Caringbah High School

Year 12 2026

Mathematics Extension 2

HSC Course

Assessment Task 4 – Trial HSC Examination

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 100

Section I **10 marks**

Attempt Questions 1-10

Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **90 marks**

Attempt Questions 11-16

Write your answers in the relevant section of the answer booklet provided. Ensure your name or student number is clearly visible.

Name: _____ **Class:** _____

Marker's Use Only							
Section I	**Section II**						**Total**
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**	**Q15**	**Q16**	
/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1. What is the value of i^{767} ?

(A) $-i$

(B) -1

(C) 1

(D) i

2. Which of the following is the negation of the statement:

'All solutions to the equation lie below the real axis on the complex plane.'

(A) All solutions to the equation lie above the real axis

(B) At least one solution to the equation lies above the real axis

(C) All solutions to the equation lie on or above the real axis

(D) At least one solution to the equation lies on or above the real axis

Section I continues next page

3. Consider the equation $z = 1 + i\sqrt{3}$.

Which of the following is equal to z^4 ?

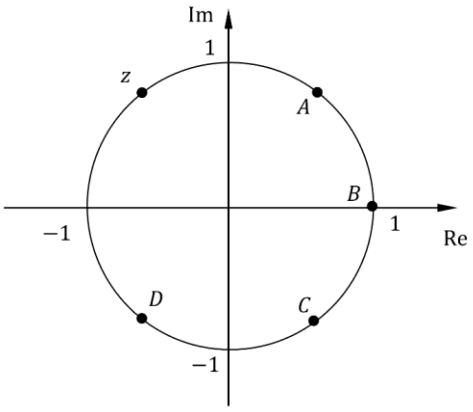
(A) $8 + 8i\sqrt{3}$

(B) $8 - 8i\sqrt{3}$

(C) $-8 + 8i\sqrt{3}$

(D) $-8 - 8i\sqrt{3}$

4. A complex number z lies on the unit circle in the second quadrant of the complex plane, as shown in the diagram



Which of the points marked *A, B, C* and *D* best shows the position of $z\bar{z}$?

(A) *A*

(B) *B*

(C) *C*

(D) *D*

Section I continues next page

5. The line $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ \sqrt{7} \end{pmatrix} + \lambda \begin{pmatrix} \sqrt{7} \\ -2\sqrt{7} \\ c \end{pmatrix}$ is a tangent to a sphere centred at $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

The point of contact between the tangent and the sphere is $\begin{pmatrix} 1 \\ -1 \\ \sqrt{7} \end{pmatrix}$.

What is the value of c ?

(A) $c = -3$

(B) $c = -1$

(C) $c = 1$

(D) $c = 3$

6. Which of the following expressions would have an anti-derivative which includes $\sin^{-1}(f(x))$?

(A) $\dfrac{2x}{\sqrt{x^4 - 2x^2 + 1}}$

(B) $\dfrac{2x}{\sqrt{-x^4 - 4x^2 + 5}}$

(C) $\dfrac{2x}{x^4 - 2x^2 + 1}$

(D) $\dfrac{2x}{-x^4 - 4x^2 + 5}$

Section I continues next page

7. A particle is moving along the x-axis. At time t seconds it has displacement x metres from the origin O, velocity $v\ ms^{-1}$, given by $v^2 = 10x - x^2$, and acceleration $a\ ms^{-2}$. What is the value of $|a|$ when $|v| = 3$?

(A) 1

(B) 2

(C) 3

(D) 4

8. Let $f(x)$ be a continuous function for all real x. It is given that $\int_0^{\pi} f(\sin x)dx = k$.

What is the value of $\int_0^{\pi} x f(\sin x)dx$ in terms of k ?

You may use the fact that $\int_0^a g(x)dx = \int_0^a g(a-x)dx$

(A) 0

(B) $\frac{k}{2}$

(C) $\frac{\pi k}{2}$

(D) πk

Section I continues next page

9. A particle of unit mass is released from rest at $x = 1$ and moves in a straight line due to the force $F = \dfrac{5 - 7x}{x^3}$.

Which of the following equations represents the particle's velocity?

(A) $V = \pm \dfrac{3}{x}\sqrt{x^2 - 7x + 5}$

(B) $V = \pm \dfrac{1}{x}\sqrt{-9x^2 + 14x - 5}$

(C) $V = \pm 3x\sqrt{x^2 - 7x + 5}$

(D) $V = \pm x\sqrt{-9x^2 + 14x - 5}$

10. The complex number $z = x + iy$ satisfies $|z| \leq y + 1$ and $|z| \leq x + 1$.

What is the maximum possible value of $|z|$?

(A) $3 + \sqrt{2}$

(B) $1 + \sqrt{2}$

(C) $2 - \sqrt{2}$

(D) $2 + \sqrt{2}$

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer the questions in the Section II Writing Booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the **relevant section** in writing booklet.

(a) Write $1 - i\sqrt{3}$ in modulus-argument form. **2**

(b) Consider the complex numbers $w = 4 - 3i$ and $z = 1 - 2i$.

(i) Evaluate $w + z$. **1**

(ii) Evaluate $\frac{w}{z}$ in the form of $a + ib$. **2**

(c) From an origin O, given vectors $\overrightarrow{OA} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. **2**

Find the angle between the two vectors, correct to the nearest minute.

(d) Find $\int xe^{5x}\,dx$. **2**

(e) Prove by contradiction that $\sqrt{3} - \sqrt{2}$ is irrational. **2**

(f) The vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\,\underset{\sim}{k}$ are parallel.

