BAULKHAM HILLS HS PERSEVERE

2026 **YEAR12**
TRIAL
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen.
- Calculators approved by NESA may be used.
- Answer all questions in the booklet provided.
- Marks may be deducted for careless or badly arranged work
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.
- A reference sheet is provided at the back of this paper

Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II – 90 marks (pages 6 – 13)

- Attempt Questions 11 – 16
- Allow about 2 hour 45 minutes for this section.

Section I
10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer page in your booklet to answer Questions 1 – 10.

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1) Consider the following statement:

"If n is a perfect square, then n is not 2 more than a multiple of 3."

Which of the following is the negation of this statement?

A) If n is not 2 more than a multiple of 3, then n is a perfect square.

B) n is a perfect square and n is 2 more than a multiple of 3.

C) If n is 2 more than a multiple of 3, then n is not a perfect square.

D) n is not a perfect square and n is 2 more than a multiple of 3.

2) Which of the following expressions is equivalent to $\int \frac{x^3 dx}{x^2 + x + 1}$?

A) $\frac{x^2}{2} - x + \tan^{-1}\frac{2x+1}{\sqrt{3}} + c$

B) $\frac{x^2}{2} - x + \tan^{-1}\frac{4x+2}{3} + c$

C) $\frac{x^2}{2} - x + \frac{4}{3}\tan^{-1}\frac{4x+2}{3} + c$

D) $\frac{x^2}{2} - x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + c$

3) A polynomial $P(z)$ has real coefficients. It is known that:

$P(1+2i) = 0$

$P'(1+2i) = 0$

$P(-3) = 0$

What is the minimum possible degree of $P(z)$?

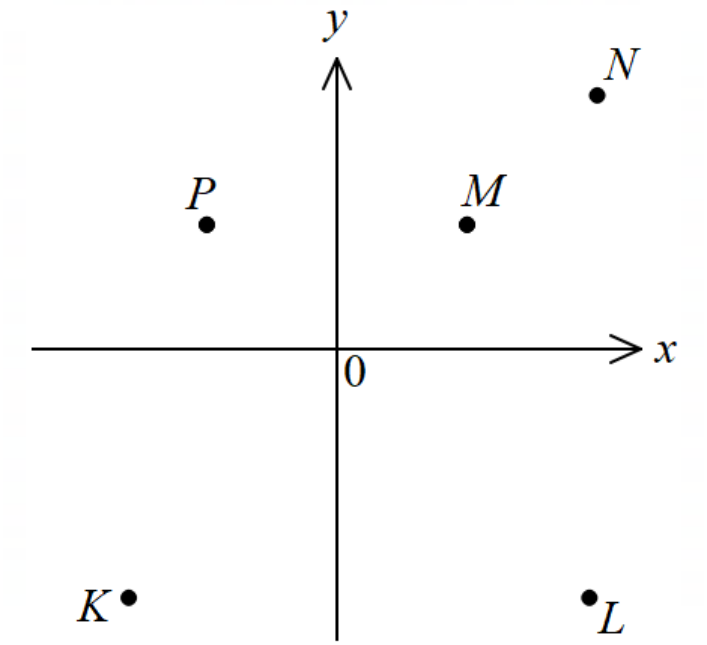
A) 3

B) 4

C) 5

D) 6

4) In the Argand diagram the point P represents a complex number. When this number is multiplied by $\frac{(1+i)^4}{(1-i)^2}$ it gave a new complex number.



Which of the following points represents the new complex number?

A) K

B) L

C) M

D) N

5) Consider three vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{c} = 13\underset{\sim}{i} + 10\underset{\sim}{j} + \underset{\sim}{k}$. Which of the following statements is true?

A) $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are perpendicular to $\underset{\sim}{a}$

B) $\underset{\sim}{b}$ and $\underset{\sim}{a}$ are perpendicular to $\underset{\sim}{c}$

C) $\underset{\sim}{c}$ is only perpendicular to $\underset{\sim}{a}$

D) $\underset{\sim}{c}$ is only perpendicular to $\underset{\sim}{b}$

6) Consider the position vector of a particle $\underset{\sim}{r} = t^2\underset{\sim}{i} + 2\cos t\,\underset{\sim}{j} + 2\sin t\,\underset{\sim}{k}$.
Which of the following statements best describes the motion of the particle?

A) A spiral about the z axis in an anticlockwise direction.

B) A spiral about the x axis in an anticlockwise direction.

C) A spiral about the z axis in a clockwise direction.

D) A spiral about the x axis in a clockwise direction.

7) The points A and B have coordinates $(4,3,-5)$ and $(2,-7,6)$ respectively. Point P lies on the interval AB such that $PA:PB = 1:2$. What are the coordinates of P?

A) $\left(\frac{10}{3},-\frac{1}{3},-\frac{4}{3}\right)$

B) $\left(\frac{14}{3},\frac{19}{3},-\frac{26}{3}\right)$

C) $\left(\frac{4}{3},-\frac{31}{3},-\frac{29}{3}\right)$

D) $\left(\frac{8}{3},-\frac{11}{3},\frac{7}{3}\right)$

8) Which of the following is a necessary and sufficient condition for function $f(x)$ to satisfy for all positive values of a?
$\int_0^a \left[f(x-a)+f(a-x)\right]dx = 0$.

A) $f(x)$ is increasing.

B) $f(x)$ is decreasing.

C) $f(x)$ is even.

D) $f(x)$ is odd.

9) A particle starts moving along a straight line from rest at the origin. At time *t*, its velocity is *v* and its displacement from a fixed origin is *x*.

If $\frac{dv}{dx} = \frac{2}{3v}$, which of the following best describes the particle's acceleration and velocity?

A) Constant acceleration and constant velocity.

B) Constant acceleration and decreasing velocity.

C) Constant acceleration and increasing velocity.

D) Increasing acceleration and increasing velocity.

10) A complex number ω satisfies $\omega^2 + \frac{1}{1+\omega^2} + \omega = 0$. Which of the following is the correct statement about ω^{2026}?

A) $\omega^{2026} = \omega$

B) $\omega^{2026} = \omega^2$

C) $\omega^{2026} = \omega^3$

D) $\omega^{2026} = \omega^4$

End of Section I

Section II (90 marks)
Attempt Questions 11 - 16
Allow about 2 hour 45 minutes for this section

Answer each question on the appropriate pages of your answer booklet. Extra paper is available.
Your responses should include relevant calculations and/or mathematical reasoning.

Question 11 (15 marks)

a) Given that z and w are two complex numbers, where $z = 1 + ai$ and $w = a + i,\ a \in \mathbb{R}$. Find:

i) $z + \overline{w}$. **1**

ii) $\text{Arg}(zw)$. **2**

iii) For what value(s) of a, will $\frac{z}{w}$ be purely real? **2**

b) If $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{i}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$, what is the value of $\left|\underset{\sim}{a}\right|$? **2**

c) Find $\int \frac{x^3}{x^2 + 9}\,dx$. **2**

d) Consider the statement P.

P: For all integers n, if $n^2 - 6n + 5$ is even, then n is odd.

i) Write down the contraposition of P. **1**

ii) Hence prove that the proposition P is true. **2**

e) Solve $z^2 - z + (4 - 2i) = 0$, giving your answers in the form of $x + yi$ where x and y are real numbers. **3**

Question 12 (15 marks)

a) The polynomial $P(z) = z^4 - 8z^3 + pz^2 + qz - 80$ has a root $3+i$, where p and q are real numbers.

i) Find all the roots of $P(z)$. **3**

ii) Write $P(z)$ as a product of two real quadratic factors. **1**

b) Show that $\dfrac{a^2+b^2}{a-b} \geq 2\sqrt{2}$ for $a > b$ and $ab > 1$. **2**

c) The lines l_1 and l_2 are given by the equations: **3**

$$l_1 : \underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + s \begin{pmatrix} 2 \\ \ln a \\ \ln a \end{pmatrix}, \quad l_2 : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} -3 \\ \ln a \\ 1 \end{pmatrix}, \text{ where } s \in \mathbb{R},\ t \in \mathbb{R} \text{ and } a \text{ is a positive}$$

constant.

