The Scots College

Number: ______________________

Teacher:______________________

2021 **HSC Year 12 Assessment Task 4**

Mathematics Extension 2

TRIAL EXAMINATION

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this document
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 3-6)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7-11)

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

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Section I

10 marks
Attempt Questions 1-0
Allow about 15 minutes for this section

Submit answers only on your page 1 for Questions 1-10.

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1 Let $z=\sqrt{3}+i$. The value of $\overline{\left(\frac{i}{z}\right)}$ is:

A. $1-i\sqrt{3}$

B. $\dfrac{1-i\sqrt{3}}{4}$

C. $\dfrac{-1+i\sqrt{3}}{4}$

D. $\dfrac{\sqrt{3}-i}{4}$

2 If $\dfrac{4}{(x+1)(x-1)^2}=\dfrac{a}{x+1}-\dfrac{1}{x-1}+\dfrac{b}{(x-1)^2}$, then the values of a and b are:

A. $a=1,\ b=2$

B. $a=1,\ b=-2$

C. $a=-1,\ b=2$

D. $a=-1,\ b=-2$

3 Which of the following integrals uses a correct substitution for $\displaystyle\int_0^{\sqrt{3}}\frac{\ln\left(\tan^{-1}x\right)}{1+x^2}dx$?

A. $\displaystyle\int_0^{\frac{\pi}{3}}\ln u\,du$

B. $\displaystyle\int_0^{\frac{\pi}{3}}\frac{\ln u}{1+\tan^2 u}du$

C. $\displaystyle\int_0^{\sqrt{3}}\ln u\,du$

D. $\displaystyle\int_0^{\sqrt{3}}\frac{\ln u}{1+\tan^2 u}du$

4 The scalar projection of $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{k}$ onto $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$ is:

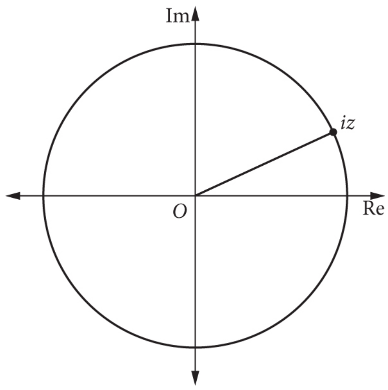
A. $\frac{8}{\sqrt{10}}$

B. $\frac{8}{\sqrt{10}}\left(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

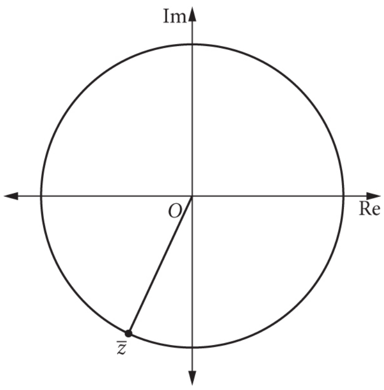
C. $\frac{4}{5}\left(3\underset{\sim}{i} - \underset{\sim}{k}\right)$

D. $\frac{8}{3}$

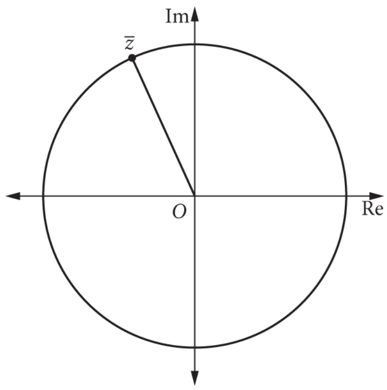
5 The complex number iz is shown in the Argand diagram below. The scale on all axes is the same.



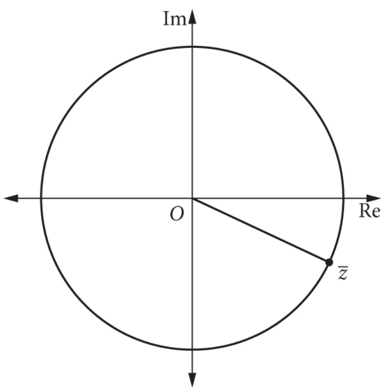
Which of the following diagrams represents $\overline{z}$?



