Northern Beaches Secondary College

Manly Campus

MATHEMATICS EXTENSION 2

2025 HSC Assessment Task 4
Thursday, 14 August 2025

General instructions

- Working time – 3 hours.
  (plus 10 minutes reading time)
- Write using black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt all questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or sloppy working.

NESA STUDENT #: .... **# BOOKLETS USED:** ....

Marker's use only

QUESTION	1-10	11	12	13	14	15	16	Total	%
MARKS	$\overline{}$ 10	$\overline{}$ 15	$\overline{}$ 15	$\overline{}$ 15	$\overline{}$ 15	$\overline{}$ 15	$\overline{}$ 15	$\overline{}$ 100	

Section I

10 marks
Attempt Questions 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided.

Questions

1. If $w = a + 2ai$, which of the following is equivalent to $i^3 w$?

A. $2a - ai$

B. $a + 2ai$

C. $a - 2ai$

D. $2a + ai$

2. The sign on a fence reads:

"If you are not authorised,
you may not enter."

Which of the following statements is logically equivalent to the one above?

A. If you may enter, then you are authorised

B. If you are authorised, then you may enter

C. If you are not authorised, then you may enter

D. If you may not enter, then you are not authorised

3. Consider the following statement: . $\forall x, y \in \mathbb{R}, xy \leq 9 \Rightarrow x \leq 3$ or $y \leq 3$.

Which of the following is the negation of this statement?

A. $\forall\, x , y \in \mathbb{R}$, $x \leq 3$ or $y \leq 3 \Rightarrow xy \leq 9$

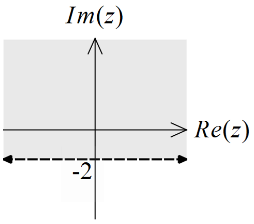
B. $\forall\, x , y \in \mathbb{R}$, $x > 3$ and $y > 3 \Rightarrow xy > 9$

C. $\exists\, x , y \in \mathbb{R}$, with $x > 3$ or $y > 3$ such that $xy \leq 9$

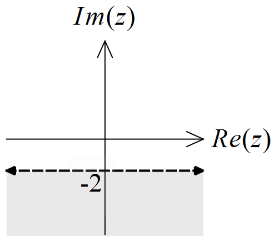
D. $\exists\, x , y \in \mathbb{R}$, with $x > 3$ and $y > 3$ such that $xy \leq 9$

4. Which of the following is the region in the Argand diagram specified by $iz - i\bar{z} > 4$?

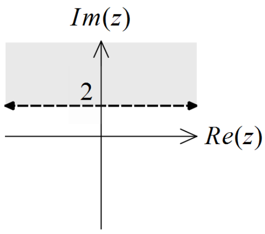
A.



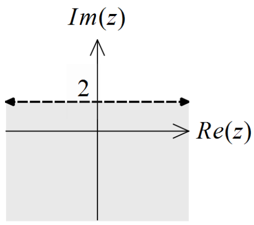
B.



C.



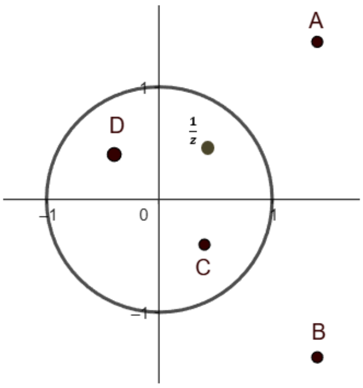
D.



5. A vector makes equal angles of $\frac{\pi}{4}$ with the positive directions of the $x-$axis and the $y-$ axis. What angle does the vector make with the positive direction of the $z-$ axis?

A. 0 B $\frac{\pi}{4}$ C. $\frac{\pi}{2}$ D. $\frac{3\pi}{4}$

6. The complex number $\frac{1}{z}$ is shown on the Argand diagram below:



Which one of the points could correspond to the complex number z?

A. A B. B C. C D. D

7. Given that $2 - i$ is a root of $P(z) = z^4 - 2z^3 + az^2 + bz + 50$, where $a, b \in \mathbb{R}$, which one of the following is a factor of $P(z)$?

A. $z^2 + 4z - 5$

B. $z^2 + 4z + 5$

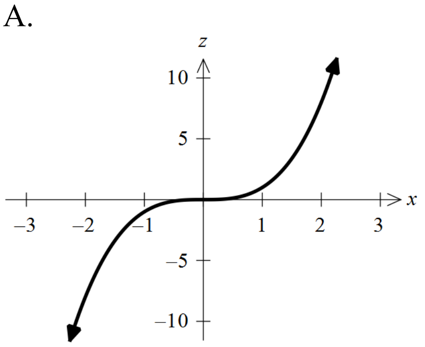
C. $z^2 + 2z - 10$

D. $z^2 + 2z + 10$

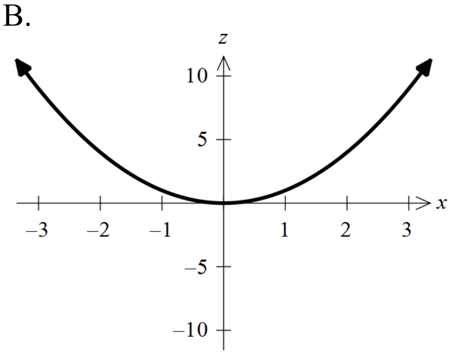
8. The curve $r(t) = (t^2, t^4, t^6)$ is defined for all real values of t.

Which of the following graphs best represents its **projection onto the xz-plane?**

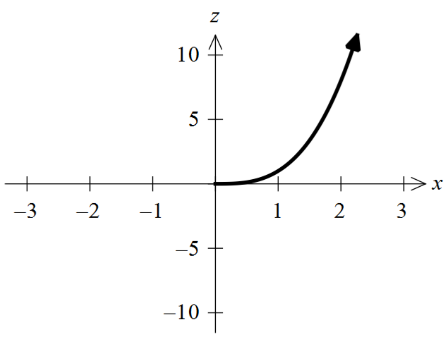
A.



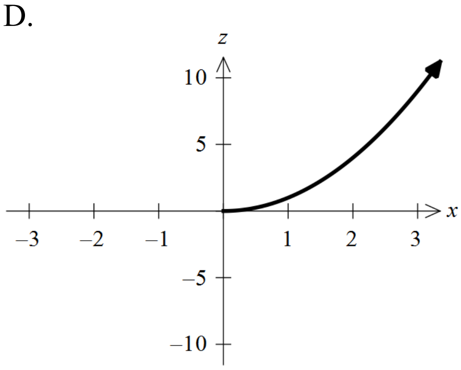
B.



C.



D.



9. Which of the following is equal to the integral below?

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta+\cos\theta}\,d\theta$$

A. $\displaystyle\int_0^1 \frac{1}{1+\theta}\,d\theta$

B. $\displaystyle\int_0^1 \frac{2}{2+2\theta-\theta^2}\,d\theta$

C. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1+\theta^2}{1+\theta}\,d\theta$

D. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1-\sin\theta-\cos\theta}{2-2\sin\theta\cos\theta}\,d\theta$

10. The point P with position vector $\overrightarrow{OP} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}$ lies on a sphere with centre O. Which one of the following points also lies on the sphere?