Given that l_1 and l_2 are perpendicular, what are the values of a?

d) By using the substitution $t = \tan\theta$, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{4}{5 - 3\cos 2\theta}\, d\theta$. **3**

e) A particle moves along the x axis in a simple harmonic motion. Its displacement x cm from the origin at time t seconds is given by $x = 5 - 4\cos^2 2\pi t$.

i) Find the centre and the amplitude of the motion. **2**

ii) Find the distance travelled by the particle in the first 5 seconds. **1**

Question 13 (14 marks)

a) Prove by contradiction that $P(x) = 120 + 120x + 60x^2 + 20x^3 + 5x^4 + x^5$ has no double roots. **2**

b) Let $I_n = \int_0^1 \frac{dx}{\left(1+x^2\right)^n}$, for $n \geq 0$.

i) Show that $I_{n+1} = \frac{1}{2n}\left[\frac{1}{2^n} + (2n-1)I_n\right]$, for $n \geq 1$. **3**

ii) Hence, evaluate I_3. **2**

c) A particle of mass 1kg is projected from the origin with initial speed V ms^{-1} at an angle α to the horizontal plane.

The forces acting on the particle are weight force and air resistance of $5\underset{\sim}{v}$, where $\underset{\sim}{v}$ is the velocity vector of the particle at time t seconds after the particle is projected.

Let the acceleration due to gravity be $g = 10\,\text{ms}^{-2}$.

i) Show that $\underset{\sim}{v}(t) = \begin{pmatrix} Ve^{-5t}\cos\alpha \\ (V\sin\alpha + 2)e^{-5t} - 2 \end{pmatrix}$. **3**

ii) Given that $\underset{\sim}{v}(1) = \begin{pmatrix} 250e^{-5} \\ (250\sqrt{3} + 2)e^{-5} - 2 \end{pmatrix}$, find V and α. **2**

iii) Show that the ratio of the horizontal velocity at the origin to the horizontal velocity at the maximum height is $\left(1 + 125\sqrt{3}\right) : 1$. **2**

Question 14 (17 marks)

a) Let a, b and c be three positive real numbers. It is known that $a+b+c=1$.

i) Prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 9$. **2**

ii) Prove that $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)\geq 8$. **3**

b) A particle P of mass m kg is released from rest and falls under its own weight.
The particle is subject to air resistance of magnitude mkv^2 newtons, where v ms^{-1} is the velocity of the particle after it has travelled x metres and k is a positive constant.

i) Show that $v^2=\left(1-e^{-2kx}\right)\frac{g}{k}$. **3**

ii) Determine the terminal velocity V_T of the particle. **1**

iii) Show that the particle attains a speed of $\frac{1}{2}V_T$ after it has fallen a distance of $\frac{1}{2k}\ln\left(\frac{4}{3}\right)$ metres. **2**

c) i) Show that $\frac{1}{x\sqrt{x^{2n}-1}}=\frac{x^{-n-1}}{\sqrt{1-x^{-2n}}}$. **1**

ii) Hence, or otherwise, find $\int_2^4 \frac{dx}{x\sqrt{x^{2n}-1}},\ n=1,2,3,\ldots.$ **3**

iii) Show that $\int_2^4 \frac{dx}{x\sqrt{x^{2n}-1}}>\frac{1}{n}\left(\frac{2^n-1}{4^n}\right)$. **2**

Question 15 (14 marks)

a) The acceleration of a particle is $\ddot{x} = 4x^3 - 16x$, where x is the displacement of the particle from a fixed point O after t seconds, in metres.

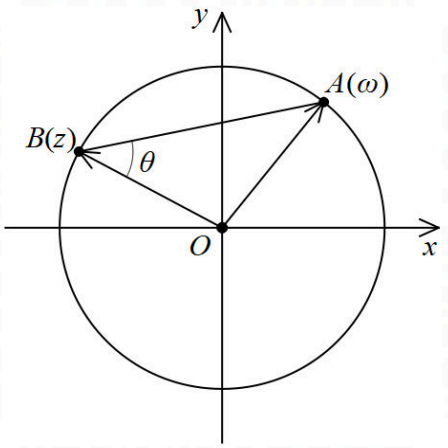
Initially the particle is at O and has a velocity of $v_\circ$ ms^{-1}.

i) Show that the quantity $(\dot{x})^2 - 2x^4 + 16x^2$ does not change with time. **2**

ii) Show that when $v_\circ = \sqrt{\frac{31}{8}}$, the particle remains in the region $-\frac{1}{2} \le x \le \frac{1}{2}$. **2**

iii) State, with reason, whether the particle is moving in simple harmonic motion. **1**

b) The diagram shows two distinct points A and B that represent the complex numbers w and z respectively. These points lie on the circle with centre O and radius r. Let $\angle OBA = \theta$.

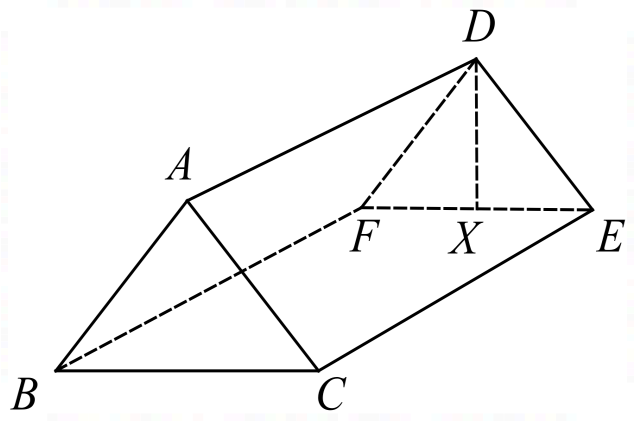


i) Show that $z = \omega(-\cos 2\theta + i \sin 2\theta)$. **2**

ii) Write down the complex number represented by $\overrightarrow{AB}$ and deduce that $|z + w| = \tan\theta\,|z - \omega|$. **2**

Question 15 continues on the next page

c) The diagram shows the triangular prism *ABCDEF,* where *BCEF* is a rectangle. The triangles *ABC* and *DEF* are equilateral.



Let $\left|\overrightarrow{AB}\right| = m$ and $\left|\overrightarrow{CE}\right| = n$. X is the midpoint of $\overrightarrow{EF}$.

i) Use vector methods to show that $\overrightarrow{XD}$ is perpendicular to $\overrightarrow{EF}$. **3**

ii) Show that $\left|\overrightarrow{BX}\right| + \left|\overrightarrow{BF}\right|$ is less than the perimeter of the rectangle *BCEF*. **2**

End of Question 15

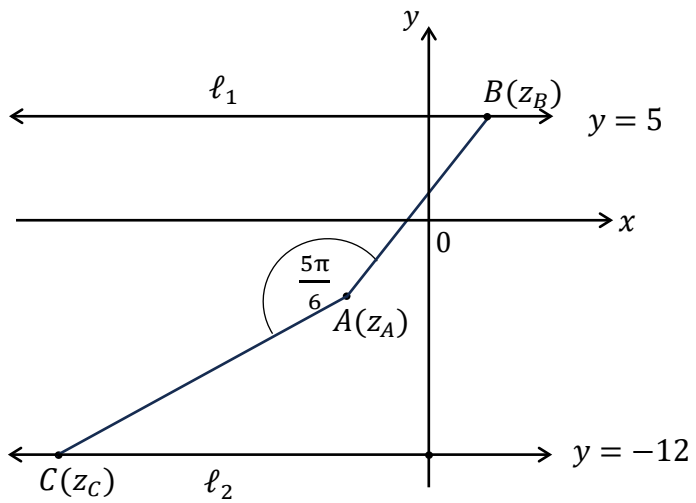
Question 16 (15 marks)

a) Use a suitable substitution to evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\cos x\sqrt{\sec x - \tan x}}\,dx$. **4**

b) The point A, which represents the complex number $z_A = -5 - 4i$, lies between two parallel lines ℓ_1 and ℓ_2. **3**

The equations of ℓ_1 and ℓ_2 are $y = 5$ and $y = -12$ respectively.