(i) Find $|\underset{\sim}{u}|$. **1**

(ii) Given that $|\underset{\sim}{v}| = 6$, find the values of a, b and c. **3**

End of Question 11

Question 12 (15 marks) Use the **relevant section** in writing booklet.

(a) Evaluate **2**

$$\int \frac{2x-1}{x^2+2x+3}\,dx.$$

(b) A particle moves in simple harmonic motion along the x-axis. The central point of motion of the particle is the origin. The particle is initially at its right-hand extremity, 3 metres to the right of the origin, and takes 0.5 seconds to first reach the origin. **2**

Find an equation for the displacement, x, of the particle in terms of t.

(c) Consider the function $f(x) = \dfrac{3kx-18}{(x+4)(x-2)}$, where k is a positive constant.

(i) Express $f(x)$ as partial fractions in terms of k. **2**

(ii) Hence, find the exact value of k, such that $\int_{-3}^{1} f(x)dx = 21$. **2**

(d) A particle is moving in simple harmonic motion given by

$$\ddot{x} = -\,2\pi^2(x+1)$$

Initially the particle is at the centre of motion with velocity $3\pi\sqrt{2}\ ms^{-1}$.

(i) Find the equation of v^2 . **2**

(ii) Find the particle's exact period **and** the values of x at the endpoints of motion. **2**

(e) For $n > 0$ fixed consider distinct constants a, b, $c \in (n, 2n)$.

(i) Show that $a + b - c > 0$. **1**

(ii) Hence, or otherwise, show that $a > \sqrt{a^2 + 2cb - b^2 - c^2}$. **2**

End of Question 12

Question 13 (15 marks) Use the **relevant section** in writing booklet.

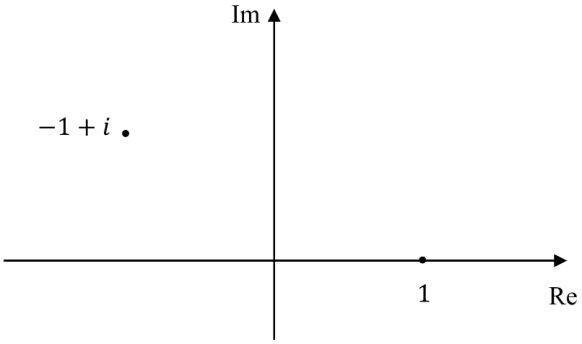
(a) For all $k \in \mathbb{Z}$, let P be the proposition: **2**

'If $5k^2 + 3k + 4$ is odd, then k is even.'

Prove by contrapositive that P is true.

(b) A particle starts at the origin at rest with acceleration, a, given by $a = -v^2 + 2v$, **3**
where v is the velocity of the particle.
Find the expression for v in terms of x, the displacement of the particle.

(c) A cubic polynomial, $P(z) = z^3 + bz^2 + cz + d$, where b, c and d are real numbers, has three distinct roots. Two of the roots are shown on the complex plane below.



(i) What is the third root of $P(z)$? **1**

(ii) Find the values of b, c and d. **3**

(iii) What are the roots of $P(z-1)$? **1**

(d) Show that $\int e^{6x}\sin(e^{3x})dx = \dfrac{\sin(e^{3x}) - e^{3x}\cos(e^{3x})}{3} + c$ **2**

Question 13 continues next page

(e) Let $f(x) = \ln\left(\frac{1+x}{1-x}\right) - 2x - \frac{2x^3}{3}$. **3**

Use differentiation to prove that $\ln\left(\frac{1+x}{1-x}\right) > 2x + \frac{2x^3}{3}$ for $0 < x < 1$.

End of Question 13

Question 14 (15 marks) Use the **relevant section** in writing booklet.

(a) Prove using mathematical induction that $a_n = \dfrac{n-1}{n+1}$ satisfies the recurrence relation **3**

$$a_{n+1} = a_n + \frac{2}{n^2 + 3n + 2}$$

with the initial condition $a_1 = 0$, for all integers $n \geq 1$.

(b) Using a suitable substitution, show that **3**

$$\int_0^{\ln(2)} \frac{e^{2x}}{\sqrt{e^x + 1}}\, dx = \frac{2\sqrt{2}}{3}$$

(c) Sketch the region of the complex plane consisting of all $z \in \mathbb{C}$ such that **3**

$$|Arg(z^2)| < \frac{\pi}{6}$$

(d) **(i)** Use De Moivre's theorem to show that **2**

$$\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1.$$

(ii) Hence, or otherwise, show that the equation $16x^4 - 16x^2 + 1 = 0$ has the roots $x = \cos\dfrac{\pi}{12}, \cos\dfrac{5\pi}{12}, \cos\dfrac{7\pi}{12}$ and $\cos\dfrac{11\pi}{12}$. **2**

(iii) Hence, show that $\cos\dfrac{5\pi}{12} = \dfrac{\sqrt{2-\sqrt{3}}}{2}$. **2**

End of Question 14

Question 15 (15 marks) Use the **relevant section** in writing booklet.

(a) The point of intersection of the line through $A\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $B\begin{pmatrix}4\\5\\0\end{pmatrix}$ with the line through $C\begin{pmatrix}0\\9\\10\end{pmatrix}$ and $D\begin{pmatrix}1\\7\\7\end{pmatrix}$ is E.