A. A with position vector $\overrightarrow{OA} = 2\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k}$

B. B with position vector $\overrightarrow{OB} = 5\underset{\sim}{i} + 5\underset{\sim}{j} + 5\underset{\sim}{k}$

C. C with position vector $\overrightarrow{OC} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 8\underset{\sim}{k}$

D. D with position vector $\overrightarrow{OB} = 2\underset{\sim}{i} + 6\underset{\sim}{j} + 6\underset{\sim}{k}$

End of Section I

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet

Marks

(a) Consider the complex numbers $w = 2 - i$ and $z = 1 + 3i$.

Find the following in the form $a + bi$, where a and b are real. Show your working.

(i) w^2 **1**

(ii) $\dfrac{1}{\overline{w} + z}$ **2**

(b) (i) Find A, B and C such that: **2**

$$\frac{8x+3}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

(ii) Hence or otherwise, find **2**

$$\int \frac{8x+3}{(x-2)(x^2+1)}\, dx.$$

(c) Find the square roots of $12 - 16i$, giving your answers in the form $a + bi$,

where a and b are real numbers. **2**

Question 11 continues on the next page.

Question 11(continued)

(d) Consider the vectors $\underset{\sim}{u} = 2\sqrt{2}\,\underset{\sim}{i} + 2\sqrt{2}\,\underset{\sim}{j} + 8\underset{\sim}{k}$ and $\underset{\sim}{v} = 6\underset{\sim}{i} + 6\underset{\sim}{j} + 2\sqrt{2}\,\underset{\sim}{k}$

(i) Find the angle between vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

(ii) Find the projection of the vector $\underset{\sim}{u}$ onto the vector $\underset{\sim}{v}$. **2**

(e) Find $\int \tan^4 x \, dx$. **2**

End of Question 11

Question 12 (15 marks) Start a new writing booklet

Marks

(a) Consider the statement: " for $x \in \mathbb{R}$: $x^2 - x > 0 \Rightarrow x < 0$"

(i) Show that the statement is false. **1**

(ii) State the converse of the statement. **1**

(iii) Determine, with proof, whether the converse is true. **2**

(b) (i) Express the complex number $-2 + 2i\sqrt{3}$ in complex exponential form with principal argument. **2**

(ii) Hence, or otherwise, find the fourth roots of $-2 + 2i\sqrt{3}$ in simplest complex exponential form with principal arguments. **2**

(c) Show that $Re\left(\dfrac{1-z}{1+z}\right) = 0$ for any $z \in \mathbb{C}$ with $|z| = 1$ **2**

(d) An object is moving so that its displacement, x metres, is given by $x = 5\cos^2 t + 4\sin^2 t$ where t is time in seconds.

(i) Show that the motion is simple harmonic. **2**

(ii) State the centre, amplitude and period of the motion **3**

End of Question 12

Question 13 (15 marks) Start a new writing booklet

Marks

(a) Consider the line l_1 with cartesian equation $\dfrac{x-4}{7}=\dfrac{y+3}{2}=-\dfrac{z}{6}$

and the line l_2 with parametric vector equation $\underset{\sim}{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$ $(\lambda \in \mathbb{R})$

(i) Find the parametric vector equation of l_1 **1**

(ii) Show that the lines l_1 and l_2 do not intersect. **2**

(b) Use a proof by contradiction to prove that there are no positive integers p and q such that $p^2-q^2=2$ **2**

(c) Find $\displaystyle\int \frac{1-3x}{\sqrt{6x-x^2}}\,dx$ **3**

(d) Sketch the region defined by $|z-i|<|z+i|$ and $-\dfrac{\pi}{6}\leq Arg(z-i)\leq\dfrac{\pi}{6}$ on a **carefully drawn fully labelled** Argand diagram (1/4 page). **3**

(e) Let $I_n=\displaystyle\int_0^{\pi} x^n \sin x\,dx$.

(i) Show that $I_n=\pi^n-n(n-1)I_{n-2}$ for $n\geq 2$. **2**

(ii) Evaluate I_4 **2**

End of Question 13

Question 14 (15 marks) Start a new writing booklet

Marks

(a) Use the substitution $t = \tan x$ to find $\displaystyle\int \frac{1}{\sin x \cos x}\,dx$ **3**

(b) A particle moves in simple harmonic motion such that its acceleration is $\ddot{x} = -25(x-4)$

Initially the particle is located at $x = 6$ metres from the origin and has a velocity of $-10\ ms^{-1}$.

Determine the range of motion of the particle. **4**

(c) Consider the lines with parametric vector equations:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \quad (\lambda \in \mathbb{R}) \text{ and}$$

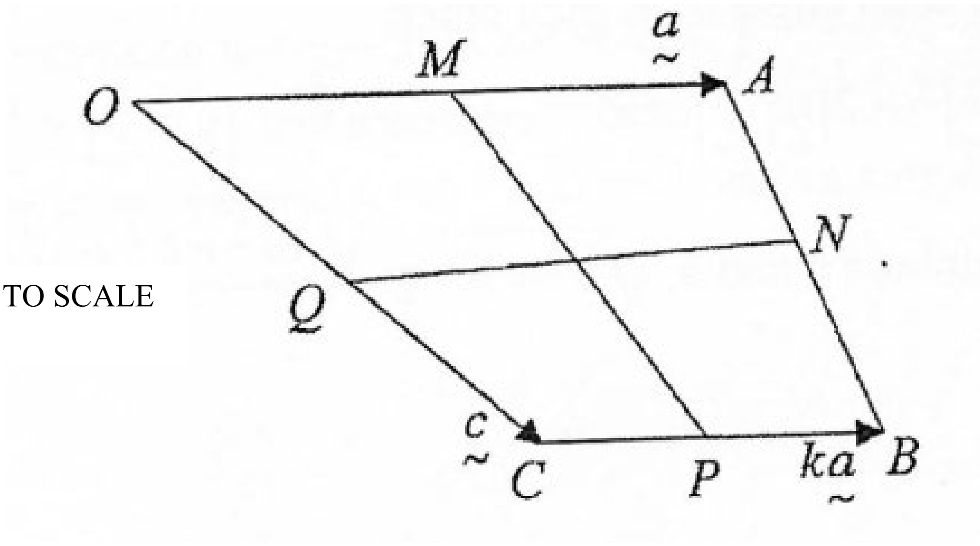
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (\mu \in \mathbb{R})$$

Find the point of intersection of the two lines. **3**

Question 14 continues on the next page

Question 14 (continued)

(d)



NOT TO SCALE

In the diagram $OABC$ is a trapezium in which $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{c}$ and $\overrightarrow{CB} = k\overrightarrow{OA}$ for some constant $0 < k < 1$.

The points M, N, P, and Q are the midpoints of OA, AB, BC, and CO respectively.