The point B, which represents the complex number z_B, lies on ℓ_1.The point C, which represents the complex number z_C, lies on ℓ_2 such that $AC = 2AB$ and $\angle BAC = \frac{5\pi}{6}$.



Find the exact value of the complex number z_C.

Question 16 continues on the next page

c) Let $z_0, z_1, z_2, ..., z_n$ be complex numbers such that

$(k+1)z_{k+1} - i(n-k)z_k = 0$ for all $k \in \{0,1,2,...,n-1\}$.

i) Prove, by mathematical induction that $z_r = i^r \binom{n}{r} z_0$ for all integers $r \geq 0$. **3**

It is known that the complex numbers form the identity $z_0 + z_1 + z_2 + ... + z_n = 2^n$.

ii) Hence, show that $z_0 = (1-i)^n$. **2**

It is known that $\sum_{m=0}^{n} \binom{n}{m}^2 = \binom{2n}{n}$. (Do NOT prove this)

iii) Hence or otherwise show that $|z_0|^2 + |z_1|^2 + |z_2|^2 + ... |z_n|^2 \leq \frac{(3n+1)^n}{n!}$. **3**

End of Exam

X2 TRIAL 2026 Solutions

Section I - Multiple Choice Questions (1 - 10)

1) B
2) D
3) C
4) D
5) B
6) B
7) A
8) D
9) C
10) A

Suggestion Solutions

1) $\neg(P \Rightarrow Q)$ is $P \wedge \neg Q$

P: n is a perfect square

Q: n is not 2 more than a multiple of 3.

$P \wedge \neg Q$: n is a perfect square and n is 2 more than a multiple of 3.

$\therefore$ B is correct answer.

2) $\int \frac{x^3\,dx}{x^2+x+1}$

$\therefore \int \left((x-1) + \frac{1}{x^2+x+1}\right) dx$

$$\int x\,dx - \int dx + \int \frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}}$$

$$\frac{x^2}{2} - x + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\frac{x^2}{2} - x + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$\therefore$ D is correct answer.

$$\begin{array}{r} x-1 \\ x^2+x+1 \overline{)\,x^3} \\ x^3+x^2+x \\ \hline -x^2-x \\ -x^2-x-1 \\ \hline 1 \end{array}$$

3) Since $P(1+2i) = 0$ & $P'(1+2i) = 0$

$\therefore (1+2i)$ is double root.

Polynomial $P(z)$ has real coefficients

$\therefore (1-2i)$ is also double root.

-3 is real root of $P(z)$

& $P(0) \neq 0$

$\therefore$ Minimum possible degree of $P(z)$ is 5.

Hence C is correct answer.

4) $(1+i) = \sqrt{2}\,\text{Cis}\frac{\pi}{4}$

$\therefore (1+i)^4 = 4\,\text{Cis}\,\pi$

$(1-i) = \sqrt{2}\,\text{Cis}\left(-\frac{\pi}{4}\right)$

$\therefore (1-i)^2 = 2\,\text{Cis}\left(-\frac{\pi}{2}\right)$

Hence $\frac{(1+i)^4}{(1-i)^2} = 4\,\text{Cis}\,\pi \div 2\,\text{Cis}\left(-\frac{\pi}{2}\right)$

$= -2i$

$\therefore$ New complex number will be twice as far from the origin as P with clockwise rotation by $\frac{\pi}{2}$.

Hence D is correct answer.

5) $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix} 13 \\ 10 \\ 1 \end{pmatrix}$

For perpendicular vectors dot product is zero.

$\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = 2(-3) - 3(4) + 4(-1)$

$= -6 - 12 - 4 = -22$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = -3(13) + 4(10) - 1(1)$

$= -39 + 40 - 1 = 0$

$\underset{\sim}{c} \cdot \underset{\sim}{a} = 13(2) + 10(-3) + 1(4)$

$= 26 - 30 + 4 = 0$

$\therefore \underset{\sim}{b}$ and $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{c}$.

Hence B is correct answer.

6) The position of the particle for certain values of t:

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$x = t^2$	0	$\frac{\pi^2}{4}$	π^2	$\frac{9\pi^2}{4}$	$4\pi^2$
$y = 2\cos t$	2	0	-2	0	2
$z = 2\sin t$	0	2	0	-2	0

$\therefore$ particle is rotating in anticlockwise direction in yz plane.

starting from $(2,0) \to (0,2) \to (-2,0) \to (0,-2)$. &

At the same time its x-coordinate is increasing at an increasing rate.

$\therefore$ Motion of the particle is a spiral about the x-axis in an anticlockwise direction.

Hence B is correct answer.

7) P lies on AB in the ratio AP : PB = 1 : 2

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-2, -10, 11)$$

$$\therefore \overrightarrow{OP} = \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$$

$$= (4, 3, -5) + \frac{1}{3}(-2, -10, 11)$$

$$= \left(\frac{10}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$$

$\therefore$ A is correct answer.

8) $\int_0^a [f(x-a) + f(a-x)]\,dx = 0$

Let $I = \underbrace{\int_0^a f(x-a)\,dx}_{I_1} + \underbrace{\int_0^a f(a-x)\,dx}_{I_2}$

$$I_1 = \int_0^a f(x-a)\,dx$$

Let $x - a = u$; $x = 0 \Rightarrow u = -a$

$dx = du$ $\quad x = a \Rightarrow u = 0$

$$\therefore I_1 = \int_{-a}^0 f(u)\,du$$

$$I_2 = \int_0^a f(a-x)\,dx$$

Let $a - x = v$; $x = 0 \Rightarrow v = a$

$-dx = dv$ $\quad x = a \Rightarrow v = 0$

$$\therefore I_2 = \int_a^0 -f(v)\,dv = -\int_a^0 f(v)\,dv = \int_0^a f(v)\,dv$$

$$\therefore I = \int_{-a}^0 f(u)\,du + \int_0^a f(v)\,dv = \int_{-a}^a f(x)\,dx$$

Since $I = 0 \quad \forall \ a > 0$

$$\therefore \int_{-a}^{a} f(x)\,dx = 0 \quad \forall \ a > 0$$

$\Rightarrow f(x)$ is odd function.

- Integral of an odd function over a symmetric interval $[-a, a]$ is zero
- Conversely, if the integral over every symmetric interval is zero, the function must be odd.

$\therefore$ D is correct answer.

9) Since $\frac{dv}{dx} = \frac{2}{3v}$

$\therefore$ acceleration $= v\frac{dv}{dx} = v \cdot \frac{2}{3v} = \frac{2}{3}$ (constant)

acceleration is constant.

For velocity: $\frac{dv}{dt} = \frac{2}{3}$

$$dv = \frac{2}{3}\,dt$$

$$v = \frac{2}{3}t + C$$

The particle starts from rest at the origin

$\therefore$ when $t = 0$, $v = 0$ then $C = 0$

$$\therefore v = \frac{2}{3}t$$

As t increases, v increases

$\therefore$ The particle has constant acceleration and increasing velocity.

Hence C is correct answer.

10) $\omega^2 + \frac{1}{1+\omega^2} + \omega = 0$

$\omega^2(1+\omega^2) + 1 + \omega(1+\omega^2) = 0$

$\omega^2 + \omega^4 + 1 + \omega + \omega^3 = 0$

$\omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$

$\frac{\omega^5 - 1}{\omega - 1} = 0$

$\omega^5 - 1 = 0 \quad (\because \omega \neq -1)$

$\therefore \omega^5 = 1$

$\omega^{2026} = (\omega^5)^{405} \cdot \omega$

$= (1)^{405} \cdot \omega$

$= \omega$

$\therefore$ A is correct answer.