(i) Prove that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$. **1**

(ii) Find the coordinates of E. **2**

(b) Find $\int x\cos^{-1}x\,dx$. **3**

(c) A particle is projected from the origin with initial velocity u at an angle α to the horizontal. The particle experiences a resistive force proportional to its velocity in both the horizontal and vertical directions, such that the equations of motion are: **3**

$$\ddot{x} = -k\dot{x} \text{ and } \ddot{y} = -k\dot{y} - g$$

where k is a positive constant and g is the acceleration due to gravity.

Let θ be the angle that the trajectory makes with the horizontal at any time t.

Prove that

$$\tan\theta = \tan\alpha - \frac{g(e^{kt}-1)}{ku\cos\alpha}.$$

(d) Let $I_n = \int \sec^n(x)dx$ for $n \geq 0$.

(i) For $n \geq 2$, show that $I_n = \frac{\tan(x)\sec^{n-2}(x)}{n-1} + \frac{n-2}{n-1} I_{n-2}$. **3**

(ii) Hence, using an appropriate substitution, show that **3**

$$\int_1^2 \sqrt{x^2-1}\,dx = \sqrt{3} - \frac{1}{2}\ln\left(2+\sqrt{3}\right)$$

End of Question 15

Question 16 (15 marks) Use the **relevant section** in writing booklet.

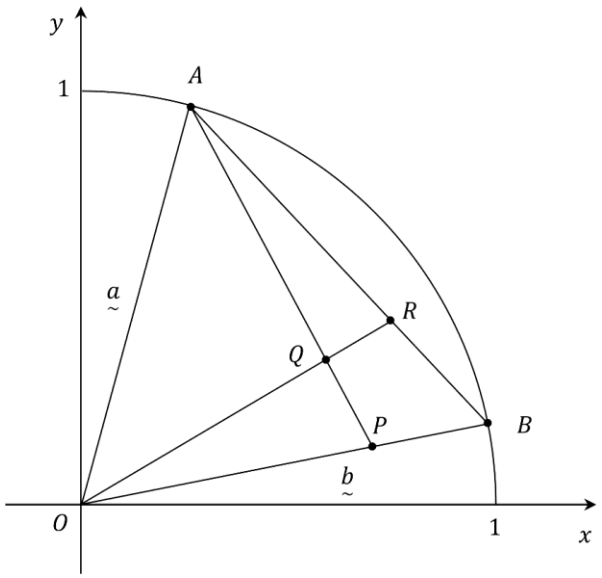
(a) Prove by mathematical induction that **3**

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+...+\sqrt{n} \geq \frac{2}{3}n\sqrt{n}, \text{ for all } n \geq 1$$

(b) In triangle OAB, A and B lie on the unit circle in the first quadrant as shown.

P lies on OB such that $\overrightarrow{OP} = 3\overrightarrow{PB}$, Q lies on AP such that $\overrightarrow{AQ} = 2\overrightarrow{QP}$, and

R lies on AB such that O, Q and R are collinear. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



(i) It is given $\overrightarrow{AR} = k\overrightarrow{AB}$. Find the value of k. **3**

(ii) Given that $\overrightarrow{OQ} \cdot \overrightarrow{AP} = 0$, find the size of $\angle AOB$, correct to the nearest minute. **2**

(c) A particle of unit mass is projected vertically upwards from the ground with an initial speed of u. **3**

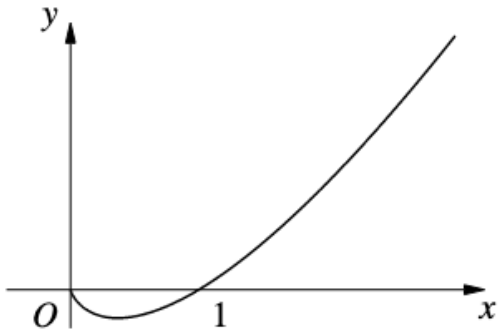
It experiences a resistive force, kv, and a gravitational force, g.

When the particle lands back on the ground, it has a speed of w.

Show that the particle takes **in total** $T = \dfrac{u+w}{g}$ seconds to land back on the ground.

Question 16 continues next page

(d) Define the sequence a_n by $a_1 = 1$ and $a_{k+1} = a_k \times (k+1)^{k+1}$. **4**



By considering the area under the curve $y = x\ln x$ shown above, prove that

$$a_n > e^{\frac{n^2}{4}(2\ln(n)-1)}.$$

End of Examination

Solutions

MC

1) $i^{767} = (i^4)^{191} \times i^3$ $\therefore$ (A)

$= i^{191} \times (-i)$

$= -i$

2) (D)

3) $(1+i\sqrt{3})^4 = \left[2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)\right]^4$

$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ $\therefore$ (D)

$= 16\left(-\frac{1}{2} + \left(\frac{-i\sqrt{3}}{2}\right)\right)$

$= -8 - 8i\sqrt{3}$

4) $z\bar{z} = |z|^2 = 1$ $\therefore$ (B)

5) $r \perp$ tangent:

$\begin{pmatrix} 1 \\ -1 \\ \sqrt{7} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{7} \\ -2\sqrt{7} \\ c \end{pmatrix} = 0$ $\therefore$ (A)

$\sqrt{7} + 2\sqrt{7} + c\sqrt{7} = 0$

$3\sqrt{7} + c\sqrt{7} = 0$

$c = -3$

6) The integrand has to be written in form

$$\frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}$$

(B)

