^+$ such that $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{m}$ **Negation** $\forall\, n \in \mathbb{Z}^+ \;\; \exists\, m \in \mathbb{Z}^+$ such that $\frac{1}{n} - \frac{1}{n+1} \geq \frac{1}{m}$	1 – correct answer
a-ii	$\forall\, m \in \mathbb{Z}^+ \;\; \exists\, n \in \mathbb{Z}^+$ such that $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{m}$ let $m = n$ and $n =$ *any* number $LHS = \frac{1}{n} - \frac{1}{n+1}$ $= \frac{(n+1)-n}{n(+1)}$ $= \frac{1}{n(n+1)}$ $< \frac{1}{n}$ $\therefore \; < \frac{1}{m}$ $\therefore$ $\forall\, m \in \mathbb{Z}^+ \;\; \exists\, n \in \mathbb{Z}^+$ such that $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{m}$	2 marks – correct answer demonstrating true for all m 1 mark – correct demonstration of a value of m but not all values.

b	$\int \frac{dx}{1 + \cos x - \sin x}$ $t = \tan\frac{x}{2}$ $\tan^{-1}(t) = \frac{x}{2}$ $\frac{1}{1+t^2}\,dt = \frac{1}{2}\,dx$ $\frac{2}{1+t^2}\,dt = dx$ $\int \frac{1}{\left(1 + \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt$ $= \int \frac{1+t^2}{(1+t^2+1-t^2-2t)} \times \frac{2}{1+t^2}\,dt$ $= \int \frac{2}{2-2t}\,dt$ $= \ln\lvert 1-t\rvert + C$ $= \ln\left(\left\lvert 1 - \tan\frac{x}{2}\right\rvert\right) + C$	3 marks – correct solution 2 marks – forms correct integrand to $\frac{2}{2-2t}$ 1 mark – - Obtains a correct transformation of dx - Correctly integrates an incorrect expression of equivalent merit.
c	$8^n + 2 \times 7^n - 1$ Let $n = 1$ $\;8^1 + 2 \times 7^1 - 1 = 8 + 14 - 1 = 21 = 3 \times 7$ Assume true for $n = k$ ie. $8^k + 2 \times 7^k - 1 = 7P$ where P element of $\mathbb{Z}$ $\therefore$ $8^k = 7P - 2 \times 7^k + 1$ RTP true for $n = k + 1$ $8^{k+1} + 2 \times 7^{k+1} - 1 = 7M$ where M element of $\mathbb{Z}$ $LHS = 8^{k+1} + 2 \times 7^{k+1} - 1$ $= 8 \times 8^k + 2 \times 7 \times 7^k - 1$ $= 8 \times (7P - 2 \times 7^k + 1) + 2 \times 7 \times 7^k - 1$ $= 8 \times 7P - 2 \times 7^k + 7$ $= 7(8P - 2 \times 7^{k-1} + 1)$ $= 7M$ where M element of $\mathbb{Z}$ as $P \in \mathbb{Z}$ QED by Mathematical Induction	3 marks – correct solution 2 marks – correctly established S(1), stated S(k) and reduces exponents in S(k+1), *(line 2 from LHS)*. 1 mark - correctly established S(1), correctly stated S(k) and S(k+1)

<table>
<tr>
<td>d-i</td>
<td>$$\arg(z-2-i)=\frac{3\pi}{4}$$
$$\arg(z-(2+i))=\frac{3\pi}{4}$$
Solution is red line with open circle and correct argument.

</td>
<td>2 marks – correct solution

1 mark – correct solution without open circle.</td>
</tr>
<tr>
<td>d-ii</td>
<td>Minimum value = perp distance
$$x=\frac{3\sqrt{2}}{2}$$</td>
<td>1 mark – correct answer.</td>
</tr>
</table>

<table>
<tr>
<td>e</td>
<td>$$\int_0^2 x^3\sqrt{4-x^2}\ dx$$
$$u^2 = 4-x^2$$
$$x=0 \Rightarrow u=2$$
$$x=2 \Rightarrow u=0$$
$$2udu = -2xdx$$
$$udu = -xdx$$
$$x^2 = 4-u^2$$
$$\int_0^2 x^2\sqrt{4-x^2}\ xdx$$
$$= -\int_0^2 x^2\sqrt{4-x^2}\ -xdx$$
$$= -\int_2^0 (4-u^2)\sqrt{u^2}\ u\ du$$
$$= \int_0^2 4u^2-u^4\ du$$
$$= \left[\frac{4u^3}{3}-\frac{u^5}{5}\right]_0^2$$
$$= \frac{32}{3}-\frac{32}{5}$$
$$= \frac{64}{15}$$</td>
<td>4 marks – correct solution

3 marks – correct transformation of differential, integrand and limits

2 marks – correct transformation of any two differential, limits and integrand.