END OF SECTION I

Section II

Question 11

a) $z = 1 + ai$, $w = a + i$, $a \in R$

i) $z + \bar{w} = 1 + ai + \overline{(a+i)}$ — 1 Correct Solution

$= 1 + ai + a - i$

$= (1+a) + i(a-1)$

ii) $zw = (1+ai)(a+i)$ — 2 Correct Solution

$= a + i + a^2 i + ai^2$ — 1 Find expression for zw.

$= a + i + a^2 i - a$

$= i(1 + a^2)$

Since $a^2 + 1 > 0$, and zw is purely imaginary,

$\therefore \text{Arg}(zw) = \frac{\pi}{2}$

iii) $\frac{z}{w} = \frac{1+ai}{a+i}$

$= \frac{1+ai}{a+i} \times \frac{(a-i)}{(a-i)}$

$= \frac{(a-a^2) + i(a^2-1)}{a^2+1}$

$= \frac{2a + i(a^2-1)}{a^2+1}$

For purely real; $\text{Im}\left(\frac{z}{w}\right) = 0$ — 2 Correct Solution

$\therefore \frac{a^2-1}{a^2+1} = 0$ — 1 Recognise $\text{Im}\left(\frac{z}{w}\right) = 0$

$a^2 - 1 = 0$

$a^2 = 1$

$a = \pm 1$

11 b) $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{i}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

2 Correct Solution

1 Correctly find $\underset{\sim}{a}$.

$$\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{b}$$

$$= \underset{\sim}{i} - 3\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$$

$$= -2\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$$

$$|\underset{\sim}{a}| = \sqrt{(-2)^2 + (-2)^2 + (2)^2}$$

$$= \sqrt{4 + 4 + 4}$$

$$= 2\sqrt{3}$$

11 c) $\int \frac{x^3}{x^2+9}\,dx$

$$\int \frac{x^3}{x^2+9}\,dx = \int \frac{x(x^2+9) - 9x}{x^2+9}\,dx$$

$$= \int \left(x - \frac{9x}{x^2+9}\right)dx$$

$$= \int x\,dx - \frac{9}{2}\int \frac{2x}{x^2+9}\,dx$$

$$= \frac{x^2}{2} - \frac{9}{2}\ln(x^2+9) + C$$

2 Correct Solution

1 Correctly express the given integral in terms of x by suitable substitution or equivalent merit.

11d) i) If n is not odd then n^2-6n+5 is not even. 1 correct solution.

ii) Let $n = 2k$, $k \in \mathbb{Z}$

then $n^2-6n+5 = (2k)^2-6(2k)+5$ 2 Correct solution.

$= 4k^2-12k+5$ 1 Use proof by contrapositive or equivalent merit.

$= 2(2k^2-6k+2)+1$

$= 2M+1$ (Since $k \in \mathbb{Z} \therefore M \in \mathbb{Z}$)

$\therefore n^2-6n+5$ is odd, if n is even, and $M = 2k^2-6k+2 \in \mathbb{Z}$

Hence, P has been proven by contraposition.

11e) $z^2 - z + (4-2i) = 0$

By quadratic formula: 3 Correct Solution

$$z = \frac{1 \pm \sqrt{1-4(4-2i)}}{2}$$ 2 Uses quadratic formula and $\sqrt{-15+8i} = \pm(1+4i)$

$$= \frac{1 \pm \sqrt{1-16+8i}}{2}$$

$$= \frac{1 \pm \sqrt{-15+8i}}{2}$$ 1 Uses quadratic formula or finds the $\Delta = -15+8i$

To find sq. roots of $-15+8i$

Let $(a+ib)^2 = -15+8i$

$a^2-b^2 = -15$ & $2ab = 8$

Solve simultaneously; $a = \pm 1$ & $b = \pm 4$

$\therefore \sqrt{-15+8i} = \pm(1+4i)$

$$\therefore z = \frac{1 \pm (1+4i)}{2}$$

$$= \frac{2+4i}{2}, \frac{-4i}{2}$$

$$= 1+2i, -2i$$

Question-12

a) $P(z) = z^4 - 8z^3 + pz^2 + qz - 80$

As the coefficients of this polynomial are real, the conjugate of the root $(3+i)$, which is $(3-i)$ is also a root.

Let the roots of $P(z)$ be:

$$(3+i),\ (3-i),\ \alpha,\ \beta$$

Sum of roots: $(3+i)+(3-i)+\alpha+\beta = 8$

$$6+\alpha+\beta = 8$$

$$\alpha+\beta = 2. \quad \text{——} (1)$$

Product of roots: $(3+i)(3-i)\alpha\beta = -80$

$$10\alpha\beta = -80$$

$$\alpha\beta = -8 \quad \text{——} (2)$$

The quad. eq. with α and β is

$$z^2 - 2z - 8 = 0$$

$$(z-4)(z+2) = 0$$

$$\therefore z = 4 \text{ or } -2$$

Hence four roots are: $3+i,\ 3-i,\ 4,\ -2$

3 Correct Solution.

2 Significant progress to find all four roots.

1 Recognise $(3-i)$ as second root.

ii) $P(z) = (z-3-i)(z-3-i)(z-4)(z+2)$

$$= ((z-3)^2 - i^2)(z^2-2z-8)$$

$$= (z^2-6z+9+1)(z^2-2z-8)$$

$$= (z^2-6z+10)(z^2-2z-8)$$

1 Correct Solution.

b) $a^2+b^2 = (a-b)^2 + 2ab$

$> (a-b)^2 + 2 \quad (\because ab > 1)$

$\frac{a^2+b^2}{a-b} \geq (a-b) + \frac{2}{(a-b)} \quad (\because a > b)$

$\geq 2\sqrt{(a-b)\frac{2}{(a-b)}} \quad (\because \text{AM/GM})$

$= 2\sqrt{2}$

Hence proved.

2 Correct Solution.

1 Significant progress towards proof by using given conditions $a > b$ and $ab > 1$ or AM/GM.

c) Since $l_1 \perp l_2$

$\therefore$ Dot product of their direction vector is zero.

i.e. $\begin{pmatrix} 2 \\ \ln a \\ \ln a \end{pmatrix} \cdot \begin{pmatrix} -3 \\ \ln a \\ 1 \end{pmatrix} = 0$

$-6 + (\ln a)^2 + \ln a = 0$

$(\ln a)^2 + (\ln a) - 6 = 0$

$(\ln a + 3)(\ln a - 2) = 0$

$\ln a = -3 \quad \text{or} \quad \ln a = 2$

$a = e^{-3} \quad \text{or} \quad a = e^2$

3 Correct Solution

2 Solve the quadratic equation in $\ln a$.

1 Use dot product

12d) $\int_0^{\frac{\pi}{4}} \frac{4}{5-3\cos\theta}\, d\theta$

Let $t = \tan\theta$

$dt = \sec^2\theta\, d\theta$

$dt = (1+\tan^2\theta)\, d\theta$ At $\theta = 0$, $t = 0$

$dt = (1+t^2)\, d\theta$ & $\theta = \frac{\pi}{4}$, $t = 1$

$$d\theta = \frac{dt}{1+t^2}$$

$$\therefore \int_0^{\pi/4} \frac{4\, d\theta}{5-3\cos\theta} = \int_0^1 \frac{4}{5-3\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{dt}{(1+t^2)}$$

$$= \int_0^1 \frac{4\, dt}{5(1+t^2)-3(1-t^2)}$$

$$= \int_0^1 \frac{4\, dt}{2+8t^2}$$

$$= \int_0^1 \frac{2\, dt}{1+4t^2}$$

$$= \left[\tan^{-1}(2t)\right]_0^1$$

$$= \tan^{-1}2 - \tan^{-1}0$$

$$= \tan^{-1}2$$

3 Correct Solution.

2 Finds Primitive function

1 Express integrand in terms of t.

12e) i) $x = 5 - 4\cos^2 2\pi t$

$x = 5 - 2(2\cos^2 2\pi t)$

$= 5 - 2(1 + \cos 4\pi t)$ $(\because 2\cos^2\theta = 1 + \cos 2\theta)$

$= 3 - 2\cos 4\pi t$

2 Correct Solution

$\therefore$ Centre of motion $= 3$

Amplitude $= 2$

1 Correct centre of motion or amplitude or express $x = 3 - 2\cos 4\pi t$

ii) Period $(T) = \frac{2\pi}{4\pi} = \frac{1}{2}$

1 Correct solution.

Since Amplitude $= 2$ cm

$\therefore$ In every $\frac{1}{2}$ second particle travel 8cm

$\therefore$ In 5 seconds particle will travel $8 \times 2 \times 5 = 80$ cm.