A: $\frac{2x}{\sqrt{(x^2-1)^2}}$ ✗

B: $\frac{2x}{\sqrt{-(x^4+4x^2+4-4)+5}} = \frac{2x}{\sqrt{-(x^2+2)^2+9}}$

$= \frac{2x}{\sqrt{9-(x^2+2)^2}}$

C: ✗

D: ✗

7) $a = \frac{1}{2}\frac{d}{dx}(v^2)$

$a = \frac{1}{2}(10-2x)$

$a = 5-x$

$|v| = 3 \Rightarrow |v|^2 = 9$

$-x^2+10x = 9$

$x^2-10x+9 = 0$

$(x-9)(x-1) = 0$

$x = 9$ or $x = 1$

When $x = 9 \Rightarrow a = 4$

$x = 1 \Rightarrow a = -4$ $\Big\} \Rightarrow |a| = 4$ (D)

8) Let $I = \int_0^{\pi} x f(\sin x)\,dx$

Since $\int_0^a g(x)\,dx = \int_0^a g(a-x)\,dx$

$$I = \int_0^{\pi} (\pi - x) f(\sin(\pi - x))\,dx$$

* $\sin(\pi - x) = \sin x$

$$\therefore \quad I = \int_0^{\pi} (\pi - x) f(\sin x)\,dx$$

$$I = \pi \int_0^{\pi} f(\sin x)\,dx - \int_0^{\pi} x f(\sin x)$$

$$I = \pi k - I$$

$$2I = \pi k$$

$$I = \frac{\pi k}{2}$$

(C)

9) $f = ma$, unit mass $\Rightarrow m = 1$ so $a = f$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{5 - 7x}{x^3}$$

$$\frac{1}{2}v^2 = \frac{5x^{-2}}{-2} - \frac{7x^{-1}}{-1}$$

$$v^2 = \frac{-5}{x^2} + \frac{14}{x} + c$$

at $v = 0$, $x = 1$

$$0 = -5 + 14 + c$$

$$\Rightarrow c = -9$$

$$\Rightarrow V^2 = -\frac{5}{x^2} + \frac{14}{x} - 9$$

$$V^2 = -\frac{5 + 14x - 9x^2}{x^2} \quad \therefore (B)$$

$$V = \pm \frac{1}{x}\sqrt{-9x^2 + 14x - 5}$$

10) Multiple ways to do this;

the inequality is symmetric in x & y, so max occurs when $x = y$

$\cdot |z| = \sqrt{x^2 + x^2}$

$= \sqrt{2x^2}$

$\cdot \sqrt{2x^2} = x + 1$

$2x^2 = x^2 + 2x + 1$

$x^2 - 2x - 1 = 0$

$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2}$

$= \frac{2 \pm \sqrt{2}}{2}$

$x = 1 \pm \sqrt{2}$

$\Rightarrow y = 1 \pm \sqrt{2}$

$|z|_{max}$ when $x = y = 1 + \sqrt{2}$ as $|1+\sqrt{2}| > |1-\sqrt{2}|$

$|z| = \sqrt{2}(1+\sqrt{2})$

$= 2 + \sqrt{2}$

$|z| \leq y + 1$

$|z| \leq x + 1$

$y \geq |z| - 1$

$x \geq |z| - 1$

$x^2 + y^2 \geq 2(|z| - 1)^2$

$|z|^2 \geq 2(|z|^2 - 2|z| + 1)$

$|z|^2 - 4|z| + 2 \leq 0$

$|z| = \frac{4 \pm \sqrt{16 - 4(2)}}{2}$

$|z| = 2 \pm \sqrt{2}$

Max $\Rightarrow$

$|z| = 2 + \sqrt{2}$.

(D)

(D)

Question 11

a) $1-i\sqrt{3} = 2\left(\cos\left(-\frac{\pi}{3}\right) + \sin\left(-\frac{\pi}{3}\right)\right)$ — (1) (1)

b) i) $w+z = 5-5i$

ii) $\frac{w}{z} = \frac{(4-3i)(1+2i)}{(1-2i)(1+2i)}$ — (1)

$= \frac{4+8i-3i+6}{1+4}$

$= \frac{10+5i}{5}$

$= 2+i$ — (1)

c) $\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$= \frac{-1-6+3}{\sqrt{11}\times\sqrt{14}}$

$= \frac{-4}{\sqrt{154}}$ — (1)

$\theta \simeq 108^\circ 48'$ — (1)

d) $\int xe^{5x}\,dx$

$u = x \qquad v' = e^{5x}$

$u' = 1 \qquad v = \frac{e^{5x}}{5}$

$= \left[\frac{xe^{5x}}{5}\right] - \frac{1}{5}\int e^{5x}\,dx$ — (1)

$$= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + c \quad \text{—(1)}$$

e) Assume $\sqrt{3} - \sqrt{2}$ is rational

so $\exists\, m, n \in \mathbb{N}$ coprime such that

$$\sqrt{3} - \sqrt{2} = \frac{m}{n}$$

$$\sqrt{3} = \frac{m}{n} + \sqrt{2}$$

$$3 = \frac{m^2}{n^2} + \frac{2\sqrt{2}m}{n} + 2$$

$$1 - \frac{m^2}{n^2} = \frac{2\sqrt{2}m}{n} \quad \text{— (1)}$$

$$2\sqrt{2} = \frac{n}{m} - \frac{m^2 n}{n^2 m}$$

$$\sqrt{2} = \frac{n^2 - m^2}{2mn} \in \mathbb{Z}, \in \mathbb{Q}$$

but $\sqrt{2}$ is irrational so contradiction —(1)

f) i) $|\underset{\sim}{u}| = \sqrt{4 + 4 + 1} = 3$ —(1)

ii) $|\underset{\sim}{v}| = 6 = 2|\underset{\sim}{u}|$ —(1)