1 mark – correct transformation of any <u>one</u> differential, integrand or limits.</td>
</tr>
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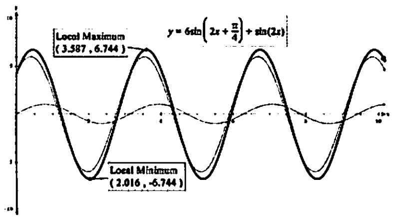
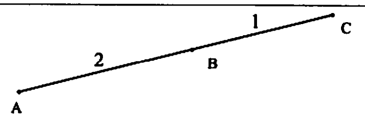
Question 12

a-i	$\int \frac{2}{x^3+x^2+x+1}\,dx$ $\frac{2}{x^3+x^2+x+1}$ $=\frac{2}{x^2(x+1)+(x+1)}$ $=\frac{2}{(x^2+1)(x+1)}$ $\frac{2}{(x^2+1)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$ $(Ax+B)(x+1)+C(x^2+1)=2$ $Ax^2+Ax+Bx+B+CX^2+C=2$ $\therefore$ $A+C=0 \;\Rightarrow\; C=-A$ $A+B=0 \;\Rightarrow\; B=-A$ $B+C=2 \;\Rightarrow\; -2A=2$ $A=-1$ $B=1$ $C=1$ $\int \frac{2}{x^3+x^2+x+1}\,dx$ $=\int \frac{-x+1}{x^2+1}+\frac{1}{x+1}\,dx$ $=\int -\frac{x}{x^2+1}+\frac{1}{x^2+1}+\frac{1}{x+1}\,dx$ $=-\frac{1}{2}\ln\lvert x^2+1\rvert+\tan^{-1}x+\ln\lvert x+1\rvert+C$ $=\ln\left\lvert\frac{1}{\sqrt{x^2+1}}\right\rvert+\tan^{-1}x+\ln\lvert x+1\rvert+C$ $=\ln\left\lvert\frac{x+1}{\sqrt{x^2+1}}\right\rvert+\tan^{-1}x+C$	3 marks – correct solution 2 marks – finds correct values of A, B, C and forms a correct integrand expression of three terms. 1 mark – finds correct values of A, B, C - forms a correct integrand expression from incorrect value(s) of A,B, C
a-ii	$\left[\ln\left\lvert\frac{x+1}{\sqrt{x^2+1}}\right\rvert+\tan^{-1}x\right]_{\frac{1}{2}}^{2}$ $=\left(\ln\left\lvert\frac{2+1}{\sqrt{2^2+1}}\right\rvert+\tan^{-1}2\right)-\left(\ln\left\lvert\frac{\frac{1}{2}+1}{\sqrt{\frac{1}{4}+1}}\right\rvert+\tan^{-1}\frac{1}{2}\right)$ $=\left(\ln\left\lvert\frac{3}{\sqrt{5}}\right\rvert+\tan^{-1}2\right)-\left(\ln\left\lvert\frac{\frac{3}{2}}{\sqrt{\frac{5}{4}}}\right\rvert+\tan^{-1}\frac{1}{2}\right)$ $=\left(\ln\left\lvert\frac{3}{\sqrt{5}}\right\rvert+\tan^{-1}2\right)-\left(\ln\left\lvert\frac{3}{\sqrt{5}}\right\rvert+\tan^{-1}\frac{1}{2}\right)$ $=\tan^{-1}2-\tan^{-1}\frac{1}{2}$ $\tan^{-1}2-\tan^{-1}\left(\frac{1}{2}\right)$ $\tan\left[\tan^{-1}2-\tan^{-1}\left(\frac{1}{2}\right)\right]$ $=\frac{\tan(\tan^{-1}2)-\tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right)}{1+\tan(\tan^{-1}2).\tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right)}$ $=\frac{2-\frac{1}{2}}{1+2\times\frac{1}{2}}$ $=\frac{\frac{3}{2}}{2}=\frac{3}{4}$ $\therefore$ $\tan\left[\tan^{-1}2-\tan^{-1}\left(\frac{1}{2}\right)\right]=\tan^{-1}\left(\frac{3}{4}\right)$	3 marks – correct solution 2 marks – obtains $\left(\tan^{-1}2-\tan^{-1}\frac{1}{2}\right)$ 1 mark - correct substitution of limits and attempts to simplify expression in exact form. - Obtains an incorrect expression involving two inverse functions.