Question- 13

a) $P(x) = 120 + 120x + 60x^2 + 20x^3 + 5x^4 + x^5$

By Contradiction:

Let $P(x)$ has double root at $x = \alpha$

$\therefore P(\alpha) = 0$ and $P'(\alpha) = 0$

$P'(x) = 120 + 120x + 60x^2 + 20x^3 + 5x^4$

$P(x) - P'(x) = x^5$

$\therefore P(\alpha) - P'(\alpha) = \alpha^5$

Since $P(\alpha) = P'(\alpha) = 0$

$\therefore \alpha^5 = 0$

$\Rightarrow \alpha = 0$

$\therefore$ Double root is at $x = 0$

However $P(0) = 120 \neq 0$

which is a contradiction.

Hence $P(x)$ has no double root.

2 Correct Solution.

1 Obtains $P(x)$ has double root at $x = \alpha$ & $P(\alpha) = 0 = P'(\alpha)$

b) $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$, $n \geq 0$

Let $u = (1+x^2)^{-n}$ $\quad dv = dx$

$du = -2nx(1+x^2)^{-n-1}$ $\quad v = x$

$\therefore I_n = \left[x(1+x^2)^{-n}\right]_0^1 + 2n\int_0^1 x^2(1+x^2)^{-n-1}\,dx$

$= \left[1(1+1)^{-n} - 0\right] + 2n\int_0^1 \frac{x^2+1-1}{(1+x^2)^{n+1}}\,dx$

$= 2^{-n} + 2n\int_0^1 \left(\frac{x^2+1}{(1+x^2)^{n+1}} - \frac{1}{(1+x^2)^{n+1}}\right)dx$

$= 2^{-n} + 2n\int_0^1 \left(\frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}}\right)dx$

$= \frac{1}{2^n} + 2n(I_n - I_{n+1})$

$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$

$2nI_{n+1} = \frac{1}{2^n} + 2nI_n - I_n$

$2nI_{n+1} = \frac{1}{2^n} + (2n-1)I_n$ 3 Correct Solution

$I_{n+1} = \frac{1}{2n}\left[\frac{1}{2^n} + (2n-1)I_n\right]$ 2 Uses an appropriate technique to find I_n or I_{n+1} after applying integration by parts

Hence shown.

1 Applies integration by parts

ii) Sub $n=2$ in part i)

$$I_3 = \frac{1}{4}\left[\frac{1}{4} + 3I_2\right]$$

$$= \frac{1}{16} + \frac{3}{4}I_2$$

$$I_2 = \frac{1}{2}\left[\frac{1}{2} + I_1\right]$$

$$= \frac{1}{4} + \frac{1}{2}I_1$$

$$I_1 = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1$$

$$= \tan^{-1}1 - \tan^{-1}0$$

$$= \frac{\pi}{4}$$

$$\therefore I_2 = \frac{1}{4} + \frac{\pi}{8}$$

& $$I_3 = \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right)$$

$$= \frac{1}{16} + \frac{3}{16} + \frac{3\pi}{32}$$

$$= \frac{1}{4} + \frac{3\pi}{32}$$

2 Correct solution.

1 Attempts to use previous result as a recurrence formula to find I_2 or I_3 or find $I_1 = \frac{\pi}{4}$

c) i) Mass of particle = 1kg.

Forces are:

Gravitation force $\underset{\sim}{F_g} = (0, -10)$

Air resistance $= -5\underset{\sim}{v}$

Hence $m\underset{\sim}{a} = m\underset{\sim}{F_g} - 5\underset{\sim}{v}$

$$\underset{\sim}{a} = \begin{pmatrix} 0 \\ -10 \end{pmatrix} - 5\underset{\sim}{v} \quad (\because m=1)$$

$$\therefore \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -5\dot{x} \\ -10 - 5\dot{y} \end{pmatrix}$$

Horizontal Component:

$$\ddot{x} = -5\dot{x}$$

$$\text{or } \frac{d\dot{x}}{dt} = -5\dot{x}$$

$$\int \frac{d\dot{x}}{\dot{x}} = -5\int dt$$

$$\ln \dot{x} = -5t + C$$

$$\dot{x} = e^{-5t+C}$$

$$\dot{x} = Ae^{-5t} \quad (A = e^C)$$

At $t = 0$, $\dot{x} = V\cos\alpha$

$\therefore A = V\cos\alpha$

Hence $\dot{x} = Ve^{-5t}\cos\alpha$

Vertical Component

$$\ddot{y} = -5\dot{y} - 10$$

$$\frac{d\dot{y}}{dt} = -5(\dot{y}+2)$$

$$\frac{d\dot{y}}{\dot{y}+2} = -5\,dt$$

$$\int \frac{d\dot{y}}{\dot{y}+2} = -5\int dt$$

$$\ln(\dot{y}+2) = -5t + C$$

$$\dot{y} + 2 = e^{-5t+C}$$

$$\dot{y} = Be^{-5t} - 2 \quad (B = e^C)$$

At $t = 0$, $\dot{y} = V\sin\alpha$

$\therefore B = V\sin\alpha + 2$

Hence $\dot{y} = (V\sin\alpha + 2)e^{-5t} - 2$

Hence $\underset{\sim}{v}(t) = \begin{pmatrix} Ve^{-5t}\cos\alpha \\ (V\sin\alpha + 2)e^{-5t} - 2 \end{pmatrix}$

3 Correct Solution

2 Correctly find $\dot{x}$ or $\dot{y}$

1 Express $\underset{\sim}{a} = \underset{\sim}{F_g} - 5\underset{\sim}{v}$ or $\ddot{x} = -5\dot{x}$ & $\ddot{y} = -5\dot{y} - 10$

ii) Sub $t=1$ in part i)

$$\therefore \underset{\sim}{v}(1) = \begin{pmatrix} V\bar{e}^{5}\cos\alpha \\ (V\sin\alpha + 2)\bar{e}^{5} - 2 \end{pmatrix} \quad \text{——— (1)}$$

However, given that $\underset{\sim}{v}(1) = \begin{pmatrix} 250\,e^{-5} \\ (250\sqrt{3}+2)\bar{e}^{5} - 2 \end{pmatrix}$ ——— (2)

Equate (1) & (2) ; $V\cos\alpha = 250$ & $V\sin\alpha = 250\sqrt{3}$

$$\therefore V^2 = (250)^2 + (250\sqrt{3})^2 \Rightarrow V = 500 \text{ m/s}$$

$$\& \quad \tan\alpha = \sqrt{3} \Rightarrow \alpha = 60^\circ$$

2 Correct Solution

1 Correct value for V or α or finding two equation $V\cos\alpha = 250$ & $V\sin\alpha = 250\sqrt{3}$

iii) Sub $V = 500$ and $\alpha = 60^\circ$ in $\underset{\sim}{v}(t)$

$$\underset{\sim}{v}(t) = \begin{pmatrix} 500\,\bar{e}^{5t}\cos 60^\circ \\ (500\sin 60^\circ + 2)\bar{e}^{5t} - 2 \end{pmatrix}$$

$$= \begin{pmatrix} 250\,\bar{e}^{5t} \\ (250\sqrt{3}+2)\bar{e}^{5t} - 2 \end{pmatrix}$$