(i) Use vector methods to show that MP and QN bisect each other. **2**

(ii) If $MP \perp QN$ use vector methods to show that $\left|\overrightarrow{OC}\right|^2 = \left|\overrightarrow{AB}\right|^2$ **3**

End of Question 14

Question 15 (15 marks) Start a new writing booklet

Marks

(a) Consider the sequence of propositions P_n, $n = 1, 2, 3 \ldots$ defined by

P_n : 14^n *can be expressed as the sum of the squares of three different positive integers*

(i) Show that P_1 and P_2 are true. **2**

(ii) By considering P_k and P_{k+2} , use Mathematical Induction to prove that P_n is true for all positive integers $n \geq 1$ **3**

(b) (i) Show that $\dfrac{1+z}{1+z^{-1}} = z$ for $z \neq -1$ **1**

(ii) Hence, find the imaginary part of $\dfrac{1+\cos\alpha + i\sin\alpha}{1+\cos\alpha - i\sin\alpha}$ **2**

(iii) Let $\alpha \in \mathbb{R}$, $n \in \mathbb{Z}^+$ and $B_n = \left(\dfrac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n$

Show that $B_n = \cos n\left(\dfrac{\pi}{2}-\alpha\right) + i\sin n\left(\dfrac{\pi}{2}-\alpha\right)$ **2**

(c) Let $I_n = \displaystyle\int_0^{\frac{\pi}{3}} \tan^{n+1}x\sec x\,dx$. Given that $\displaystyle\int \tan x\sec x\,dx = \sec x$,

(i) Show that $I_n = \dfrac{2\times 3^{\frac{n}{2}}}{n+1} - \dfrac{n}{n+1} I_{n-2}$ **3**

(ii) Hence find I_2 **2**

End of Question 15

Question 16 (15 marks) Start a new writing booklet

Marks

(a) Find $\int \operatorname{cosec}^3 x \sec x \, dx$ **3**

(b) (i) If $x \geq y > 0$, prove that $x^x y^y \geq x^y y^x$. **1**

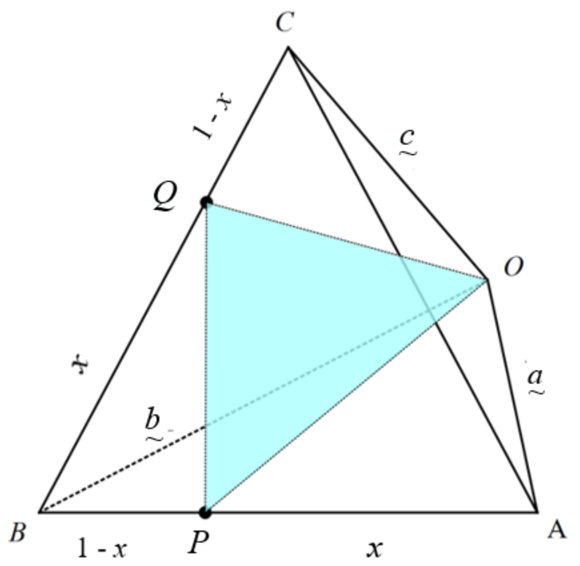
(ii) Hence, or otherwise, prove that for real numbers $a \geq b \geq c > 0$,

$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$ **3**

Question 16 continues on the next page

Question 16 (continued)

(c) A regular tetrahedron $OABC$ has sides of length 1, where O is the origin.



NOT TO SCALE

Let P and Q be the points that divide sides AB and BC in the ratio $x : 1 - x$ as shown, where $0 < x < 1$.

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

It is given that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = \dfrac{1}{2}$

(i) Show that $\left|\overrightarrow{OP}\right| = \sqrt{1 - x + x^2}$. **2**

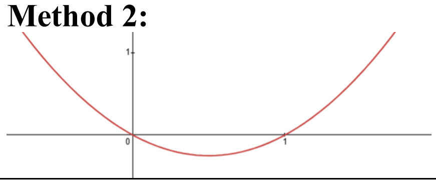
(ii) Let $f(x) = \dfrac{1 + x - x^2}{1 - x + x^2}$ where $0 < x < 1$.

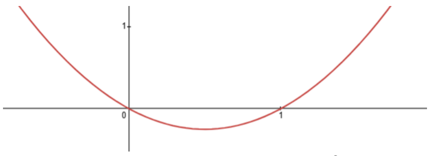
Without using calculus prove that $1 < f(x) \leq \dfrac{5}{3}$ **3**