b-i	$x = 6\sin\left(2t+\frac{\pi}{4}\right)+\sin 2t$ $\dot{x} = 12\cos\left(2t+\frac{\pi}{4}\right)+2\cos 2t$ $\ddot{x} = -24\sin\left(2t-\frac{\pi}{4}\right)-4\sin 2t$ $= -4\left[6\sin\left(2t+\frac{\pi}{4}\right)+\sin 2t\right]$ $= -(2)^2 x$	2 marks – correct solution 1 mark – obtains a correct expression for $\dot{x}$
b-ii	$\text{Period} = \frac{2\pi}{n}$ $= \frac{2\pi}{2} = \pi$	1 mark – correct answer
b-iii	 $x = 6\sin\left(2t+\frac{\pi}{4}\right)+\sin 2t$ $= 6\sin 2t\cos\frac{\pi}{4}+6\cos 2t\sin\frac{\pi}{4}+\sin 2t$ $= \frac{6}{\sqrt{2}}\sin 2t+\frac{6}{\sqrt{2}}\cos 2t+\sin 2t$ $= (1+3\sqrt{2})\sin 2t+3\sqrt{2}\cos 2t$ From auxillary angles $R = \sqrt{(1+3\sqrt{2})^2+(3\sqrt{2})^2}$ $= \sqrt{1+6\sqrt{2}+18+18}$ $= \sqrt{37+6\sqrt{2}}$ $\cong 6.744$	2 marks – correct solution 1mark - Obtains a correct exprsssion for x in terms of *sin 2t* and *cos 2t* - Correctly evaluates R from an incorrect expression. For x.

c-i	$z = 2e^{\frac{2\pi i}{3}} \qquad w = \sqrt{2}\,e^{\frac{i\pi}{4}}$ $re^{i\theta} = r\text{cis}\frac{\theta}{2}$ $z = 2\text{cis}\frac{2\pi}{3} \qquad w = \sqrt{2}\text{cis}\frac{\pi}{4}$ $= 2\left\{-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right\} \quad w = \sqrt{2}\left\{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right\}$	1 mark – both correct.
c-ii	$\frac{z}{w} = \frac{\|z\|}{\|w\|}\text{cis}[\arg z - \arg w]$ $= \frac{2}{\sqrt{2}}\text{cis}\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)$ $= \sqrt{2}\text{cis}\left(\frac{5\pi}{12}\right)$ $= \sqrt{2}\,e^{\frac{5\pi}{12}i}$ $\frac{z}{w} = \frac{-1+\sqrt{3}i}{1+i}$ $= \frac{-1+\sqrt{3}i}{1+i}\times\frac{1-i}{1-i}$ $= \frac{(-1+\sqrt{3}i)(1-i)}{1^2-i^2}$ $= \frac{-1+i+\sqrt{3}i+\sqrt{3}}{2}$ $= \frac{-1+\sqrt{3}}{2}+\frac{1+\sqrt{3}}{2}i$	2 mark – correct solution 1 mark – one correct expression
c-iii	$\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2}$ $\sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2}$	1 mark – both expressions correct.

Q13		
a-i	 **Method 1** $B = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\overrightarrow{AB} = \frac{2}{3}\overrightarrow{AC}$ $\begin{pmatrix} 1-x \\ 2-y \\ 2-z \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1-4 \\ 2+1 \\ 2-5 \end{pmatrix}$ $1 - x = \frac{2}{3} \times -3 = -2$ $x = 3$ $2 - y = \frac{2}{3} \times 3 = 2$ $y = 0$ $2 - z = \frac{2}{3} \times -3 = -2$ $z = 4$ $B = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ **Method 2** $\frac{AB}{BC} = \frac{2}{1}$ $\begin{pmatrix} x-1 \\ y-2 \\ z-2 \end{pmatrix} = 2\begin{pmatrix} 4-x \\ -1-y \\ 5-z \end{pmatrix}$ $x - 1 = 8 - 2x \Rightarrow x = 3$ $y - 2 = -2 - 2y \Rightarrow y = 0$ $z - 2 = 10 - 2z \Rightarrow z = 4$ $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$	2 marks – correct solution 1 mark – finds <u>two</u> correct values for x, y, z.

b-i	$v^2 = 2 - x - x^2$ $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $= \frac{1}{2} \times \frac{d}{dx}\left(2 - x - x^2\right)$ $= \frac{1}{2}(-2x - 1)$ $= -x - \frac{1}{2}$ $= -1 \times \left(x - \left(-\frac{1}{2}\right)\right)$ of form $\ddot{x} = -n^2(x - b)$ $\therefore$ simple harmonic motion.	2 marks – correct solution 1 mark – obtains $\ddot{x} = \frac{1}{2}(-2x - 1)$
b-ii	$v^2 = 2 - x - x^2$ $v^2 \geq 0$ $2 - x - x^2 \geq 0$ $(2 + x)(1 - x) \geq 0$ $-2 \leq x \leq 1$ $\therefore$ maximum distance from the origin = 2 metres	2 marks – correct solution 1 mark – forms $2 - x - x^2 \geq 0$ and attempts to solve inequality.
b-iii	Initially at centre of motion $x = -\frac{1}{2}$ $n = 1$, Period $= \frac{2\pi}{n} = 2\pi$ time taken to travel from centre and return to centre $= \frac{1}{2} \times$ period $\therefore t = \pi$ seconds	1 mark – correct answer

