Name:……

Teacher:……

Mathematics Extension 2
HSC Trial Examination
Term 3 2022

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided with the Answer Booklet
- For questions in Section II, show relevant mathematical reasoning or calculations

Total marks 100

Section 1 – 10 marks (pages 1-4)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 5-11)

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section
- Answer each question in the appropriate space in the Answer Booklet. Extra writing pages are included at the end of each question.

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Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. The equation $P(x) = x^3 - 5x^2 + 9x + c = 0$ (where c is real) has a root $x = 2 - i$.
Which of the following is a correct factorisation of $P(x)$?

(A) $P(x) = (x^2 - 4x + 5)(x + 1)$

(B) $P(x) = (x^2 - 4x + 5)(x - 1)$

(C) $P(x) = (x^2 - 4x - 5)(x + 1)$

(D) $P(x) = (x^2 - 4x - 5)(x - 1)$

2. What is $\int \frac{dx}{x^2 + 4x + 9}$?

(A) $\frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$

(B) $\frac{1}{5} \tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$

(C) $\frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$

(D) $\frac{1}{5} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$

3. For the complex polynomial $P(z) = z^3 + az^2 + bz + c$ with real coefficients a, b and c, $P(-2) = 0$ and $P(3i) = 0$.
What are the values of a, b and c are respectively?

(A) $-2, 9, -18$

(B) $2, 9, 18$

(C) $-3, -4, 12$

(D) $2, -9, -18$

4. Consider the 3 lines given below in parametric form:

$$\underset{\sim}{r}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + a\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

$$\underset{\sim}{r}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\underset{\sim}{r}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + c\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Which pair of lines are skew?

(A) $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$

(B) $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_3$

(C) $\underset{\sim}{r}_2$ and $\underset{\sim}{r}_3$

(D) There are no pairs of skew lines.

5. What is $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\cos x \; dx$?

(A) $\frac{\pi}{2} - 1$

(B) $2\int_{0}^{\frac{\pi}{2}} x\cos x \; dx$

(C) 0

(D) $\pi - 2$

6. Two particles oscillate horizontally. The displacement of the first particle is given by $x = -6\sin 4t$ and the displacement of the second one is given by $x = a\cos nt$.
In one complete oscillation, the second particle covers half the distance of the first particle, but in double the time. What are the values of a and n?

(A) $a = 3$ and $n = 8$

(B) $a = 6$ and $n = 8$

(C) $a = 6$ and $n = 2$

(D) $a = 3$ and $n = 2$

7. A rock of unit mass falls vertically from rest from the top of a cliff, in a medium whose air resistance is proportional to the velocity of the rock. If the rock falls to ground level under the influence of *g*, the acceleration due to gravity, which of the following is the correct expression for the velocity of the rock, given that downwards is taken to be the positive direction?

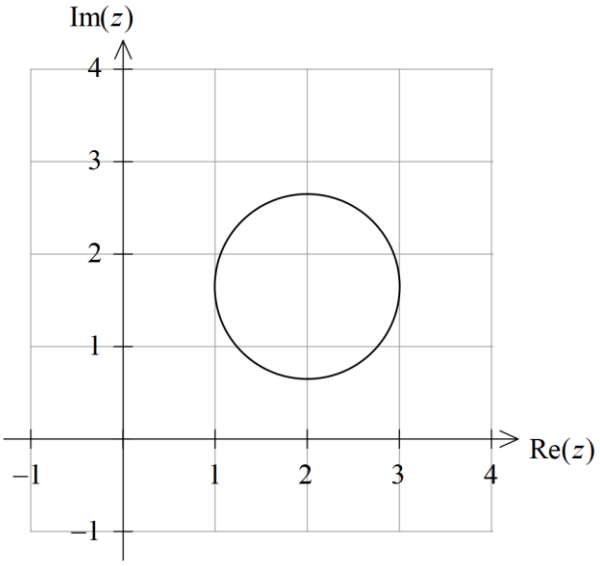
(A) $v = \frac{g}{k}\left(1+e^{-kt}\right)$

(B) $v = \frac{g}{k}\left(1-e^{-kt}\right)$

(C) $v = \frac{-g}{k}\left(e^{-kt}+1\right)$

(D) $v = \frac{g}{k}\left(e^{-kt}-1\right)$

8. The graph of the circle given by $\left|z-2-\sqrt{3}i\right|=1,$ where $z \in C,$ is shown below.



For points on this circle, what is the maximum value of $|z|$?