(iii) Determine the range of values of $\angle POQ$ **3**

End of assessment

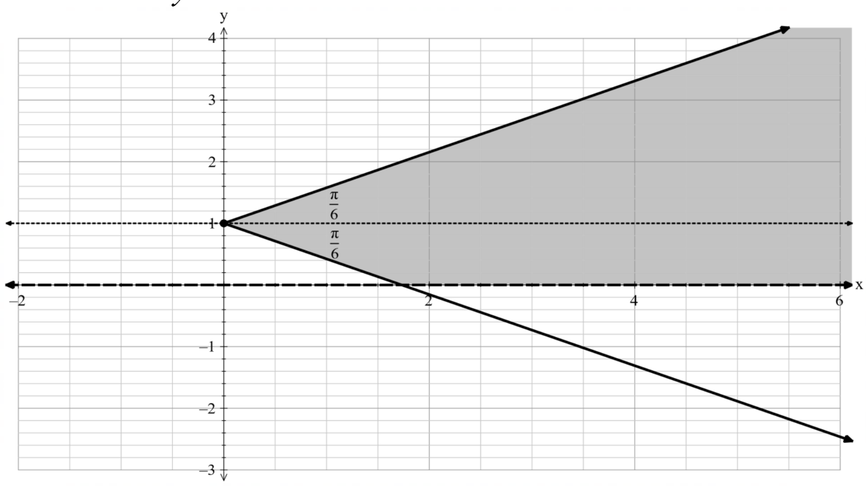
1	$i^3(a+2ai) = -i(a+2ai) = -ai - 2ai^2 = 2a - ai$	A
2	The contrapositive is: If you may enter, then you are authorised.	A
3	The negation of the given statement must contradict the implication by stating the existence of two real numbers greater than 3 with a product less than or equal to 9	D
4	$i(x+iy) - i(x-iy) = -2y > 4$ $\Rightarrow y < -2$	B
5	If α, β, γ are the direction cosines, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. $\alpha = \beta = \dfrac{\pi}{4} \Rightarrow \cos^2\gamma = 0 \Rightarrow \gamma = \dfrac{\pi}{2}$	C
6	Let $\frac{1}{z} = R\ cis\ \theta$, where $0 < R < 1$ and $0 < \theta < \frac{\pi}{2}$. $z = \dfrac{1}{\frac{1}{z}} = \dfrac{1}{Rcis(\theta)} = \dfrac{1}{R}cis(-\theta)$ $\frac{1}{R} > 1$ and given the argument of $-\theta$ which is a reflection in the x-axis, the point B best represents the complex number z.	B
7	Since the polynomial has real coefficients, any complex roots occur in conjugate pairs. Hence, $2+i$ is also a root. The two roots give the quadratic factor $\big(z-(2-i)\big)\big(z-(2+i)\big) = (z^2-4z+5)$. The other quadratic factor takes the form (z^2+pz+q), for some real numbers p and q. Since we require the constant term to be 50, the required factor is $z^2+2z+10$.	D
8	$r_{proj_{xz}}(t) = (t^2, 0, t^6) = (x, 0, x^3)$ where $x \geq 0$	C
9	Let $t = \tan\frac{\theta}{2} \Rightarrow d\theta = \frac{2dt}{1+t^2}$ When $\theta = \frac{\pi}{2}, t = 1; \theta = 0, t = 0$ $\therefore \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta+\cos\theta} = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}dt$ $\displaystyle\int_0^1 \frac{2\,dt}{1+t^2+2t+1-t^2} = \int_0^1 \frac{2\,dt}{2t+2} = \int_0^1 \frac{dt}{t+1} = \int_0^1 \frac{d\theta}{\theta+1}$	A
10	Radius is $\lvert\overrightarrow{OP}\rvert = \lvert\overrightarrow{OC}\rvert = \sqrt{77}$ but $\lvert\overrightarrow{OA}\rvert = \sqrt{78}$, $\lvert\overrightarrow{OB}\rvert = \sqrt{75}$ and $\lvert\overrightarrow{OD}\rvert = \sqrt{76}$.	C
11a i	$w^2 = (2-i)^2 = 4 - 4i + i^2 = 3 - 4i$	1 mark
11 a ii	$\dfrac{1}{\overline{w}+z} = \dfrac{1}{(2+i)+(1+3i)} = \dfrac{1}{3+4i} = \dfrac{1}{3+4i} \times \dfrac{3-4i}{3-4i} = \dfrac{3-4i}{25} = \dfrac{3}{25} - \dfrac{4}{25}i$	2 marks – correct solution 1 mark – attempts to realise denominator
11b i	$8x + 3 \equiv A(x^2+1) + (Bx+C)(x-2)$ Let $x = 2$, then $A = \frac{19}{5}$ Let $x = 0$, then $3 = \frac{19}{5} - 2C \Rightarrow C = \frac{2}{5}$ Let $x = 1$, then $11 = 2\left(\frac{19}{5}\right) - B - \frac{2}{5} \Rightarrow B = -\frac{19}{5}$	2 marks correct solution 1 mark for finding A, B or C

11b ii	$\int \frac{8x+3}{(x-2)(x^2+1)}\,dx = \frac{19}{5}\int \frac{1}{x-2}\,dx + \int \frac{\frac{-19}{5}x+\frac{2}{5}}{x^2+1}dx$ $= \frac{19}{5}\ln\lvert x-2\rvert - \frac{19}{5}\int \frac{x}{x^2+1}dx + \frac{2}{5}\int \frac{1}{x^2+1}dx$ $= \frac{19}{5}\ln\lvert x-2\rvert - \frac{19}{10}\ln(x^2+1) + \frac{2}{5}\tan^{-1}x + C$	2 marks correct solution 1 mark for integrating at least one term correctly
11 c	Let $z = x + iy,$ where $x, y \in \mathbb{R}$. Hence, solve the equation $z^2 = 12 - 16i$. $(x+iy)^2 = 12 - 16i$ $x^2 - y^2 + 2xyi = 12 - 16i$ Equating real and imaginary parts, $x^2 - y^2 = 12$ and $xy = -8$ Solving equations simultaneously, $x^2 - \left(\frac{-8}{x}\right)^2 = 12$ $x^4 - 12x^2 - 64 = 0$ Using the quadratic formula, $x = \pm 4, \therefore y = \mp 2$. Hence, the square roots are $4 - 2i, -4 + 2i$.	2 marks correct solution 1 mark for correct equation in x or y
11d i	$\cos\theta = \frac{\underset{\sim}{u}.\underset{\sim}{v}}{\lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert} = \frac{(12+12+16)\sqrt{2}}{\sqrt{80}\times\sqrt{80}} = \frac{1}{\sqrt{2}}$ $\quad\therefore\ \theta = \frac{\pi}{4}$	2 marks 1 mark for correct numerator or denominator
11 d ii	$proj_{\underset{\sim}{v}}\,\underset{\sim}{u} = \lvert\underset{\sim}{u}\rvert\cos\theta\,\hat{\underset{\sim}{v}} = \sqrt{80}\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{80}}\underset{\sim}{v} = 3\sqrt{2}\,\underset{\sim}{i} + 3\sqrt{2}\,\underset{\sim}{j} + 2\underset{\sim}{k}$	2 marks 1 mark for partially correct expression for the projection
11e	$\int \tan^4 x\,dx = \int \tan^2 x(sec^2 x - 1)dx$ $= \int \tan^2 x\ sec^2 x - (sec^2 x - 1)dx$ $= \frac{1}{3}\tan^3 x - \tan x + x + c$	2 marks 1 mark for correctly obtaining one of the terms of the primitive function
12a i	**Method 1:** A suitable counterexample is $x = 2$ since $2^2 - 2 > 0$ without requiring a negative value of x. **Method 2:** 	1 mark

	$x^2 - x > 0$ $x(x-1) > 0$ $x < 0$ or $x > 1$ (from graph) $\therefore$ $x < 0$ may not be the case	
12 a ii	Converse is: "*If* $x < 0$, then $x^2 - x > 0$".	1 mark correct solution
12 a iii	**Method 1:** Let $x < 0$. Then $x^2 > 0$ and $-x > 0$. Adding these gives $x^2 + (-x) > 0 \rightarrow x^2 - x > 0$ **Method 2:** Let $x < 0$. Then $x - 1 < 0$. Multiplying these gives $x(x-1) > 0 \rightarrow x^2 - x > 0$ **Method 3:** Consider the graph of $y = x^2 - x$.  We can see that for $x < 0$, $x^2 - x > 0$.	2 marks – correct solution 1 mark – a correct method with unclear reasoning
12 b i	$-2 + 2i\sqrt{3} = 4e^{i\frac{2\pi}{3}}$	2 marks – correct 1 mark – correct modulus or argument
12 b ii	Let $z = re^{i\theta}$ be a fourth root. Thus, $z^4 = 4e^{i\frac{2\pi}{3}}$ $z = \sqrt{2}e^{i\frac{\left(\frac{2\pi}{3}+2k\pi\right)}{4}}$ $z = \sqrt{2}e^{i\frac{2\pi(1+3k)}{12}}$, $k = 0,1,2,3$ That is, $z = \sqrt{2}e^{i\frac{\pi}{6}}, \sqrt{2}e^{i\frac{2\pi}{3}}, \sqrt{2}e^{-i\frac{\pi}{3}}, \sqrt{2}e^{-i\frac{5\pi}{6}}$	2 marks – correct solution 1 mark – finds one correct root