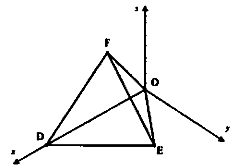
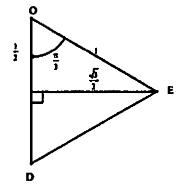
d-ii	$a_1 = 2\ ,\ a_n = \dfrac{a_{n-1}}{n}$ for $n \geq 2 \Rightarrow a_n = \dfrac{2}{n!}$ for $n \geq 1$ $a_1 = \dfrac{2}{1!} = 2$ $a_2 = \dfrac{a_1}{2} = \dfrac{2}{2} = \dfrac{2}{2!}$ Assume true for $n = k$ ie . => $a_k = \dfrac{2}{k\,!}$ RTP for $n = k+1$ ie . => $a_{k+1} = \dfrac{2}{(k+1)!}$ $a_{k+1} = \dfrac{a_k}{k+1}$ $= \dfrac{2}{k!} \times \dfrac{1}{(k+1)}$ $= \dfrac{2}{(k+1)k!}$ $= \dfrac{2}{(k+1)!}$ QED by process of Mathematical induction.	3 marks – correct solution. 2 marks- correct solution for S(1), correct statement for S(k) and S(k+1) and attempts to show S(k)=>S(k+1) 1 mark -correct solution for S(1), correct statement for S(k)
d-i	If composite then Let $n = pq$ where $p < n$, $q < n$ for $n > 1$ If $p \leq \sqrt{n}$ then proposition is true If $p > \sqrt{n}$ then $pq = n \Rightarrow q < \sqrt{n}$ $\therefore$ proposition is true	2 marks – correct solution. 1 mark – one correct statement for $p \leq \sqrt{n}$ or $p > \sqrt{n}$
d-ii	Initial proposition P = n is a composite number Q = a factor $\leq \sqrt{n}$ $P \Rightarrow Q$ Converse n has a factor less than $\sqrt{n}$ => n is composite	1 mark – correct answer

d-iii	Contrapositive: $\neg Q \Rightarrow \neg P$ If n does not have factor $\leq \sqrt{n}$ then not composite (ie prime) If A then B If n is composite, then there exists a factor of n not equal to 1 or itself, and at most equal to $\sqrt{n}$. Contrapositive If NOT B then NOT A If a factor exists greater than $\sqrt{n}$ then n is NOT composite.	2 marks – correct solution 1 mark – attempts to formulate a statement 7Q =>

Question 14

a	$I = \int_0^1 \tan^{-1} dx$ $= \int_0^1 1 \times \tan^{-1} dx$ $u = \tan^{-1}x \qquad dv = 1$ $du = \dfrac{1}{1+x^2} \qquad v = x$ $I = \Big[x\tan^{-1}x\Big]_0^1 - \int_0^1 \dfrac{x}{1+x^2}\,dx$ $= \Big[x\tan^{-1}x\Big]_0^1 - \dfrac{1}{2}\Big[\ln\lvert 1+x^2\rvert\Big]_0^1$ $= [\tan^{-1} 1 - 0] - 1.2[\ln 2 - \ln 1]$ $= \dfrac{\pi}{4} - \ln(\sqrt{2})$	3 marks – correct solution 2 marks – finds a correct primitive function for $\int \tan^{-1} x\ \ dx$ 1 mark – finds correct expressions for *du* and *dv*.
b-i	$\dfrac{x}{y} + \dfrac{y}{x} \geq 2$ $(x-y)^2 \geq 0$ $x^2 - 2xy + y^2 \geq 0$ $x^2 + y^2 \geq 2xy$ $x \Rightarrow \sqrt{\dfrac{x}{y}}$ $y \Rightarrow \sqrt{\dfrac{y}{x}}$ $\left(\sqrt{\dfrac{x}{y}}\right)^2 + \left(\sqrt{\dfrac{y}{x}}\right)^2 \geq 2\sqrt{\dfrac{x}{y}}\sqrt{\dfrac{y}{x}}$ $\left(\sqrt{\dfrac{x}{y}}\right)^2 + \left(\sqrt{\dfrac{y}{x}}\right)^2 \geq 2\sqrt{\dfrac{x}{y} \times \dfrac{y}{x}}$ $\left(\sqrt{\dfrac{x}{y}}\right)^2 + \left(\sqrt{\dfrac{y}{x}}\right)^2 \geq 2$ Or any similar method.	2 marks – correct solution 1mark – states or derives $x^2 + y^2 \geq 2xy$ and indentifies $x \Rightarrow \sqrt{\dfrac{x}{y}}$ and/or $y \Rightarrow \sqrt{\dfrac{y}{x}}$