$\Rightarrow \underset{\sim}{v} = \pm 2\underset{\sim}{u}$

$= \pm 2(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$ —(1)

$= \pm (4\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k})$

$\underset{\sim}{v} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$ or $\underset{\sim}{v} = -4\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$ —(1)

Question 12

a) $\int \frac{2x-1}{x^2+2x+3}\,dx$

$= \int \frac{2x+2-3}{x^2+2x+3}\,dx$

$= \int \frac{2x+2}{x^2+2x+3}\,dx - \int \frac{3}{x^2+2x+3}\,dx$

$= \ln|x^2+2x+3| - 3\int \frac{1}{(x+1)^2+2}\,dx$

$= \ln|x^2+2x+3| - \frac{3}{\sqrt{2}}\tan^{-1}\left|\frac{x+1}{\sqrt{2}}\right| + C$

(1) (1)

b) $T = 4 \times a.\bar{5}$

$= 2\,s$

$\frac{2\pi}{n} = 2 \Rightarrow n = \pi$ } (1)

* starts extreme right $\Rightarrow x = A\cos(nt)$

$A = 3 - 0 = 3$

$x = 3\cos \pi t$ — (1)

c) i) $\frac{3kx-18}{(x+4)(x-2)} = \frac{a}{x+4} + \frac{b}{x-2}$

$\frac{3kx-18}{(x+4)(x-2)} = \frac{a(x-2)+b(x+4)}{(x+4)(x-2)}$

$3kx-18 = a(x-2)+b(x+4)$

$x \to 2$ $\quad 6k-18 = 6b$

$b = k-3$ — (1)

$x \to -4$ $\quad -12k-18 = -6a$

$a = 2k+3$ — (1)

$f(x) = \frac{2k+3}{x+4} + \frac{k-3}{x-2}$

ii) $\int_{-3}^{1} f(x)\,dx = 21$

$\int_{-3}^{1} \left(\frac{2k+3}{x+4} + \frac{k-3}{x-2}\right) dx = 21$

$\left[(2k+3)\ln|x+4| + (k-3)\ln|x-2|\right]_{-3}^{1} = 21$ — (1)

$(2k+3)\ln 5 + (k-3)\ln 1 - (2k+3)\ln 1 - (k-3)\ln 5 = 21$

$(k+6)\ln 5 = 21$

$k = \frac{21}{\ln 5} - 6$ — (2)

d) i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2n^2(x+1)$

$\frac{1}{2}v^2 = -2n^2\left(\frac{x^2}{2}+x\right)+C$

$v^2 = -4n^2\left(\frac{x^2}{2}+x\right)+C_1$

$v^2 = -2n^2x^2 - 4n^2x + C_1$ — (1)

$x=-1$, $v = 3n\sqrt{2}$

$18n^2 = -2n^2(1) - 4n^2(-1) + C_1$

$C_1 = 18n^2 - 2n^2$

$C_1 = 16n^2$

$v^2 = -2n^2x^2 - 4n^2x + 16n^2$ — (1)

ii) $v^2 = -2n^2(x^2+2x-8)$

$v^2 = -2n^2(x+4)(x-2)$

$x+4=0$ or $x-2=0$

$\boxed{x=-4}$ or $\boxed{x=2}$ (1)

$n^2 = 2n^2$

$n = \sqrt{2}n$

$P = \frac{2n}{\sqrt{2}n} = \sqrt{2}$ — (1)

e) i) $a > n$

$b > n$

$a < 2n \Rightarrow -c > -2n$

$a + b - c > n + n - 2n$ — (1)

$a + b - c > 0$

ii) by same logic of i)

$a + c - b > 0$ — (1)

by AM-GM

$$\frac{(a+b-c)(a+c-b)}{2} > \sqrt{(a+b-c)(a+c-b)}$$

$a > c - b$ & $a > b - c$

so

$$\frac{2a}{2} > \sqrt{a^2 - (b-c)^2}$$

$$a > \sqrt{a^2 - b^2 - c^2 + 2bc}$$ — (1)

Question 13

a) If k is odd then $5k^2+3k+4$ is even — (1)

Let $k = 2m+1$, $m \in \mathbb{Z}$

$$5k^2+3k+4 = (2m+1)^2 + (2m+1) + 4$$
$$= 5(4m^2+4m+1) + 6m + 7$$
$$= 20m^2 + 26m + 12$$
$$= 2(10m^2 + 13m + 6) \quad \text{— (1)}$$

where $(10m^2 + 13m + 6) \in \mathbb{Z}$

$\Rightarrow$ even

b) $a = -v^2 + 2v$

$$v\frac{dv}{dx} = -v^2 + 2v$$

$$\frac{dv}{dx} = -v + 2$$

$$\frac{dx}{dv} = \frac{1}{-v+2} \quad \text{— (1)}$$

$$\int_0^x dx = \int_0^v \frac{dv}{-v+2}$$

$$x = -\ln|-v+2|\Big]_0^v$$

$$x = \ln(2) - \ln(-v+2)$$

$$\ln(-v+2) = \ln 2 - x \quad \text{— (1)}$$

$-v+2 = 2e^{-x}$

$v = 2 - 2e^{-x}$

$v = 2(1 - e^{-x})$ — (1)

c) i) $\overline{-1+i} = -1-i$

ii) $1 - 1+i-1-i = -b$

$\boxed{b = 1}$ — (1)