12 c	Let $z = x + iy$ $Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-(x+iy)}{1+(x+iy)}\right)$ $= Re\left(\frac{(1-x)-iy}{(1+x)+iy}\right)$ $= Re\left(\frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy}{(1+x)-iy}\right)$ $= \left(\frac{1-x^2-y^2}{(1+x)^2+y^2}\right)$ $= \left(\frac{1-(x^2+y^2)}{(1+x)^2+y^2}\right)$ $= \left(\frac{1-\|z\|^2}{(1+x)^2+y^2}\right)$ $= \left(\frac{1-1}{(1+x)^2+y^2}\right)$ as $\|z\| = 1$ $= 0$ **Method 2:** A complex number w is purely imaginary if $w + \overline{w} = 0$ $\frac{1-z}{1+z} + \overline{\left(\frac{1-z}{1+z}\right)}$ $= \frac{1-z}{1+z} + \frac{1-\overline{z}}{1+\overline{z}}$ $= \frac{1+\overline{z}-z-z\overline{z}+1-\overline{z}+z-z\overline{z}}{1+z+\overline{z}+z\overline{z}}$ $= \frac{2-2\|z\|^2}{1+2Re(z)+\|z\|^2}$ $= \frac{2-2}{2+2Re(z)}$ $= 0$ $\therefore \frac{1+z}{1-z}$ is purely imaginary **Method 3:** Let	2 marks 1 mark for rewriting $\frac{1-z}{1+z}$ using conjugate properties
12 d i	$x = 5cos^2t + 4sin^2t$ $x = 4cos^2t + 4sin^2t + cos^2t$ $x = 4 + cos^2t$ $x = 4 + \frac{\cos 2t+1}{2}$ $x = \frac{9}{2} + \frac{\cos 2t}{2}$ $\dot{x} = -\sin 2t$ $\ddot{x} = -2\cos 2t$ $\ddot{x} = -4\left(x - \frac{9}{2}\right)$ Since acceleration is proportional to displacement, the motion is simple harmonic.	2 marks 1 mark for finding a correct expression for the acceleration

12 d ii	Centre = 4.5, Amplitude = ½ Period = π	3 marks [1 mark each]
13 a i	$\frac{x-4}{7}=\frac{y+3}{2}=-\frac{z}{6}=\mu$ $x=4+7\mu$ $y=-3+2\mu$ $z=-6\mu$ $l_1=\begin{pmatrix}4\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}7\\2\\-6\end{pmatrix}$	
13 a ii	If they meet it would be where (x,y,z) equal on both lines. equating x, y, z components gives $4+7\mu=1+2\lambda$ ① $-3+2\mu=\lambda$ ② $-6\mu=-1+\lambda$ ③ sub ② into ③ $-6\mu=-1-3+2\mu$ $-8\mu=-4$ $\mu=\frac{1}{2}$ $\lambda=-2$ sub into ① $4+\frac{7}{2}=1-4$ $7.5=-3$ not true ∴ lines do not intersect	
13 b	Suppose that there exist positive integers p and q such that $p^2-q^2=2\Rightarrow(p-q)(p+q)=2$ Since p and q are positive integers, $p-q<p+q$ are also integers. Now consider all pairs of positive integers whose product is 2: the only possibility is $p-q=1$ and $p+q=2\Rightarrow p=\frac{3}{2}$, $q=\frac{1}{2}$ which contradicts p and q being integers. Hence the assumption is wrong and no such positive integers p and q exist.	2 marks - Derives contradiction clearly (e.g., $p=3/2$) and concludes correctly 1 mark – sets up contradiction and shows $(p-q)(p+q)=2$ with correct factor pair up
13 c	Completing the square in the denominator, $\int\frac{1-3x}{\sqrt{9-(x-3)^2}}\,dx$	

	Let $u = x - 3$, $du = dx$ $\int \frac{1-3(u+3)}{\sqrt{9-u^2}}\,du = \int \frac{-3u-8}{\sqrt{9-u^2}}\,du = -3\int \frac{u}{\sqrt{9-u^2}}\,du - 8\int \frac{du}{\sqrt{9-u^2}}$ $= 3\sqrt{9-u^2} - 8\sin^{-1}\frac{u}{3} + C$ $= 3\sqrt{9-(x-3)^2} - 8\sin^{-1}\frac{x-3}{3} + C$ $= 3\sqrt{6x-x^2} - 8\sin^{-1}\frac{x-3}{3} + C$	
13 d	let $z = x + iy$ $\lvert z-i\rvert < \lvert z+i\rvert$ $\lvert x+iy-i\rvert < \lvert x+iy+i\rvert$ $\lvert x+i(y-1)\rvert < \lvert x+i(y+1)$ $x^2+(y-1)^2 < x^2+(y+1)^2$ $x^2+y^2-2y+1 < x^2+y^2+2y+1$ $-2y < 2y$ $0 < 4y$ $y > 0$ (the perpendicular bisector of the interval from -*i* to *i*) 	

13 e i	$I_n = \int_0^{\pi} x^n \sin x \, dx$ $u = x^n \qquad dv = \sin x \, dx$ $du = nx^{n-1} \qquad v = -\cos x$ $I_n = [-x^n \cos x](\pi,0) - \int_0^{\pi} -\cos x \, nx^{n-1} \, dx$ $= [-\pi^n \times -1 - 0] + n\int_0^{\pi} x^{n-1} \cos x \, dx$ $[\pi^n] + n\int_0^{\pi} x^{n-1} \cos x \, dx$ $u = x^{n-1} \qquad dv = \cos x \, dx$ $du = (n-1)x^{n-2} \, dx \qquad v = \sin x$ $I_n = \pi^n + n\left([x^{n-1} \sin x](\pi,0) - \int_0^{\pi} \sin x \, (n-1)x^{n-2} \, dx \right)$ $I_n = \pi^n + n\left([x^{n-1} \sin x](\pi,0) - (n-1)\int_0^{\pi} x^{n-2} \sin x \, (n-1) \, dx \right)$ $I_n = \pi^n + n([0-0] - (n-1)I_{n-2})$ $I_n = \pi^n - n(n-1)I_{n-2}$	
13 e ii	$I_4 = \pi^4 - 4 \times 3 \times I_2 = \pi^4 - 12I_2$ $I_2 = \pi^2 - 2 \times 1 \times I_0 = \pi^2 - 2I_0$ $I_0 = \int_0^{\pi} x^0 \sin x dx = \int_0^{\pi} \sin x \, dx = -[\cos x](\pi,0) = -(-1-1) = 2$ $I_2 = \pi^2 - 4$ $I_4 = \pi^4 - 12(\pi^2 - 4) = \pi^4 - 12\pi^2 + 48$	