b-ii	Hence method $(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)$ $= 1+\frac{x}{y}+\frac{y}{x}+1$ $= 2+\frac{x}{y}+\frac{y}{x}$ $\geq 2+2$ ≥ 4 Alternate method $x+y \geq 2\sqrt{xy}$ $(x+y)^2 \geq 4xy$ $\frac{(x+y)^2}{xy} \geq 4$ $\frac{1}{x}+\frac{1}{y}=\frac{y+x}{xy}$ $\therefore (x+y)\frac{x+y}{xy}$ $=\frac{(x+y)^2}{xy}$ $\therefore (x+y)\left(\frac{1}{x}+\frac{1}{y}\right) \geq 4$	1 mark – a correct solution.
b-iii	$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ $=(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)+(x+y)\frac{1}{z}+z\left(\frac{1}{x}+\frac{1}{y}\right)+z\times\frac{1}{z}$ $\geq 4+1+(x+y)\frac{1}{z}+z\left(\frac{1}{x}+\frac{1}{y}\right)$ $\geq 5+\left(\frac{x}{z}+\frac{y}{z}\right)+\left(\frac{z}{x}+\frac{z}{y}\right)$ $\geq 5+\left(\frac{x}{z}+\frac{z}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)$ $\geq 5+2+2$ ≥ 9	2 marks – correct solution 1 mark – expands expression and attempts to simplify using part (i)

c	$\lvert e^{2i\theta}-1\rvert=\sqrt{3}$ for $-\pi<\theta<\pi$ $\lvert \operatorname{cis}(2\theta)-1\rvert=\sqrt{3}$ $(\cos 2\theta-1)^2+\sin^2 2\theta=3$ $\cos^2 2\theta-2\cos 2\theta+1+\sin^2\theta=3$ $-2\cos 2\theta=1$ for $-2\pi<2\theta<2\pi$ $\cos 2\theta=-\frac{1}{2}$ $2\theta=\pm\frac{2\pi}{3},\pm\frac{4\pi}{3}$ $\theta=\pm\frac{\pi}{3},\pm\frac{2\pi}{3}$	3 marks - Four correct solutions 2 marks - Determines $\cos 2\theta=-\frac{1}{2}$ 1 mark - Conversion to mod/arg form -
d-i	Therefore using exact triangle ratios $E=\left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right)$	1 mark – acceptable demonstration
d-ii	Method 1 In $\Delta\ FOD$ $\angle FOD=\angle FOE=\frac{\pi}{3}$ $\cos\angle FOD=\frac{(OF).(OD)}{\lvert OF\rvert\lvert OD\rvert}$ $\cos\frac{\pi}{3}=\frac{(x,y,z)(1,0,0)}{1\times 1}$ $\frac{1}{2}=x$ In $\Delta\ FOE$ $\cos\angle FOE=\frac{(x,y,z).\left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right)}{1\times 1}$ $\frac{1}{2}=\frac{x}{2}+\frac{\sqrt{3}y}{2}+0$ $x+\sqrt{3}y=1$ $\frac{1}{2}+y\sqrt{3}=1$ $y=\frac{1}{2\sqrt{3}}=\frac{\sqrt{3}}{6}$	3 marks – x, y, z correctly derived 2 marks 2 of x, y, z correctly derived 1 mark One of x, y, z correctly derived

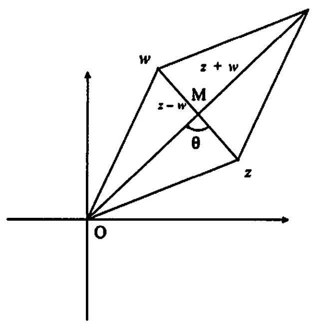
	$\lvert\overrightarrow{OF}\rvert = \sqrt{x^2 + y^2 + z^2}$ $1 = \sqrt{\frac{1}{4} + \frac{1}{12} + c^2}$ $c^2 = 1 - \frac{1}{4} - \frac{1}{2} = \frac{2}{3}$ $c = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$ $F = \left(\frac{1}{2}, \frac{\sqrt{3}}{5}, \frac{\sqrt{6}}{3}\right)$	
d-ii	Alternate acceptable vector approach.	