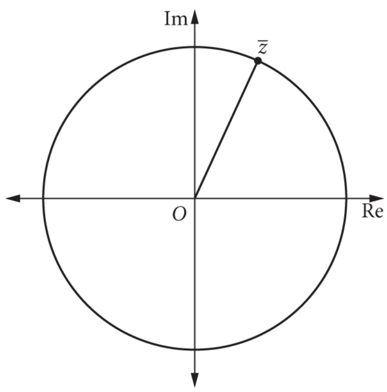
A.



B.



C.



D.

6 A particle moves 6 cm either side of a central point with Simple Harmonic Motion. The period of the motion is 6 seconds.

What is its maximum speed?

A. 0.5 cm/s

B. 1 cm/s

C. π cm/s

D. 2π cm/s

7 The polynomial $P(z)$ has real coefficients. Four of the roots are $z = 1$, $z = 1 + 2i$, $z = 1 - 2i$ and $z = 2i$. The minimum number of roots that the equation could have is:

A. 4

B. 5

C. 6

D. 7

8 Consider the statement: n is a multiple of 3 and n is divisible by 7.
What is the negation of this statement?

A. n is not a multiple of 3 and n is not a multiple of 7

B. n is either a multiple of 3 or a multiple of 7

C. n is not a multiple of 3 or n is not a multiple of 7

D. n is a multiple of 7 that is not divisible by 3

9 If $\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -m\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, where m is a real constant, the vector $\underset{\sim}{a} - \underset{\sim}{b}$ will be perpendicular to vector $\underset{\sim}{b}$ where m equals:

A. 0 only

B. 2 only

C. 0 or 2

D. 0 or -2

10 The acceleration of a particle moving in a straight line with velocity v is given by $\ddot{x} = \sqrt{v}$. Initially $v = 1$. What is v as a function of t?

A. $\left(\frac{t+2}{2}\right)^2$

B. $\left(\frac{2-t}{2}\right)^2$

C. $\left(\frac{1}{1+t}\right)^2$

D. $\left(\frac{2}{t-2}\right)^2$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

In questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Evaluate $\int_0^1 \frac{1+x}{1+x^2}dx$. **3**

(b) If $z = a + ib$, $b \neq 0$, simplify:

(i) $\left|\frac{\bar{z}}{z}\right|$ **1**

(ii) $\frac{i(\text{Re}(z) - z)}{\text{Im}(z)}$ **1**

(iii) $\arg z + \arg\left(\frac{1}{z}\right)$ **1**

(iv) $\frac{z\bar{z}}{|z|^2}$ **1**

(c) Use de Moivre's theorem to find the values of n for which $(\sqrt{3}+i)^n - (\sqrt{3}-i)^n = 0$, where n is a positive integer. . **4**

(d) (i) Draw a neat sketch of the locus defined by $|z|^2 - 2iz + 2t(1+i) = 0$, where $z = x + iy$ and x, y and t are real numbers. **3**

(ii) For what values of t can x and y be found so that z satisfies the given equation? **1**

Question 12 (15 marks)

(a) Prove by contradiction that $\log_4 6$ is irrational. **3**

(b) Prove that $3x^2 - 4xy + 3y^2 > 0$ for all real $x, y \neq 0$. **2**

(c) Consider the vector lines $\underset{\sim}{a} = \begin{bmatrix} 8 \\ 16 \\ 6 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 14 \\ 12 \\ 10 \end{bmatrix} + \phi \begin{bmatrix} 12 \\ 8 \\ 10 \end{bmatrix}$.

(i) Find the point where the lines intersect. **2**

(ii) Find the acute angle between the lines (to the nearest degree). **3**

(d) The diagonals of a parallelogram are given by the vectors:

$$\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j} - \underset{\sim}{k} \text{ and } \underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$$

(i) Show that the parallelogram is a rhombus. **1**

(ii) Find the length of the sides. **2**

(iii) Calculate the internal angles between the sides (nearest degree). **2**

Question 13 (15 marks)

(a) Evaluate $\int_1^2 x^2 \log_e x \, dx$. **3**

(b) The speed v m s^{-1} of a particle moving along the x-axis is given by $v^2 = 18 + 32x - 8x^2$, where x is the distance of the point from the origin.