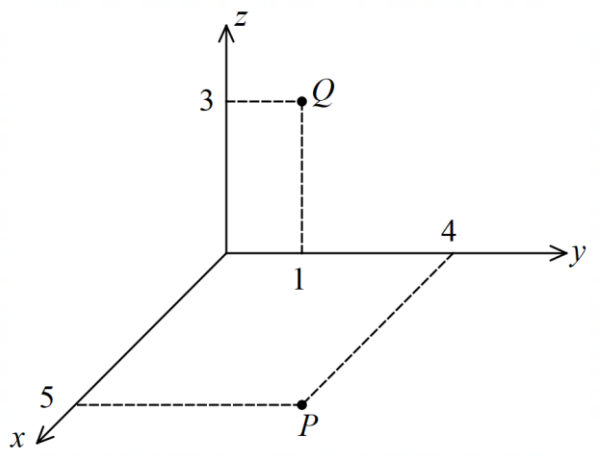
(A) $\sqrt{3}+1$

(B) 3

(C) $\sqrt{13}$

(D) $\sqrt{7}+1$

9.



Consider the points P and Q given on the diagram. If M is the midpoint of PQ, what is the position vector, $\overrightarrow{OM}$?

(A) $\frac{5}{2}\underset{\sim}{i}+\frac{5}{2}\underset{\sim}{j}+\frac{3}{2}\underset{\sim}{k}$

(B) $\frac{5}{2}\underset{\sim}{i}+2\underset{\sim}{j}+\frac{3}{2}\underset{\sim}{k}$

(C) $\frac{5}{2}\underset{\sim}{i}+\frac{3}{2}\underset{\sim}{j}+\frac{3}{2}\underset{\sim}{k}$

(D) $\frac{5}{2}\underset{\sim}{i}+\frac{3}{2}\underset{\sim}{j}+\underset{\sim}{k}$

10. For $z \in C$, if $\text{Im}(z) > 0$, then what is $\arg\left(\frac{z\,\overline{z}}{z-\overline{z}}\right)$?

(A) $-\frac{\pi}{2}$

(B) 0

(C) $\frac{\pi}{2}$

(D) π

Section II

Name:……

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Teacher:……

Answer these questions in the Answer Book provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) z is the complex number $\dfrac{1+3i}{1-2i}$.

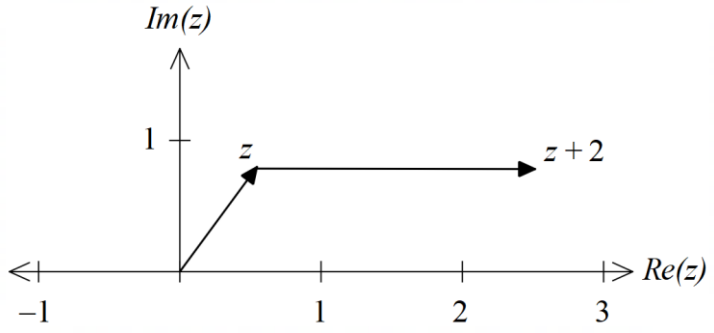
(i) Find z in the form $x+iy$, where x and y are real numbers. **2**

(ii) Find $|z|$. **1**

(iii) Find $\arg(z)$. **1**

(iv) Find z^4. **1**

(b) The Argand diagram below shows the complex numbers z and $z+2$ where $z = cis\left(\dfrac{\pi}{3}\right)$.



Determine the exact value for:

(i) $\arg(-z)$. **1**

(ii) $|z+2|$. **2**

Question 11 continues on page 6

(c) Express $\dfrac{x}{(x+1)(x^2+1)}$ in the form $\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}$ and hence find **4**

$$\int \frac{x}{(x+1)(x^2+1)}\,dx.$$

(d) Evaluate $\displaystyle\int_0^1 xe^{2x}\,dx.$ **3**

End of Question 11

Question 12 (15 marks) **Marks**

(a) A particle is moving in a straight line according to the equation $x = 5 + 6\cos 2t + 8\sin 2t$, where x is the displacement in metres and t is the time in seconds.

(i) Prove that the particle is moving in simple harmonic motion. **2**

(ii) When is the displacement of the particle zero for the first time? **3**

(b) Consider the vector equation $\underset{\sim}{r} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \lambda_1 \in \mathbb{R}$.

(i) Find the vector equation of the line $\underset{\sim}{q}$ which is parallel to $\underset{\sim}{r}$ and passes through the point $C(5, 4, 2)$. **2**

(ii) Show that the vector $\underset{\sim}{p} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ is perpendicular to $\underset{\sim}{r}$. **1**

(iii) Find the point of intersection between $\underset{\sim}{r}$ and the line $\underset{\sim}{l} = \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix} + \lambda_2 \underset{\sim}{p}, \quad \lambda_2 \in \mathbb{R}$. **3**

(c) The complex number $u = 5 + mi$ has $|u| = 6$. **2**

Given that $0 < \arg(u) < \dfrac{\pi}{2}$, find the exact value of the real number m.