Q15		
a	$l_1 : r = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad l_2 : r = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ Comparing k components $-1 = 6 - \mu$ $\mu = 7$ Comparing j components $\lambda = 3 + \mu$ $\lambda = 10$ as $\mu = 7$ Comparing i components $1 + \lambda = 1 + 2\mu$ $\lambda = 2\mu$ $10 \neq 14$ Therefore, vectors do not intersect.	3 marks –correct solution 2 marks – - Using μ to find a value for λ 1 mark - Finding value for μ - Forming simultaneous equations but not solving
b	$\forall a(\forall b, a^{\log_e b} = b^{\log_e a})$ Let $a^{\log_e b} = m$ $\log_e (a^{\log_e b}) = \log_e m$ $\log_e b \log_e a = \log_e [a^{\log_e b}]$ $\log_e b \log_e a = \log_e b \log_e a$ $\log_e b \log_e a = \log_e a \log_e b$ $\log_e (a^{\log_e b}) = \log_e \left(b^{\log_e (a)}\right)$ $(a^{\log_e b}) = \left(b^{\log_e (a)}\right)$	2 marks – correct solution 1 mark – takes logs of both sides and completes some simplification of both sides.

c-i	$P(x) = x^4 + 16x^3 + 108x^2 + 400x + 800$ Real coefficients therefore roots in integer pairs Therefore roots are in conjugate pairs Roots are $(a + bi), (a - bi), (3a + bi), (3a - bi)$ $\Sigma \alpha = -\frac{b}{a} = -16$ $\Sigma \alpha \beta = \frac{c}{a} = 108$ $\Sigma \alpha \beta \gamma = -\frac{d}{a} = -400$ $\Sigma \alpha \beta \gamma \theta = \frac{e}{a} = 800$ $(a + 2bi) + (a - 2bi) + (3a + bi) + (3a - bi) = -16$ $8a = -16$ $a = -2$ $(a + 2bi) \times (a - 2bi) \times (3a + bi) \times (3a - bi) = 800$ $(a^2 + 4b^2)(9a^2 + b^2) = 800$ $(4 + 4b^2)(36 + b^2) = 800$ $144 + 148b^2 + 4b^4 = 800$ $36 + 37b^2 + b^4 = 200$ let $u = b^2$ $u^2 + 37u - 164 = 0$ $u = \frac{-37 \pm \sqrt{37^2 + 4 \times (-164)}}{2}$ $b^2 = \frac{-37 \pm 45}{2}$ $b^2 = 4$ or -82 as b is real $b^2 = 4$ $b = \pm 2$ $\therefore$ roots are $(2 - 4i), (2 + 4i), (-6 + 2i)(-6 - 2i)$	3 marks – correct solution 2 marks – finds $a = -2$ and forms a second equation to find b 1 mark - Finds $a = -2$ or $b = -2$ or $b = 2$

c-ii	$P(x) = (x - z)(x - \bar{z}) = x^2 - 2Re(z) + \|z\|^2$ $Re(-2 \pm 4i) = -2 \quad \|(-2 \pm 4i)\|^2 = 20$ $Re(-6 \pm 2i) = -6 \quad \|(-6 \pm 2i)\|^2 = 40$ $P(x) = (x^2 - 2(-2)x + 20)(x^2 - 2(-6)x + 40)$ $= (x^2 + 4x + 20)(x^2 + 12x + 40)$	2 marks – correct solution 1 mark - Forms one correct quadratic factor - Correctly forms two correct quadratic factors from incorrect roots.