(i) Prove that the motion is simple harmonic. **2**

(ii) For this motion, find:

a. the centre of motion. **1**

b. the period. **1**

c. the amplitude. **2**

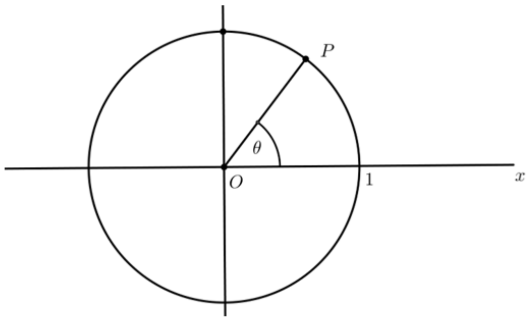
(c) Let $I_n = \int_0^2 \left(4 - x^2\right)^n dx$ where n is an integer and $n \geq 0$.

(i) Show that $I_n = \frac{8n}{2n+1} I_{n-1}$. **4**

(ii) Hence find I_3. **2**

Question 14 (15 marks)

(a) In the Argand diagram below, point P lies in the first quadrant on the unit circle. P represents the complex number ω and $z = \omega$ is a root of $z^5 - 1 = 0$.
Let $\angle POx = \theta$.



(i) Show that $\theta = 72°$. **1**

(ii) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. **2**

(iii) If $a = \omega + \omega^4$ and $b = \omega^2 + \omega^3$ show that $a + b = -1$ and $ab = -1$. **2**

(iv) Show that $(a - b)^2 = 5$. **1**

(v) Given $(a - b) > 0$, find the exact value of $\cos 72°$. **3**

(b) (i) Show that $\int_0^a f(x)dx = \int_0^a f(a - x)dx$. **2**

(ii) Hence show that $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan x}\right)dx$. **2**

(iii) Hence evaluate $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)dx$. **2**

Question 15 (15 marks)

(a) A particle of mass m kg is falling from rest and experiences air resistance of mkv^2 Newtons, where k is a positive constant and v m/s is the velocity of the particle. Acceleration due to gravity is g m/s^2.

(i) Draw the force diagram and use it to show that the equation of motion of the particle is $\ddot{x} = g - kv^2$, where x metres is the distance the particle fell from its original position. **1**

(ii) Explain how the value of the terminal velocity, W m s^{-1}, of the particle can be obtained and state its value in terms of k and g. **1**

(iii) Show that the velocity of the particle, v m s^{-1}, at t seconds is given by

$v = W\left(\frac{e^{2kWt}-1}{e^{2kWt}+1}\right)$. **3**

(iv) Show that the position of the particle, x metres, in terms of v is given by

$x = \frac{1}{2k}\log_e\left|\frac{g}{g-kv^2}\right|$. **2**

(b) Prove by induction that $\displaystyle\sum_{n=1}^{N}\frac{1}{(2n+1)(2n-1)} = \frac{N}{2N+1}$. **3**

(c) Using the substitution $t = \tan\frac{x}{2}$ find the primitive of $\dfrac{1}{1+\cos x - \sin x}$ **3**

(d) Consider the complex numbers $u = 1+2i$, $v = -2+6i$ and z. What is the minimum possible value of $|z-u|+|v-z|$? Give reasons for your answer. **2**

Question 16 (15 marks)

(a) The point $\underset{\sim}{v}$ is a point on the surface of a sphere with centre $P(1,3,1)$ and radius 14 units.

(i) Write down the vector equation of the sphere. **1**

(ii) Find the **Cartesian** equation of the tangent plane to the sphere at $Q(2,1,4)$ **3**

(You may assume that the radius drawn to the point of contact of the tangent is perpendicular to the tangent.)

(b) When a projectile is fired with velocity $V\,ms^{-1}$ at an angle θ above the horizontal the horizontal and vertical displacements (in metres) from the point of projection at time t seconds are given by $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$ respectively (where g is the acceleration due to gravity).

(i) A projectile is fired horizontally with velocity $V\,ms^{-1}$ from the top of a cliff. The projectile hits a fixed target in the water after T seconds.

Show that the height of the cliff is given by $H = \frac{1}{2}gT^2$ metres. **1**

(ii) It is found that the target can also be hit by firing a projectile from the top of the cliff at an angle α above the horizontal with the same initial speed $\left(V\,ms^{-1}\right)$.

Show that $\tan\alpha = \dfrac{2V}{gT}$. **4**

(c) The function $F(p)$ is defined as $F(p) = \lim\limits_{t\to\infty}\int_0^t x^{p-1}e^{-x}\,dx$, for $p > 0$.

(i) Show that $F(1) = 1$. **2**

(ii) Use integration by parts to show $F(p+1) = pF(p)$. **2**

(iii) Hence find $F(n)$ for integers $n \geq 1$. **2**

End of paper