At $t=0$; horizontal velocity is $\dot{x} = 250$

Let at time $t=T$, the particle reaches at max height.

$$\therefore \text{At } t=T; \quad \dot{y} = 0$$

$$\therefore (250\sqrt{3}+2)\bar{e}^{5T} - 2 = 0$$

$$\bar{e}^{5T} = \frac{2}{250\sqrt{3}+2}$$

$$= \frac{1}{125\sqrt{3}+1}$$

$\therefore$ At $t = T$; Horizontal velocity is $\dot{x} = 250e^{-5T}$

$$\therefore \dot{x} = \frac{250}{125\sqrt{3}+1}$$

Ratio of two horizontal velocities is

$$250 : \frac{250}{125\sqrt{3}+1}$$

$$= (125\sqrt{3}+1) : 1$$

Hence shown

2 Correct solution

1 find time at max height or horizontal velocity at max. height.

Question 14

a) Since $a, b, c \in \mathbb{R}^+$

$$\therefore \frac{a+b+c}{3} \geq \sqrt[3]{abc} \quad \text{(AM/GM Property)}$$

$$\therefore \frac{1}{3} \geq \sqrt[3]{abc} \quad (\because a+b+c=1) \quad \text{---} \quad (1)$$

Also, for $a, b, c > 0$

$$\therefore \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3\left(\sqrt[3]{\frac{1}{abc}}\right)$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3(3) \quad \text{---} \quad \left(\text{from } (1)\ \frac{1}{\sqrt[3]{abc}} \geq 3\right)$$

$$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$$

Hence proved.

2 Correct solution.

1 Uses AM/GM along with substitution of $a+b+c=1$

ii) $L.H.S = \left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$

$= \left(\frac{1-a}{a}\right)\left(\frac{1-b}{b}\right)\left(\frac{1-c}{c}\right)$

$= \frac{(1-a)(1-b)(1-c)}{abc}$

$= \frac{1-(a+b+c)+(ab+bc+ca)-abc}{abc}$

$= \frac{1-1+ab+bc+ca-abc}{abc} \qquad (\because a+b+c=1)$

$= \frac{ab+bc+ca-abc}{abc}$

$= \frac{ab}{abc}+\frac{bc}{abc}+\frac{ca}{abc}-\frac{abc}{abc}$

$= \frac{1}{c}+\frac{1}{a}+\frac{1}{b}-1$

$\geq 9-1 \qquad$ (By part i)

$= 8$

$= R.H.S$

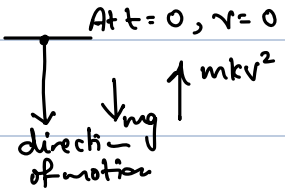
Hence proved.

3 Correct Solution.

2 Significant progress towards the solution using the result of part i)

1 Correctly expand L.H.S $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$ or equivalent merit

14 b)



Equation of motion:

$$m\ddot{x} = mg - mkv^2$$

$$\therefore \ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$x = -\frac{1}{2k}\ln(g - kv^2) + C$$

At $x = 0, v = 0 \Rightarrow 0 = -\frac{1}{2k}\ln g + C$

$$\therefore x = \frac{1}{2k}\ln g - \frac{1}{2k}\ln(g - kv^2)$$

$$= \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

$$2kx = \ln\left(\frac{g}{g - kv^2}\right)$$

$$e^{2kx} = \frac{g}{g - kv^2}$$

$$g - kv^2 = ge^{-2kx}$$

$$kv^2 = g - ge^{-2kx}$$

$$v^2 = \frac{g}{k}(1 - e^{-2kx}) \quad \text{—— (1)}$$

3 Correct solution

2 Finds x as a function of v.

1 Correct expression for the resultant force.

ii) As $x \to \infty$, $v^2 \to \frac{g}{k}$ $(\because e^{-2kx} \to 0)$

$$\therefore v_T = \sqrt{\frac{g}{k}}$$

1 Correct solution

iii) When $v = \frac{1}{2}v_T = \frac{1}{2}\sqrt{\frac{g}{k}}$

$$v^2 = \frac{g}{4k}$$

Sub in (1) $\frac{g}{4k} = \frac{g}{k}(1 - e^{-2kx})$

$$\frac{1}{4} = 1 - e^{-2kx}$$

$$e^{-2kx} = 1 - \frac{1}{4}$$

$$e^{-2kx} = \frac{3}{4}$$

$$e^{2kx} = \frac{4}{3}$$

$$2kx = \ln\frac{4}{3}$$

$$x = \frac{1}{2k}\ln\frac{4}{3}$$

Hence Shown.

2 Correct solution

1 Sub. $v = \frac{1}{2}v_T$ in the expression of part i) or equivalent merit

c) i) $L.H.S = \frac{1}{x\sqrt{x^{2n}-1}} = \frac{1}{x\sqrt{x^{2n}(1-x^{-2n})}}$

$$= \frac{1}{x \cdot x^n\sqrt{1-x^{-2n}}}$$

$$= \frac{x^{-1-n}}{\sqrt{1-x^{-2n}}}$$

$$= R.H.S$$

1 Correct Solution

c ii) $\int_2^4 \frac{1}{x\sqrt{x^{2n}-1}} = \int_2^4 \frac{x^{-n-1}}{\sqrt{1-(x^{-n})^2}}\,dx$

Let $u = x^{-n}$; $x = 2 \Rightarrow u = 2^{-n}$

$du = -nx^{-n-1}\,dx$ $\quad x = 4 \Rightarrow u = 4^{-n}$

$= -\frac{1}{n}\int_{2^{-n}}^{4^{-n}} \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{n}\left[\cos^{-1}u\right]_{4^{-n}}^{2^{-n}}$

$= \frac{1}{n}\left[\cos^{-1}(4^{-n}) - \cos^{-1}(2^{-n})\right]$

or $-\frac{1}{n}\left[\sin^{-1}(4^{-n}) - \sin^{-1}(2^{-n})\right]$

3 Correct Solution.

2 Finds the primitive function

1 Uses part i) to rearrange integrand and substitute $u = x^{-n}$ or equivalent merit

iii) $\int_2^4 \frac{dx}{x\sqrt{x^{2n}-1}} > \int_2^4 \frac{dx}{x\sqrt{x^{2n}}}$ $\quad (\because x^{2n} > x^{2n}-1)$

$= \int_2^4 \frac{dx}{x^{n+1}} = \int_2^4 x^{-(n+1)}\,dx$

$= -\frac{1}{n}\left[x^{-n}\right]_2^4$

$= -\frac{1}{n}\left[4^{-n} - 2^{-n}\right]$

$= \frac{1}{n}\left[2^{-n} - 4^{-n}\right]$

$= \frac{1}{n}\left[\frac{1}{2^n} - \frac{1}{4^n}\right]$

$= \frac{1}{n}\left[\frac{2^n-1}{4^n}\right]$

Hence shown

2 Correct solution

1 Identifies $x^{2n} > x^{2n}-1$

or equivalent merit.