14a	$I=\int \frac{1}{\sin x \cos x}\,dx=\int \frac{1}{\frac{1}{2}(\sin 2x)}\,dx$ $=\int \frac{2}{\sin 2x}\,dx$ let $t=\tan x$ then $\sin 2x = \frac{2t}{1+t^2}$ and $\operatorname{cosec} 2x=\frac{1+t^2}{2t}$ $x=\tan^{-1} t$ $dx=\frac{1}{1+t^2}\,\mathrm{dt}$ now $I=\int \frac{2(1+t^2)}{2t}\frac{1}{1+t^2}\,\mathrm{dt}$ $=\int \frac{1}{t}\,\mathrm{dt}$ $=\ln\lvert t\rvert+c$ $=\ln\lvert\tan x\rvert+c$	
14b	see below	

Criteria	Marks
• Correctly solves the inequality and determines the range of motion	4
• Substitutes into the velocity equation and sets $v^2 \geq 0$to find the limits of motion	3
• Integrates both sides and uses the initial conditions to find the constant of integration	2
• Correctly uses the chain rule to relate acceleration to velocity: $a=v\frac{dv}{dx}$ and sets up the differential equation	1

Sample answer:

$v\frac{dv}{dx}=-25(x-4) \Rightarrow \frac{1}{2}v^2=-\frac{25}{2}x^2+100x+c$

$x=6, v=-10 \Rightarrow 50=-450+600+c$

$\Rightarrow c=-100$

$v^2=-25x^2+200x-200=-25\left(x^2-8x+16\right)+200=200-25(x-4)^2$

$v^2 \geq 0 \Rightarrow 25(x-4)^2 \leq 200$

$\Rightarrow (x-4)^2 \leq 8$

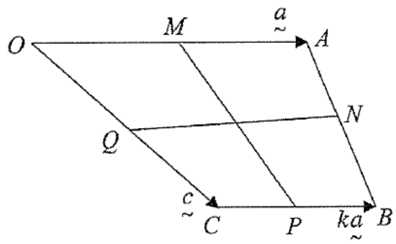
$\Rightarrow -2\sqrt{2} \leq x-4 \leq 2\sqrt{2}$

$\Rightarrow 4-2\sqrt{2} \leq x \leq 4+2\sqrt{2}$

i.e. the particle moves between $x=4-2\sqrt{2}$ and $x=4+2\sqrt{2}$

<table>
<tr><td>14c</td><td>$x : 1 = \mu$ ①
$y : -\lambda = 1$ ②
$z : 3\lambda = -1 + 2\lambda$ ③

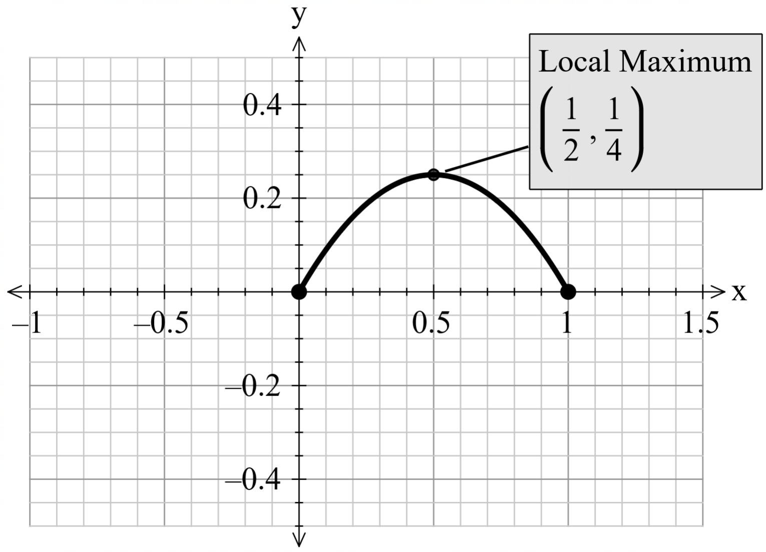
$\mu = 1$
$\lambda = -1$

point of intersection is (1, 1, -3)</td><td></td></tr>
<tr><td>14d i</td><td>Answer
N is the midpoint of AB. $\therefore\ \overrightarrow{ON} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$
$\therefore\ \overrightarrow{ON} = \frac{1}{2}\left(\underset{\sim}{a} + \left(\underset{\sim}{c} + k\underset{\sim}{a}\right)\right) = \frac{1}{2}\underset{\sim}{c} + \frac{1}{2}(k+1)\underset{\sim}{a}$
$\therefore\ \frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{ON}) = \frac{1}{2}\left(\frac{1}{2}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{c} + \frac{1}{2}(k+1)\underset{\sim}{a}\right)$
$= \frac{1}{2}\underset{\sim}{c} + \frac{1}{4}(k+1)\underset{\sim}{a}$
But $\frac{1}{2}(\overrightarrow{OM} + \overrightarrow{OP}) = \frac{1}{2}\left(\frac{1}{2}\underset{\sim}{a} + \left(\underset{\sim}{c} + \frac{1}{2}k\underset{\sim}{a}\right)\right) = \frac{1}{2}\underset{\sim}{c} + \frac{1}{4}(k+1)\underset{\sim}{a}$
Hence QN and MP have a common midpoint T where $\overrightarrow{OT} = \frac{1}{2}\underset{\sim}{c} + \frac{1}{4}(k+1)\underset{\sim}{a}$.
Hence QN and MP bisect each other in their intersection point T.
</td><td></td></tr>
<tr><td>14 d ii</td><td>Answer
$\overrightarrow{MP} = \overrightarrow{MO} + \overrightarrow{OC} + \overrightarrow{CP} = -\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{c} + \frac{1}{2}k\underset{\sim}{a} = \underset{\sim}{c} + \frac{1}{2}(k-1)\underset{\sim}{a}$
$\overrightarrow{QN} = \overrightarrow{QO} + \overrightarrow{ON} = -\frac{1}{2}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{c} + \frac{1}{2}(k+1)\underset{\sim}{a} = \frac{1}{2}(k+1)\underset{\sim}{a}$ (using the expression for $\overrightarrow{ON}$ found in (i))
Hence if $MP \perp QN$, $\left(\underset{\sim}{c} + \frac{1}{2}(k-1)\underset{\sim}{a}\right).\left(\frac{1}{2}(k+1)\underset{\sim}{a}\right) = 0$, giving. $\underset{\sim}{a}.\underset{\sim}{c} = -\frac{1}{2}(k-1)\underset{\sim}{a}.\underset{\sim}{a}$
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OC} + \overrightarrow{CB} = -\underset{\sim}{a} + \underset{\sim}{c} + k\underset{\sim}{a} = \underset{\sim}{c} + (k-1)\underset{\sim}{a}$
$\therefore |\overrightarrow{AB}|^2 = \left(\underset{\sim}{c} + (k-1)\underset{\sim}{a}\right).\left(\underset{\sim}{c} + (k-1)\underset{\sim}{a}\right)$
$= \underset{\sim}{c}.\underset{\sim}{c} + (k-1)^2\underset{\sim}{a}.\underset{\sim}{a} + 2(k-1)\underset{\sim}{a}.c$
$= \underset{\sim}{c}.\underset{\sim}{c} + \left\{(k-1)^2 + 2(k-1)\left(-\frac{1}{2}(k-1)\right)\right\}\underset{\sim}{a}.\underset{\sim}{a}$ if $MP \perp QN$
$= \underset{\sim}{c}.\underset{\sim}{c}$
$= |\overrightarrow{OC}|^2$</td><td></td></tr>
<tr><td>15 a i</td><td>$14^1 = 1^2 + 2^2 + 3^2$ hence P_1 is true. $14^2 = 12^2 + 6^2 + 4^2$ hence P_2 is true.</td><td>1 mark for each P1 and P2</td></tr>
<tr><td>15 a ii</td><td>Marking Guidelines
<table><tr><th>Criteria</th><th>Marks</th></tr><tr><td>Uses mathematical induction to establish the truth of all the propositions P_n for integers $n \geq 1$</td><td>3</td></tr><tr><td>Substantial progress eg. shows the truth of P_k implies the truth of P_{k+2}</td><td>2</td></tr><tr><td>Some progress eg. attempts to incorporate truth of P_k into consideration of P_{k+1} or P_{k+2}</td><td>1</td></tr></table>
Answer
If P_k is true: $14^k = a^2 + b^2 + c^2$ for different positive integers a, b, c. *
Consider P_{k+2}: $14^{k+2} = 14^2 \times 14^k$
$= 14^2(a^2 + b^2 + c^2)$ if P_k is true, using *
$= (14a)^2 + (14b)^2 + (14c)^2$, where $14a$, $14b$, $14c$ are different positive integers.
Hence, if P_k is true, then P_{k+2} is true. But P_1 is true, therefore by mathematical induction, P_n is true for all odd integers $n \geq 1$. Also P_2 is true, and therefore by mathematical induction, P_n is true for all even integers $n \geq 2$. Hence P_n is true for all integers $n \geq 1$.</td><td></td></tr>
<tr><td>15 b i</td><td>$\dfrac{1+z}{1+\frac{1}{z}} = \dfrac{1+z}{\frac{z+1}{z}} = z\dfrac{1+z}{1+z} = z$</td><td>1 mark – correct solution</td></tr>
</table>