(d) Shade the region on the Argand diagram defined by the two inequalities **2**

$$|z - 2| \le 2 \text{ and } 0 \le \arg(z - 2) \le \frac{\pi}{4}.$$

End of Question 12

Question 13 (14 marks) **Marks**

(a) (i) Given $t = \tan x$, show that $\dfrac{dx}{dt} = \dfrac{1}{1+t^2}$. **1**

(ii) Use the substitution $t = \tan x$ to find $\displaystyle\int \frac{1}{1+\sin 2x}\,dx$. **3**

(b) A particle is moving along the x-axis in simple harmonic motion centred at the origin. When $x = 4\,\text{m}$ the velocity of the particle is 8 m/s, and when $x = 10\,\text{m}$ the velocity of the particle is 6 m/s. Find the period of motion. **3**

(c) A solid is defined in three-dimensional space by the equation

$$(x-2)^2 + (y+1)^2 = 9 \text{ where } -1 \le z \le 3.$$

(i) Describe the solid geometrically, including, but not limiting to, the significant points on the top and bottom surfaces. **1**

(ii) Write down the equation of a plane that is tangential to the side of the solid. **1**

(iii) Given that the line $\underset{\sim}{r} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$ intersects with the top and bottom surfaces of the solid, find the coordinates of the points of intersection. **2**

(d) Prove by mathematical induction that $3^n > n^3$ for all integers $n \ge 4$. **3**

End of Question 13

Question 14 (17 marks) **Marks**

(a) Consider the two spheres S_1 and S_2 with equations given below.

$$S_1 : (x-1)^2 + (y+1)^2 + (z-2)^2 = 16$$
$$S_2 : (x+1)^2 + (y-3)^2 + (z+2)^2 = 4$$

(i) Show that S_1 and S_2 are tangential to each other. **2**

(ii) Find the coordinates of the common point. **2**

(b) Find $\int \sqrt{\frac{4-x}{2+x}}\, dx$. **3**

(c) Given that the real part of $\frac{z-2i}{z-4}$ is zero and $z \neq 4$, find the locus of points described by z. **2**

(d) Solve the equation $z^2 = i\left(|z|^2 - 4\right)$. **3**

(e) (i) A particle of mass m is projected vertically upwards under gravity. The air resistance to the motion being $\frac{mgv^2}{k^2}$ when the speed is v, where k is a constant. Let the acceleration due to gravity be g, show that during the upward motion of the particle: **2**

$$v\frac{dv}{dx} = -\frac{g}{k^2}\left(k^2 + v^2\right).$$

(ii) Show that the greatest height reached, given the speed of the projection u, is **3**

$$\frac{k^2}{2g}\ln\left(1+\frac{u^2}{k^2}\right).$$

End of Question 14

Question 15 (16 marks) **Marks**

A 2 kg particle is projected upwards at a speed of 400 ms^{-1} at an angle $30°$ to the horizontal. It experiences air resistance opposite to the direction of its motion of $\frac{2}{5}v$ Newtons, where v is the velocity of the particle. Let the acceleration g due to gravity be 10 ms^{-2}.

(a) (i) Show that $\ddot{x} = -\frac{1}{5}\dot{x}$ and $\ddot{y} = -10 - \frac{1}{5}\dot{y}$ where x is the horizontal displacement and y is the vertical displacement. **1**

(ii) Find the initial velocities in the horizontal and vertical directions. **1**

(iii) Show that $\dot{x} = 200\sqrt{3}e^{\frac{-t}{5}}$ and $x = 1000\sqrt{3}\left(1 - e^{\frac{-t}{5}}\right)$. **2**

(iv) Determine the maximum possible horizontal range of the projectile in theory (the range it can never reach). **1**

(v) Show that $\dot{y} = 250e^{\frac{-t}{5}} - 50$ and $y = 1250e^{\frac{-t}{5}} - 50t + 1250$. **3**

(vi) Find the maximum height of the projectile, correct to the nearest metre. **2**

(b) The trapezium $PQRS$ is formed by the following lines, PQ, SP, RQ and RS with vector equations given respectively.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 19 \end{bmatrix} + \lambda_1 \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 33 \\ -5 \end{bmatrix} + \lambda_3 \begin{bmatrix} -11 \\ 16 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 43 \\ -9 \end{bmatrix} + \lambda_4 \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

(i) Which lines are parallel? Justify your answer? **1**

(ii) Which lines are perpendicular? Justify your answer? **1**

(iii) Given $Q(22,11)$ and $R(33,-5)$, use vectors to find the coordinates of P and S. **2**

(iv) Find the area of the trapezium $PQRS$. **2**

End of Question 15

Question 16 (13 marks) **Marks**

(a) Given $\int \frac{dx}{\sqrt{a^2+x^2}} = \log_e \left| x+\sqrt{x^2+a^2} \right| + C$ and $I_n = \int_0^{\frac{3}{2}} \left(4+x^2\right)^{-\frac{1}{2}n} \, dx$

(i) Find the exact value of I_1. **2**

(ii) Show that $4nI_{n+2} = \frac{3}{2}\left(\frac{2}{5}\right)^n + (n-1)I_n$. **5**

(iii) Find the value of I_5. **2**

(b) The Fibonacci sequence F_n is defined by $F_0 = 0$ and $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. **4**

Use Mathematical Induction to prove that $F_1F_2 + F_2F_3 + ... + F_{2n-1}F_{2n} = \left(F_{2n}\right)^2$.

END OF PAPER