Question 15

a) $\ddot{x} = 4x^3 - 16x$

$\dot{x}\frac{d\dot{x}}{dx} = 4x^3 - 16x$

$\int_{v_0}^{\dot{x}} \dot{x}\, d\dot{x} = \int_0^x (4x^3 - 16x)\, dx$

$\left[\frac{\dot{x}^2}{2}\right]_{v_0}^{\dot{x}} = \left[x^4 - 8x^2\right]_0^x$

$\frac{1}{2}\left[\dot{x}^2 - v_0^2\right] = x^4 - 8x^2$

$\dot{x}^2 - v_0^2 = 2x^4 - 16x^2$

$\dot{x}^2 - 2x^4 + 16x^2 = v_0^2$

2 correct solution

1 Uses $\dot{x}\frac{d\dot{x}}{dx}$ or equivalent merit.

Since v_0^2 is constant, therefore $(\dot{x})^2 - 2x^4 + 16x^2$ does not change with time.

Alternatively:

To show $(\dot{x})^2 - 2x^4 + 16x^2$ does not change with time, we can show $(\dot{x})^2 - 2x^4 + 16x^2 =$ constant

$\frac{d}{dt}\left[(\dot{x})^2 - 2x^4 + 16x^2\right] = 2\dot{x}\ddot{x} - 8x^3\dot{x} + 32x\dot{x}$

$= 2\dot{x}\left[\ddot{x} - 4x^3 + 16x\right]$

$= 2\dot{x}\left[\ddot{x} - \ddot{x}\right] \quad (\because \ddot{x} = 4x^3 - 16x)$

$= 0$

2 correct solution

Since $\frac{d}{dt}\left[\dot{x}^2 - 2x^4 + 16x^2\right] = 0$

$\therefore \dot{x}^2 - 2x^4 + 16x^2 =$ constant.

1 Correctly differentiate the given quantity w.r.t. time.

Q)iii) When $v_0 = \sqrt{\frac{31}{8}}$, $(\dot{x})^2 - 2x^4 + 16x^2 = \frac{31}{8}$

2 correct solution.

$(\dot{x})^2 = 2x^4 - 16x^2 + \frac{31}{8}$

1 Uses the fact that $(\dot{x})^2 \geq 0$ to find the region.

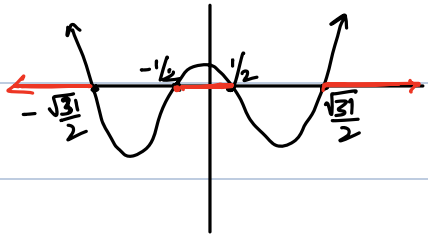
$= \frac{1}{8}(16x^4 - 128x^2 + 31)$

$= \frac{1}{8}(4x^2 - 31)(4x^2 - 1)$

Since $(\dot{x})^2 \geq 0$

$\therefore (4x^2 - 31)(4x^2 - 1) \geq 0$

$(2x - \sqrt{31})(2x + \sqrt{31})(2x - 1)(2x + 1) \geq 0$



However as $x = 0$ when $t = 0$, therefore particle must remains in the region $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Q) iii) $\ddot{x} = 4x^3 - 16x^2$

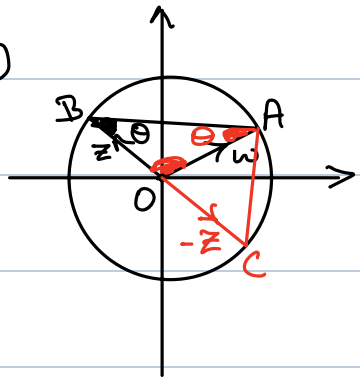
1 correct solution.

$= 4x^2(x - 4)$, which is not in the form of $\ddot{x} = -n^2x$

$\therefore$ The particle is not moving in simple harmonic motion.

15 b)

i)



In ΔOAB

Since $\angle OBA = \theta$

$\therefore \angle OAB = \theta \quad (\because OA = OB)$

$\therefore \angle BOA = \pi - 2\theta$ (Angle sum of ΔOAB)

$\therefore z = w\,\mathrm{cis}(\pi - 2\theta)$

$= w[\cos(\pi - 2\theta) + i\sin(\pi - 2\theta)]$

$= w(-\cos 2\theta + i\sin 2\theta)$

Hence shown.

2 Correct Solution.

1 Find $\angle BOA$ or equivalent merit.

ii) $\overrightarrow{AB} = z - w$

$\overrightarrow{OC} = -z$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= w + z$

In ΔABC; $\angle BAC = \frac{\pi}{2}$ (Angle in semicircle)

$\therefore \tan\theta = \frac{|\overrightarrow{AC}|}{|\overrightarrow{AB}|}$

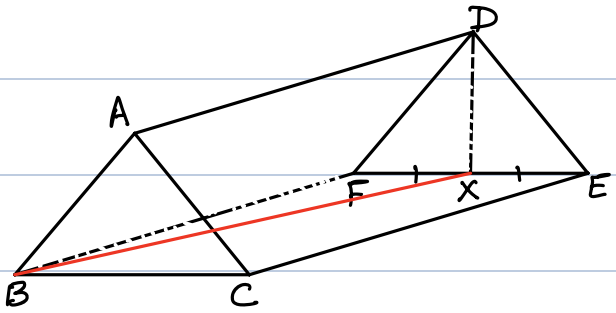
$= \frac{|w+z|}{|z-w|}$

$\therefore |z+w| = \tan\theta\,|z-w|$

2 Correct Solution

1 Find $\overrightarrow{AB}$ or $\overrightarrow{AC}$ or $\angle BAC = \frac{\pi}{2}$ or equivalent merit.

15 c)



$|\vec{AB}| = m$, $|\vec{CE}| = n$, $\vec{XF} = \frac{1}{2}\vec{EF}$, $\vec{XE} = \frac{1}{2}\vec{FE}$

i) To show $\vec{XD} \perp \vec{EF}$, prove that $\vec{XD} \cdot \vec{EF} = 0$

$\vec{XD} = \vec{XF} + \vec{FD}$

$= \frac{1}{2}\vec{EF} + \vec{FD}$

$\vec{XD} \cdot \vec{EF} = \left(\frac{1}{2}\vec{EF} + \vec{FD}\right) \cdot \vec{EF}$

$= \frac{1}{2}\vec{EF} \cdot \vec{EF} + \vec{FD} \cdot \vec{EF}$

$= \frac{1}{2}|\vec{EF}|^2 - \vec{FD} \cdot \vec{FE}$

$= \frac{1}{2}|\vec{EF}|^2 - |\vec{FD}||\vec{FE}| \cos \angle DFE$

$= \frac{1}{2}m^2 - (m \times m \times \cos 60^\circ)$ $(\because \triangle DEF$ is equilateral $\therefore \angle DFE = 60^\circ)$

$= \frac{1}{2}m^2 - \frac{1}{2}m^2 = 0$

$\therefore \vec{XD} \perp \vec{EF}$

3 Correct Solution

2 Applies the property of equilateral $\triangle$ to the dot product. or equivalent merit.

1 Express $\vec{XD} = \vec{XF} + \vec{FD}$ or attempt to use dot product $\vec{XD} \cdot \vec{EF}$.

2

ii) Join B to X ; In $\triangle BFX$

$|\vec{BX}| < |\vec{BF}| + |\vec{FX}|$ (By Triangle Inequality)

$= n + \frac{1}{2}m$ $(\because |\vec{BF}| = n, |\vec{FX}| = \frac{1}{2}|\vec{FE}|)$

$\therefore |\vec{BX}| + |\vec{BF}| < n + \frac{1}{2}m + n$

$= \frac{1}{2}m + 2n$

$< 2m + 2n$

$= 2(m+n)$, which is perimeter of rectangle BCEF.

Hence shown.