1

d-i	 Area $= \frac{1}{2}ab\sin\theta$ $\Delta\, Owz = 1$ $\Delta\, Owz = \frac{1}{2}[ab\sin\theta + ab\sin(180-\theta)]$ $= \frac{1}{2}[2ab\sin\theta]$ $1 = \frac{1}{2}\lvert z+w\rvert\frac{1}{2}\lvert z-w\rvert\sin\theta$ $4 = \lvert z+w\rvert\lvert z-w\rvert\sin\theta$	2 marks – correct solution 1 mark - Identifies $\overrightarrow{OM} = \frac{1}{2}(z+w)$ and $\overrightarrow{MZ} = \frac{1}{2}(z-w)$ - Or equivalent.
d-ii	Method 1 $c^2 = a^2 + b^2 - 2ab\cos\theta$ $\lvert z\rvert^2 = \left\lvert\frac{1}{2}(z-w)\right\rvert^2 + \left\lvert\frac{1}{2}(z+w)\right\rvert^2 - 2\left\lvert\frac{1}{2}(z-w)\right\rvert\left\lvert\frac{1}{2}(z+w)\right\rvert\cos\theta \ldots (a)$ $\lvert w\rvert^2 = \left\lvert\frac{1}{2}(z-w)\right\rvert^2 + \left\lvert\frac{1}{2}(z+w)\right\rvert^2 - 2\left\lvert\frac{1}{2}(z-w)\right\rvert\left\lvert\frac{1}{2}(z+w)\right\rvert\cos(180-\theta) \ldots (b)$ $-\cos\theta = \cos(180-\theta)$ $(a) + (b)$ $\lvert z\rvert^2 + \lvert w\rvert^2 = 2 \times \frac{1}{4}[\,\lvert(z-w)\rvert^2 + \lvert(z+w)\rvert^2\,]$ $\lvert z\rvert^2 + \lvert w\rvert = \frac{\lvert(z-w)\rvert^2 + \lvert(z+w)\rvert^2}{2}$	2 marks- correct solution 1 mark – forms a correct cosine rule expression for $\lvert z\rvert^2$ or $\lvert w\rvert^2$

d-ii	Method 2 – Dot Product method. (Shorter) $\lvert z+w\rvert^2 + \lvert z-w\rvert^2$ $= (z+w)\cdot(z+w) + (z-w)\cdot(z+w)$ $= z\cdot z + 2 \times z\cdot w + w\cdot w + z\cdot z - 2z\cdot w + w\cdot w$ $= 2\lvert z\rvert^2 + 2\lvert w\rvert^2$ $\therefore$ $\lvert z\rvert^2 + \lvert w\rvert^2 = \frac{\lvert z+w\rvert^2 + \lvert z-w\rvert^2}{2}$	2 marks – correct solution 1 mark – writes $\lvert z+w\rvert^2$ and $\lvert z-w\rvert^2$ in dot product form.
d-iii	Option 1 If $\theta = \frac{\pi}{2}$ then the parallelogram must be a rhombus therefore $\lvert z\rvert = \lvert w\rvert$ $\therefore \quad \lvert z\rvert - \lvert w\rvert = 0$ $\therefore (\lvert z\rvert - \lvert w\rvert)(\lvert z\rvert + \lvert w\rvert) = 0$	1 mark – correct answer.
d-iii	Option 2 If If $\theta = \frac{\pi}{2}$ then $(z-w)\cdot(z+w) = 0$ $(z-w)\cdot(z+w)$ $= z\cdot z + z\cdot w - z\cdot w - w\,w$ $= \lvert z\rvert^2 - \lvert w\rvert^2$ $= (\lvert z\rvert + \lvert w\rvert)(\lvert z\rvert - \lvert w\rvert)$ $\therefore$ $(\lvert z\rvert + \lvert w\rvert)(\lvert z\rvert - \lvert w\rvert) = 0$	
	Note – do NOT use binomial expansion incorrectly ie. $(\lvert z+w\rvert)(\lvert z-w\rvert) \neq \lvert z\rvert^2 - \lvert z\rvert\lvert w\rvert + \lvert z\rvert\lvert w\rvert - \lvert w\rvert^2$	

Question 16

a	$3x - 4y = 11$ $\begin{pmatrix}1\\2\end{pmatrix}$ lies on the line Gradient of the line $\frac{3}{4}$ Vector equation $\underset{\sim}{v} = \begin{pmatrix}1\\2\end{pmatrix} + \lambda\begin{pmatrix}4\\3\end{pmatrix}$	2 marks - Correct gradient vector - A correct initial vector
b-i	Vector equation $\left\lvert \vec{v} - \begin{pmatrix}1\\3\\1\end{pmatrix} \right\rvert = 14$ $\left\lvert \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}1\\3\\1\end{pmatrix} \right\rvert = 14$ (Cartesian equation) $(x-1)^2 + (y-3)^2 + (z-1)^2 = 14^2$	1 mark for a vector equation
b-ii	As radius is perpendicular to tangent then Radius vector = $(2-1, 1-3, (4-1))$ Tangent Vector = $(x-2, y-1, (z-4))$ $(x-2, y-1, (z-4)).(2-1, 1-3, (4-1)) = 0$ $(x-2) - 2(y-1) + 3(z-4) = 0$ $x - 2y + 3z - 2 + 2 - 12 = 0$ $x - 2y + 3z = 12$	3 marks – correct solution 2 marks – correctly states radius and tangent vectors - Finds and equation from one/ two incorrect equations using dot product 1 mark - Finds one correct radius or tangent - Attempts to apply dot product to two vectors.