$\alpha\beta + \beta\gamma + \beta\alpha = \frac{c}{a}$

$(-1+i) + (-1-i) + (-1-i)(-1+i) = c$

$-2 + (-1)^2 - (i)^2 = c$

$-1 + 1 = c \Rightarrow \boxed{c = 0}$ — (1)

$(1)(-1+i)(-1-i) = -\frac{d}{a}$

$\boxed{d = -2}$ — (1)

iii) $P(z-1) = \pm i, 2$ (one unit right) — (1)

d) next page

d) $\int e^{6x}\sin(e^{3x}) = \frac{1}{3}\int 3e^{3x}\times e^{3x}\sin(e^{3x})\,dx$

$u = e^{3x}$ $\quad v' = 3e^{3x}\sin(e^{3x})$

$u' = 3e^{3x}$ $\quad v = -\cos(e^{3x})$

(1)—

$$= \frac{1}{3}\left(\left[-e^{3x}\cos(e^{3x})\right] + \int 3e^{3x}\cos(e^{3x})\,dx\right)$$

$$= -\frac{1}{3}e^{3x}\cos(e^{3x}) + \frac{1}{3}\sin e^{3x} + C$$

$$= \frac{\sin(e^{3x}) - e^{3x}\cos(e^{3x})}{3} + C \quad -(1)$$

e) $f'(x) = \frac{1}{1+x} + \frac{1}{1-x} - 2 - 2x^2$

$$= \frac{1-x+1+x-2(1-x^2)-2x-(1-x^2)}{(1+x)(1-x)}$$

$$= \frac{2-2+2x^2-2x^2+2x^4}{1-x^4}$$

$$= \frac{2x^4}{1-x^2} \quad -(1)$$

$f'(x) = 0$

$\Rightarrow x = 0$

x	$-\frac{1}{2}$		0		$\frac{1}{2}$
y'	$\frac{1}{6}$		$-$		$\frac{1}{6}$
		/		$-$	/

$\Big\}$ (1)

$\Rightarrow f(x) > 0 \quad 0 < x < 1$

$\ln\left(\frac{1+x}{1-x}\right) > 2x + \frac{2x^3}{3} \quad 0 < x < 1$ $\Big\}$ (1)

Question 14

a) $n=1 \quad a_1 = \frac{1-1}{2} = 0$ ✓

Assume true for $n=k$, so $a_k = \frac{k-1}{k+1}$

To prove true for $n=k+1$

that is $a_{k+1} = \frac{k+1-1}{k+1+1} = \frac{k}{k+2}$

$$\text{LHS } a_{k+1} = \frac{k-1}{k+1} + \frac{2}{k^2+3k+2}$$

$$= \frac{k-1}{k+1} + \frac{2}{(k+1)(k+2)}$$

$$= \frac{(k-1)(k+2) + 2}{(k+1)(k+2)}$$

$$= \frac{k^2+k-2+2}{(k+1)(k+2)}$$

$$= \frac{k(k+1)}{(k+1)(k+2)}$$

$$= \frac{k}{k+2} \quad \text{✓}$$

b) $$\int_0^{\ln 2} \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \frac{2\sqrt{2}}{3}$$

Let $u = e^x + 1$

$\frac{du}{dx} = e^x \Rightarrow du = e^x dx$ } (1)

$$= \int_2^3 \frac{(u-1)}{\sqrt{u}}\,du$$

$$= \int_2^3 \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du$$

$$= \frac{2}{3}u^{\frac{1}{2}+1} - 2u^{-\frac{1}{2}+1}\Big]_2^3 \quad \text{—(1)}$$

$$= \left(\frac{2}{3}\times 3\sqrt{3} - 2\sqrt{3}\right) - \left(\frac{2}{3}\times 2\sqrt{2} - 2\sqrt{2}\right)$$

$$= \left(2\sqrt{3} - 2\sqrt{3}\right) - \left(\frac{4}{3}\sqrt{2} - 2\sqrt{2}\right)$$

$$= -\frac{4\sqrt{2}+6\sqrt{2}}{3} \quad \text{—(1)}$$

$$= \frac{2\sqrt{2}}{3}$$

c) $|Arg(z^2)| < \frac{\pi}{6}$

$-\frac{\pi}{6} < Arg(z^2) < \frac{\pi}{6}$

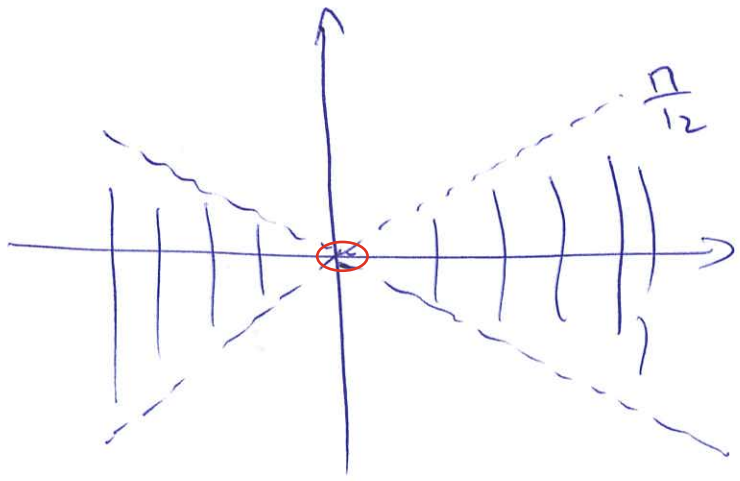
so $-\frac{\pi}{6} < 2\theta < \frac{\pi}{6}$ — (1)