2 Correct Solution

1 Use triangle inequality or equivalent merit.

Question 16

a) $$\int_0^{\frac{\pi}{4}} \frac{1}{\cos x \sqrt{\sec x - \tan x}}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec x}{\sqrt{\sec x - \tan x}}\, dx$$

At $x = 0 \Rightarrow u = \sec 0 - \tan 0 = 1$

At $x = \frac{\pi}{4} \Rightarrow u = \sec\frac{\pi}{4} - \tan\frac{\pi}{4}$

$= \sqrt{2} - 1$

Let $\sec x - \tan x = u$

$$(\sec x \tan x - \sec^2 x)\, dx = du$$

$$\sec x (\tan x - \sec x)\, dx = du$$

$$\sec x\, dx = \frac{du}{-u}$$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\sec x}{\sqrt{\sec x - \tan x}}\, dx = \int_1^{\sqrt{2}-1} \frac{1}{\sqrt{u}} \left(-\frac{du}{u}\right)$$

$$= \int_1^{\sqrt{2}-1} -u^{-3/2}\, du$$

$$= 2\left[u^{-1/2}\right]_1^{\sqrt{2}-1}$$

$$= 2\left[\frac{1}{\sqrt{\sqrt{2}-1}} - 1\right]$$

$$= \frac{2}{\sqrt{\sqrt{2}-1}} - 2$$

4 Correct solution.

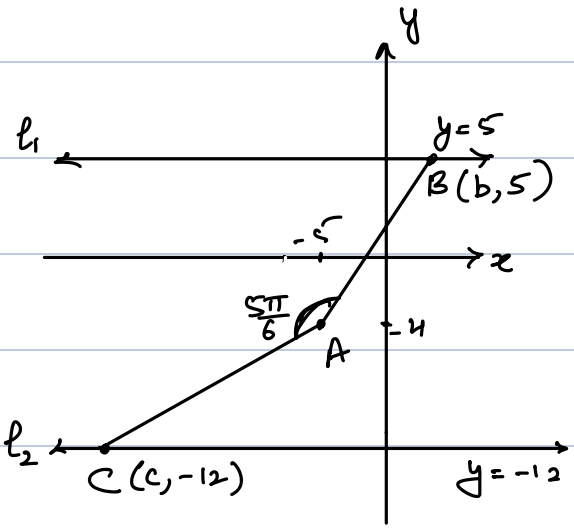
3 Correctly integrate after substitution.

2 Express x in terms of u along with both limits.

1 Express x in terms of u after substitution.

16 b) $A(-5,-4)$, Let $B(b,5)$

and $C(c,-12)$



$$\vec{AC} = 2\vec{AB}\left(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}\right)$$

$$= 2\vec{AB}\left(-\tfrac{\sqrt{3}}{2} + i\tfrac{1}{2}\right)$$

$$\vec{AC} = \vec{AB}(-\sqrt{3} + i)$$

$$\vec{AC} = (c+5) - 8i$$

$$\vec{AB} = (b+5) + 9i$$

$$\therefore (c+5) - 8i = [(b+5) + 9i](-\sqrt{3} + i)$$

$$= [-\sqrt{3}(b+5) - 9] + i[(b+5) - 9\sqrt{3}]$$

Compare real and imaginary parts;

$$-8 = (b+5) - 9\sqrt{3} \Rightarrow b = 9\sqrt{3} - 13$$

$$c+5 = -\sqrt{3}(b+5) - 9$$

$$c = -\sqrt{3}(9\sqrt{3} - 13 + 5) - 9 - 5$$

$$= -27 + 8\sqrt{3} - 14$$

$$= 8\sqrt{3} - 41$$

$$\therefore z_c = 8\sqrt{3} - 41 - 12i$$

3 Correct Solution

2 Equating real and imaginary part to find $b = 9\sqrt{3} - 13$ or significant progress towards solution.

1 Express $\vec{AC} = 2\vec{AB}\,\mathrm{Cis}\,\frac{5\pi}{6}$

16 c) i) Let Statement S(r) be $z_r = i^r \binom{n}{r} z_0$

4 key points

1. Proving true for n=0
2. Clearly stating the assumption and what to be proved.
3. Using the assumption in the proof.
4. Correctly proving the required result

Prove S(0) is true ; let r = 0

L.H.S = z_0

R.H.S = $i^0 \binom{n}{0} z_0$

$= 1(z_0)$

$= z_0$

$\therefore$ S(0) is true.

3 Correctly done all 4 key points.
2 Correctly done 3 out of 4 key points.
1 Correctly done 2 out of 4 key points.

Assume S(p) is true for $p \geq 0$

i.e $z_p = i^p \binom{n}{p} z_0$ ——— (1)

Prove S(p+1) is true

i.e $z_{p+1} = i^{p+1} \binom{n}{p+1} z_0$

Sub z = p in given condition

$\therefore (p+1) z_{p+1} - i(n-p) z_p = 0$

$(p+1) z_{p+1} = i(n-p) z_p$

$z_{p+1} = i\left(\frac{n-p}{p+1}\right) z_p$

$= i\left(\frac{n-p}{p+1}\right) \cdot i^p \binom{n}{p} z_0$ (Using (1))

$= i^{p+1} \left(\frac{n-p}{p+1}\right) \frac{n!}{p!(n-p)!} z_0$

$= i^{p+1} \frac{n!}{(p+1)!(n-p-1)!} z_0$

$= i^{p+1} \binom{n}{p+1} z_0$

Hence true for S(p+1) if true for S(p)

$\because$ S(p) true $\Rightarrow$ S(p+1) true and S(0) is true. Therefore by principle of mathematical induction S(r) is true for all integers $r \geq 0$.

16 c) ii) Using the given identity:

$$Z_0 + Z_1 + Z_2 + \dots + Z_n = 2^n$$

$$Z_0 + i\binom{n}{1}Z_0 + i^2\binom{n}{2}Z_0 + \dots + i^n\binom{n}{n}Z_0 = 2^n$$

(Using the result of part (i))

$$Z_0\left[1 + i\binom{n}{1} + i^2\binom{n}{2} + \dots + i^n\binom{n}{n}\right] = 2^n$$

$$Z_0(1+i)^n = 2^n$$

$$Z_0 = \left(\frac{2}{1+i}\right)^n$$

$$= \left(\frac{2}{1+i} \times \frac{1-i}{1-i}\right)^n$$

$$= \left[\frac{2(1-i)}{2}\right]^n$$

$$= (1-i)^n$$

2 Correct solution.

1 Uses result of part i) to get $Z_0(1+i)^n = 2^n$

Hence shown.

c) iii) $|Z_0|^2 + |Z_1|^2 + |Z_2|^2 + \dots + |Z_n|^2$

$$= |Z_0|^2 + \left|i\binom{n}{1}Z_0\right|^2 + \left|i^2\binom{n}{2}Z_0\right|^2 + \dots + \left|i^n\binom{n}{n}Z_0\right|^2$$

$$= |Z_0|^2\left[\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2\right]$$

$$= |(1-i)^n|^2\binom{2n}{n} \quad \left(\because \text{Given that } \sum_{m=0}^{n}\binom{n}{m}^2 = \binom{2n}{n}\right)$$

$$= 2^n\binom{2n}{n}$$

$$= 2^n\frac{(2n)!}{n!\,n!}$$

$$= \frac{2^n}{n!}(2n)(2n-1)(2n-2)\dots(n+1)$$

$$\leq \frac{2^n}{n!}\left[\frac{2n + (2n-1) + (2n-2) + \dots + (n+1)}{n}\right]^n \quad (\because \text{AM/GM inequality})$$

$$= \frac{2^n}{n!}\left(\frac{3n+1}{2}\right)^n \quad (\text{Taking sum of A.P})$$

$$= \frac{(3n+1)^n}{n!}$$

$2n + (2n-1) + (2n-2) + \cdots + (n+1)$

is an AP with n number of terms with common difference (-1).

$$\therefore S_n = \frac{n}{2}[a + l]$$

$$= \frac{n}{2}[2n + (n+1)]$$

$$= \frac{n(3n+1)}{2}$$

3 Correct Solution.

2 Obtains $2^n\binom{2n}{n}$ by using $z_0 = (1-i)^n$.

1 Attempts to apply the result of part (i) (OR) the sum provided.

END OF EXAM.