c	$f(x) = e^x - x$ let $x = 0$ $f(0) = e^0 - 0 = 1 > 0$ $f'(x) = e^x - 1 > 0$ for $x \geq 0$ as $e^x > 1$ for $x \geq 0$ $\therefore$ as $f(0) = 1 > 0$ and $f'(x) > 0$ $\therefore$ $e^x - x > 0 ; x \geq 0$ $e^x > x , x \geq 0$	2 marks – correct solution 1 mark – attempts to apply $f'(x) > 0$
	Let P_n represent the proposition $e^x \geq \frac{x^n}{n!}$ $n = 1$ – true – proven in part i ie. $e^x > x \Rightarrow e^x > \frac{x^1}{1!}$ Assume true for $n = k$ $e^x > \frac{x^k}{k!}$ RTP true for $n = k + 1$ Form assumption $e^x > \frac{x^k}{k!}$ $\int_0^x e^x \, dx > \int_0^x \frac{x^k}{k!} \, dx$ $\left[e^x\right]_0^x > \left[\frac{x^{k+1}}{(k+1)k!}\right]_0^x$ $e^x > \frac{x^{k+1}}{(k+1)!} - 0$ $e^x > \frac{x^{k+1}}{(k+1)!}$ $P(k) \Rightarrow P(k+1)$ QED by process of Mathematical Induction.	3 marks – correct solution 2 marks - show S(1) is true and states S(k), S(k+1) correctly and attempts to use calculus method to establish the result. 1 mark – establishes S(1) and correctly state S(k) and S(k+1)

d	Method 1. **A partial fractions approach and some observation.** $\int \frac{\log_e x}{(1+\log_e x)^2}\,dx$ $\int \frac{1+\log_e x - 1}{(1+\log_e x)^2}\,dx$ $\int \frac{1+\log_e x}{(1+\log_e x)^2} - \frac{1}{(1+\log_e x)^2}\,dx$ $\int \frac{1}{(1+\log_e x)} - \frac{1}{(1+\log_e x)^2}\,dx$ if $y = \frac{x}{1+\ln x} = x(1+\ln x)^{-1}$ (*by* observation) $\frac{dy}{dx} = \frac{1}{1+\ln x} + x \times \left(-\frac{1}{x}\right) \times \frac{1}{(1+\ln x)^2}$ $= \frac{1}{(1+\log_e x)} - \frac{1}{(1+\log_e x)^2}$ $\therefore \int \frac{\log_e x}{(1+\log_e x)^2}\,dx = \frac{x}{1+\ln x}$	**4 marks**

d	Method 2 – a more traditional approach $I = \int \frac{\ln x}{(1+\ln x)^2}\,dx$ let $u = \ln x \;=\; .x = e^u$ $du = \frac{1}{x}dx \;\Rightarrow\; x\,du = dx$ $I = \int \frac{u}{(1+u)^2}\, x$ du $= \int \frac{u}{(1+u)^2}\, e^u$ du $= \int \frac{e^u \times u}{(1+u)^2}$ du $= \int \frac{e^u \times u + e^u - e^u}{(1+u)^2}$ du $= \int \frac{(1+u)e^u - e^u}{(1+u)^2}$ du $= \int \frac{(1+u)e^u}{(1+u)^2}du - \int \frac{e^u}{(1+u)^2}$ du $= \int \frac{e^u}{1+u}$ du $- \int \frac{e^u}{(1+u)^2}$ du	4 marks- correct solution. 3 marks- $I = \int \frac{e^u}{1+u}du - \int \frac{e^u}{1+u}^2 du$ Or equivalent expression, and then attempts to use IBP for $\int \frac{e^u}{(1+u)^2}\,du$ 2 marks Obtains $I = \int \frac{e^u}{1+u}du - \int \frac{e^u}{1+u}^2 du$ But no further progress. 1 mark – obtains $I = \int \frac{ue^u}{(1+u)^2}$ du

	$I = \int \frac{e^u}{1+u}\,du - \int \frac{e^u}{(1+u)^2}\,du$ Complete second integral by parts $I_2 = \int \frac{e^u}{(1+u)^2}\,du$ $u = e^u \quad dv = (1+u)^{-2}$ $du = e^u \quad v = -\frac{1}{1+u}$ $I_2 = \frac{-e^u}{1+u} - \int \frac{-e^u}{1+u}\,\text{du}$ $\text{I} = \int \frac{e^u}{1+u}\,du - I_2$ $= \int \frac{e^u}{1+u}\,du - \left\{ \frac{-e^u}{1+u} - \int \frac{-e^u}{1+u}\,du \right\}$ $= \frac{e^u}{1+u}$ $= \frac{e^{\ln x}}{1+\ln x}$ $= \frac{x}{1+\ln x} + C$	

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