$\Rightarrow -\frac{\pi}{12} < \theta < \frac{\pi}{12}$

& to cover $2\theta + 2k\pi$ in $(-\pi, \pi)$

$-\frac{\pi}{12} < \theta < \frac{\pi}{12}$ & $\frac{11\pi}{12} < \theta \leq \pi$ & — (1)

$-\pi < \theta < -\frac{11\pi}{12}$



— (1)

d) i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

$(\cos\theta + i\sin\theta)^4 = \cos^4\theta - 4\cos^3\theta\sin\theta\, i - 6\cos^2\theta\sin^2\theta + 4\cos\theta\sin^3\theta\, i + \sin^4\theta$ (1)

$\Rightarrow \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^2\theta$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$ (1)

$= 8\cos^4\theta - 8\cos^2\theta + 1$

ii) $16x^4 - 16x^2 + 1 = 0$

$2(8x^4 - 8x^2 + 1) - 2 + 1 = 0$

$2(8x^4 - 8x^2 + 1) = 1$

$8x^4 - 8x^2 + 1 = \frac{1}{2}$ — (1)

Let $x = \cos\theta$

$8\cos^4\theta - 8\cos^2\theta + 1 = \frac{1}{2}$

$\cos 4\theta = \frac{1}{2}$

$4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$ — (1)

iii) $x^2 = \frac{16 \pm \sqrt{16^2 - 4 \times 16}}{32}$

$x^2 = \frac{16 \pm \sqrt{192}}{32}$

$= \frac{2 \pm \sqrt{3}}{4}$ — (1)

$\frac{5\pi}{12}$ is in Q1 so $\cos\left(\frac{5\pi}{12}\right) > 0$

$\Rightarrow (\cos\theta)^2 = \frac{2 \pm \sqrt{3}}{4}$ (1)

$\cos\theta = \sqrt{\frac{2 \pm \sqrt{3}}{4}}$

$\cos\theta = \frac{\sqrt{2 \pm \sqrt{3}}}{2}$

Question 15

a) i) $\vec{AB} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$ $\vec{CD} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

$\vec{AB} \cdot \vec{CD} = 3 - 12 + 9$

$= 0$

ii) $r_{\vec{AB}} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$ → can use A instead

$r_{\vec{CD}} = \begin{bmatrix} 0 \\ 9 \\ 10 \end{bmatrix} + M \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ → can use D instead

$$\begin{cases} 4 + 3\lambda = M & -① \\ 5 + 6\lambda = 9 - 2M & -② \\ 0 - 3\lambda = 10 - 3M & -③ \end{cases}$$ ——— (1)

① − ③ $\quad 4 = -2M + 10$

$-2M = -6$

$M = 3$

sub in ①

$3\lambda = 3 - 4$

$\lambda = -\frac{1}{3}$

Check in ② $\quad 5 + 6\left(-\frac{1}{3}\right) = 5 - 2 = 3$ ✓

$9 - 2M = 9 - 6 = 3$ ✓

$\Rightarrow E = \begin{bmatrix} 0 + 3 \\ 9 - 6 \\ 10 - 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ ——— (1)

b) $\int x \cos^{-1} x \, dx$

$u = \cos^{-1} x \qquad v' = x$

$u' = \frac{-1}{\sqrt{1-x^2}} \qquad v = \frac{x^2}{2}$

$= \left[\frac{x^2}{2}\cos^{-1} x\right] + \int \frac{x^2}{2\sqrt{1-x^2}} dx \quad \text{—(1)}$

* $\int \frac{x^2}{2\sqrt{1-x^2}}$ Let $x = \sin\theta$

$dx = \cos\theta \, d\theta$

$= \frac{1}{2}\int \frac{\sin^2\theta \cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$

$= \frac{1}{2}\int \sin^2\theta \, d\theta$

$= \frac{1}{4}\int (1-\cos 2\theta) d\theta$

$= \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta + C \quad \text{—(1)}$

* $\int x \cos^{-1} x \, dx = \left[\frac{x^2}{2}\cos^{-1} x\right] + \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta + C$

$= \frac{x^2}{2}\cos^{-1} x + \frac{1}{4}\sin^{-1} x - \frac{1}{4}\sin(\sin^{-1} x)\cos(\sin^{-1} x) + C$

$= \frac{x^2}{2}\cos^{-1} x + \frac{1}{4}\sin^{-1} x - \frac{1}{4} x\sqrt{1-x^2} + C$