15 b ii	let $z = \cos\alpha + i\sin\alpha$ then $z^{-1} = \cos(-\alpha) + i\sin(-\alpha)$ $= \cos\alpha - i\sin\alpha$ and $\dfrac{1+(\cos\alpha + i\sin\alpha)}{1-(\cos\alpha - i\sin\alpha)}$ $= \dfrac{1+z}{1+z^{-1}} = z = \cos\alpha + i\sin\alpha$ the imaginary part is $\sin\alpha$	
15 b iii	let $z = \sin\alpha + i\cos\alpha$ $B_n = \left(\dfrac{1+z}{1+z^{-1}}\right)^n$ $= z^n$ (from (i)) $= (\sin\alpha + i\cos\alpha)^n$ $= \left(\cos\left(\dfrac{\pi}{2}-\alpha\right) + i\sin\left(\dfrac{\pi}{2}-\alpha\right)\right)^n$ (complemetary angles) $= \cos n\left(\dfrac{\pi}{2}-\alpha\right) + i\sin n\left(\dfrac{\pi}{2}-\alpha\right)$	2 marks – correct solution 1 mark – finds Bn in cis form
15 c i	For $I_n = \int_0^{\frac{\pi}{3}} tan^{n+1}x \sec x\ dx$ Rewrite as $I_n = \int_0^{\frac{\pi}{3}} tan^n x \tan x \sec x\ dx$ Using integration by parts, $u = tan^n x,\ v' = \tan x \sec x$ $u' = ntan^{n-1}x\ sec^2 x,\ v = \sec x$ $\therefore \int_0^{\frac{\pi}{3}} tan^n x \tan x \sec x\ dx = [tan^n x \sec x]_0^{\frac{\pi}{3}} - n\int_0^{\frac{\pi}{3}} tan^{n-1}x\ sec^3\ x\ dx$ $= \left[2 \times 3^{\frac{n}{2}}\right] - n\int_0^{\frac{\pi}{3}} tan^{n-1}x\ sec^2\ x \sec x\ dx$ $= \left[2 \times 3^{\frac{n}{2}}\right] - n\int_0^{\frac{\pi}{3}} tan^{n-1}x\ (tan^2\ x + 1) \sec x\ dx$ $= \left[2 \times 3^{\frac{n}{2}}\right] - n\int_0^{\frac{\pi}{3}} tan^{n+1}x \sec x + tan^{n-1}x \sec x\, dx$ $I_n = \left[2 \times 3^{\frac{n}{2}}\right] - nI_n - nI_{n-2}$ $nI_n + I_n = \left[2 \times 3^{\frac{n}{2}}\right] - nI_{n-2}$ $I_n = \dfrac{2 \times 3^{\frac{n}{2}}}{n+1} - \dfrac{n}{n+1}I_{n-2}$	3 marks – correct solution 2 marks – substantial progress – correct integration. 1 mark – uses integration by parts and makes some progress
15 c ii	$I_2 = \frac{2\times 3^{\frac{2}{2}}}{2+1} - \frac{2}{2+1}I_0 = \frac{6}{3} - \frac{2}{3}I_0 = 2 - \frac{2}{3}I_0$	2 marls – correct solution.

	$I_0 = \int_0^{\frac{\pi}{3}} \tan x \sec x \; dx = [\sec x]_0^{\frac{\pi}{3}} = 1$ $\therefore I_2 = 2 - \frac{2}{3} \times 1 = \frac{4}{3}$	1 mark – finds I_0 or equivalent merit.
16a	$I = \int \operatorname{cosec}^3 x \sec x \, dx$ $= \int \frac{\operatorname{cosec}^2 x}{\sin x \cos x} dx$ $= \int \frac{1 + \cot^2 x}{\sin x \cos x} dx$ $= \int \frac{1}{\sin x \cos x} + \int \frac{\cot^2 x}{\sin x \cos x} \times \frac{\cos x}{\cos x} dx$ $= \int 2 \operatorname{cosec} 2x \, dx + \int \sec^2 x \, (\tan x)^{-3} \, dx$ let $I_i = 2 \int \operatorname{cosec} 2x \, dx$ let $t = \tan x$ $dx = \frac{1}{1+t^2}$ dt $\operatorname{cosec} 2x = \frac{1+t^2}{2t}$ $I_1 = \int \frac{1+t^2}{2t} \frac{2}{1+t^2} dt = \ln\lvert t \rvert + \underset{\sim}{c} = \ln\lvert \tan x \rvert + \underset{\sim}{c}$ and I = $\ln\lvert \tan x \rvert - \frac{1}{2\tan^2 x} + \underset{\sim}{c}$	3 marks – correct solution 2 marks – substantial progress – partially correct integration 1 mark – some progress. Manipulates integrand so as to make progress towards solution.
16 b i	$x \geq y > 0$ $\therefore \quad \frac{x}{y} \geq 1$ and $\left(\frac{x}{y}\right)^x \geq \left(\frac{x}{y}\right)^y$ $\frac{x^x}{y^x} \geq \frac{x^y}{y^y}$ $x^x y^y \geq x^y y^x$	1 mark – correct solution
16 b ii	From $x^x y^y \geq x^y y^x$, $a^a b^b \geq a^b b^a$ $b^b c^c \geq b^c c^b$ $c^c a^a \geq c^a a^c$ Multiplying these inequalities, $\left(a^a b^b c^c\right)^2 \geq a^{b+c} b^{a+c} c^{a+b}$	3 marks – correct solution 2 marks – multiplies the 3 inequalities correctly 1 mark – uses part i to obtain a suitable inequality

	Multiplying by $a^a b^b c^c$, $\left(a^a b^b c^c\right)^3 \geq a^{a+b+c} b^{a+b+c} c^{a+b+c}$ $\left(a^a b^b c^c\right)^3 \geq (abc)^{a+b+c}$ Therefore, $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$ as required.	
16c i	$\overrightarrow{\mathrm{OP}} = \underset{\sim}{a} + x\overrightarrow{\mathrm{AB}}$ $\overrightarrow{\mathrm{AB}} = -\underset{\sim}{a} + \underset{\sim}{b}$ so $\overrightarrow{\mathrm{OP}} = \underset{\sim}{a} + x(-\underset{\sim}{a} + \underset{\sim}{b})$ $= (1-x)\,\underset{\sim}{a} + x\,\underset{\sim}{b}$ $\left\lvert \overrightarrow{\mathrm{OP}} \right\rvert = \sqrt{\overrightarrow{\mathrm{OP}} \cdot \overrightarrow{\mathrm{OP}}}$ $\overrightarrow{\mathrm{OP}} \cdot \overrightarrow{\mathrm{OP}} = ((1-x)\,\underset{\sim}{a} + x\ \ \mathrm{b}) \cdot ((1-x)\,\underset{\sim}{a} + x\ \ \mathrm{b})$ $= (1-x)^2\,\underset{\sim}{a} \cdot \underset{\sim}{a} + 2x(1-x)\,\underset{\sim}{a} \cdot \underset{\sim}{b} + x^2\,\underset{\sim}{b} \cdot \underset{\sim}{b}$ $= (1-x)^2 + x(1-x) + x^2$ (as $\underset{\sim}{b} \cdot \underset{\sim}{b} = \lvert \underset{\sim}{b} \rvert^2 = 1$ (same for a) and a $\cdot\, \underset{\sim}{b} = \frac{1}{2}$ (given)) $= 1 - 2x + x^2 + x - x^2 + x^2$ $= 1 - x + x^2$ and $\left\lvert \overrightarrow{\mathrm{OP}} \right\rvert = \sqrt{1 - x + x^2}$	2 marks – correct solution 1 mark – correct vector expression for *OP*.
16c ii	$f(x) = \dfrac{1 + x - x^2}{1 - x + x^2} = \dfrac{1 + x(1-x)}{1 - x(1-x)}$ $= \dfrac{1 + x(1-x)}{1 - x(1-x)} \times \dfrac{1 - x(1-x)}{1 - x(1-x)}$ $= \dfrac{1^2 - (x(1-x))^2}{(1 - x(1-x))^2}$ Consider $x(1-x)$ for $0 \leq x \leq 1$	3 marks – correct solution 2 marks – progress towards second inequality 1 mark – correctly shows one side of the inequality



In the domain $0 < x < 1$

$$0 < x(1-x) \le \frac{1}{4}$$

when $x(1-x) = 0$

$$f(x) = \frac{1-0^2}{(1-0)^2} = \frac{1}{1} = 1$$

but $x(1-x) > 0 \therefore f(x) > 1$

When $x(1-x) = \frac{1}{4}$

$$f(x) = \frac{1-\left(\frac{1}{4}\right)^2}{\left(1-\frac{1}{4}\right)^2} = \frac{\frac{15}{16}}{\frac{9}{16}} = \frac{15}{9} = \frac{5}{3}$$

but $x(1-x) \le \frac{1}{4} \therefore f(x) \le \frac{5}{3}$

combining the inequalities

so $1 < f(x) \le \frac{5}{3}$

Alternate solution:

$$Now,\ f(x)-1=\frac{1+x-x^2}{1-x+x^2}-1=\frac{1+x-x^2-\left(1-x+x^2\right)}{1-x+x^2}=\frac{2x-2x^2}{1-x+x^2}$$

$$=\frac{2x(1-x)}{\frac{3}{4}+\left(x-\frac{1}{2}\right)^2}$$

$$>0, \quad \text{since } x>0 \text{ and } 1-x>0 \Rightarrow x(1-x)>0$$

$$\text{and } \frac{3}{4}+\left(x-\frac{1}{2}\right)^2 \geq \frac{3}{4}>0$$

Hence $f(x)>1$ (1)

Also, $\frac{5}{3}-f(x)=\frac{5}{3}-\frac{1+x-x^2}{1-x+x^2}=\frac{5\left(1-x+x^2\right)-3\left(1+x-x^2\right)}{3\left(1-x+x^2\right)}=\frac{2-8x+8x^2}{3\left(1-x+x^2\right)}$

$$=\frac{2(1-2x)^2}{3\left(\frac{3}{4}+\left(x-\frac{1}{2}\right)^2\right)}$$

$$\geq 0$$

Hence $f(x)\leq\frac{5}{3}$ (2)

Combining (1) and (2) gives $1<f(x)\leq\frac{5}{3}$.

16 c iii	$OQ = b + xBC$ $BC = -b + c$ $OQ = b + x(-b + c)$ $= (1-x)\, b + xc$ and as in (i) $\lvert OQ \rvert = \sqrt{1-x+x^2}$ $\cos\ \angle POQ = \dfrac{OP \cdot OQ}{\lvert OP\rvert\, \lvert OQ\rvert}$ $= \dfrac{((1-x)\, a + xb)\cdot((1-x)\, b + xc)}{\sqrt{1-x+x^2}\sqrt{1-x+x^2}}$ $= \dfrac{(1-x)^2\, a \cdot b + x(1-x)\, a \cdot c\ + x(1-x)\, b \cdot b + x^2\, b \cdot c}{1-x+x^2}$ $\dfrac{\frac{1}{2}(1-2x+x^2) + \frac{1}{2}(x-x^2) + x - x^2 + \frac{1}{2}x^2}{1-x+x^2}$ $= \dfrac{1}{2}\, \dfrac{1-2x+x^2+x-x^2+2x-2x^2+x^2}{1-x+x^2}$ $= \dfrac{1}{2}\, \dfrac{1+x-x^2}{1-x+x^2}$ so from (ii) $\dfrac{1}{2} < \cos\ \angle POQ \le \dfrac{5}{6}$ $0.585 \le \angle\, POQ < \dfrac{\pi}{3}$, or $33.6° \le \angle POQ < 60°$	3 marks – correct solution 2 marks – correctly expresses $\angle\, MON$ as a quotient of dot product and magnitude 1 mark – correct dot product fully simplified