(1)

c) $\tan\theta = \frac{\dot{y}}{\dot{x}}$

$\ddot{x} = -k\dot{x}$ $\qquad$ $\ddot{y} = -k\dot{y} - g$

exp decay so $\qquad$ exp decay

$\dot{x} = Ae^{-kt}$ $\qquad$ $\dot{y} = Ae^{-kt} - \frac{g}{k}$

at $t=0$, $\dot{x} = u\cos\alpha$ $\qquad$ at $t=0$, $\dot{y} = u\sin\alpha$

$\dot{x} = u\cos\alpha e^{-kt}$ $\qquad$ $\dot{y} = \left(u\sin\alpha + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}$ $\qquad$ — (1)

$$\tan\theta = \frac{\dot{y}}{\dot{x}}$$

$$= \frac{u\sin\alpha e^{-kt} + \frac{g}{k}e^{-kt} - \frac{g}{k}}{u\cos\alpha e^{-kt}}$$

$$\tan\theta = \frac{ku\sin\alpha e^{-kt} + ge^{-kt} - g}{ku\cos\alpha e^{-kt}} \quad — (1)$$

$$\tan\theta = \frac{ku\sin\alpha e^{-kt}}{ku\cos\alpha e^{-kt}} + \frac{ge^{-kt} - g}{ku\cos\alpha e^{-kt}}$$

$$\tan\theta = \tan\alpha + \frac{g(e^{-kt} - 1)}{ku\cos\alpha e^{-kt}} \times e^{kt}$$

$$\tan\theta = \tan\alpha + \frac{g(1 - e^{kt})}{ku\cos\alpha}$$

$$\tan\theta = \tan\alpha - \frac{g(e^{kt} - 1)}{ku\cos\alpha} \quad — (1)$$

d) $I_n = \int \sec^n(x)\,dx$

$= \int \sec^{n-2}(x)\sec^2(x)\,dx$

$u = \sec^{n-2}(x)$ $\quad v' = \sec^2(x)$ — (1)

$u' = (n-2)\sec^{n-3}(x)[\sec x \tan x]$ $\quad v = \tan x$

$I_n = \tan x \sec^{n-2}(x) - (n-2)\int \sec^{n-2}(x)\tan^2 x\,dx$

$I_n = \tan x \sec^{n-2}(x) - (n-2)\int \sec^{n-2}(x)[\sec^2 x - 1]\,dx$ (1)

$I_n = \tan x \sec^{n-2}(x) - (n-2)\left[\int \sec x\,dx - \int \sec^{n-2}(x)\,dx\right]$

$I_n = \tan x \sec^{n-2}(x) - (n-2)[I_n - I_{n-2}]$

$I_n + (n-2)I_n = \tan x \sec^{n-2}(x) + (n-2)I_{n-2}$

$(n-1)I_n = \tan x \sec^{n-2}(x) + (n-2)I_{n-2}$

$I_n = \dfrac{\tan x \sec^{n-2}(x)}{n-1} + \dfrac{(n-2)}{n-1} I_{n-2}$ (1)

ii) Let $x = \sec u$ $\quad * x = 1 \Rightarrow u = 0$

$dx = \sec u \tan u\,du$ $\quad * x = 2 \Rightarrow u = \frac{\pi}{3}$

$$\int_1^2 \sqrt{x^2 - 1}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 u - 1}\,(\sec u \tan u)\,du$$

$$= \int_0^{\frac{\pi}{3}} \sec u \sqrt{\tan^2 u} \tan u \, du$$

$$= \int_0^{\frac{\pi}{3}} \sec u \tan^2 u \, du$$

$$= \int_0^{\frac{\pi}{3}} \sec u \left[\sec^2 u - 1\right] du$$

$$= \int_0^{\frac{\pi}{3}} \left(\sec^3 u - \sec u\right) du$$

$$= \int_0^{\frac{\pi}{3}} \sec^3 u \, du - \int_0^{\frac{\pi}{3}} \sec u \, du \quad \text{— (1)}$$

$$= \left[\frac{\tan u \sec u}{2}\right]_0^{\frac{\pi}{3}} + \frac{1}{2}\int_0^{\frac{\pi}{3}} \sec u \, du - \int_0^{\frac{\pi}{3}} \sec u \, du$$

$$= \left[\frac{\tan u \sec u}{2}\right]_0^{\frac{\pi}{3}} - \frac{1}{2}\int_0^{\frac{\pi}{3}} \sec u \, du \quad \text{— (1')}$$

$$= \left[\frac{\tan u \sec u}{2}\right]_0^{\frac{\pi}{3}} - \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{\sec u (\tan u + \sec u)}{\tan u + \sec u} du$$

$$= \left[\frac{\tan u \sec u}{2}\right]_0^{\frac{\pi}{3}} - \frac{1}{2}\left[\ln|\tan u + \sec u|\right]_0^{\frac{\pi}{3}}$$

$$= \left[\frac{2\sqrt{3}}{2} - 0\right] - \frac{1}{2}\left[\ln|\sqrt{3} + 2| - \ln|0 + 1|\right]$$ (ln|0+1| → 0)

$$= \sqrt{3} - \frac{1}{2}\ln(2+\sqrt{3}) \quad \text{— (1)}$$

a) $n=1$ LHS $= \sqrt{1} = 1$

RHS $= \frac{2}{3}\sqrt{1} = \frac{2}{3}$ $\quad 1 \geq \frac{2}{3}$ ✓

$\Rightarrow$ true for $n=1$

Assume it holds true for all $n=k \geq 1$

so $\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots\sqrt{k} \geq \frac{2}{3}k\sqrt{k}$

to prove

$$\sqrt{1}+\sqrt{2}+\sqrt{3}\cdots+\sqrt{k}+\sqrt{k+1} \geq \frac{2}{3}(k+1)\sqrt{k+1}$$

$$\text{LHS} = \underbrace{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{k}}+\sqrt{k+1}$$

$$\geq \frac{2}{3}k\sqrt{k}+\sqrt{k+1} \quad \text{— (1)}$$

so to prove $\frac{2}{3}k\sqrt{k}+\sqrt{k+1} \geq \frac{2}{3}(k+1)\sqrt{k+1}$

Since everything is positive, same as proving square of both sides

to prove: $2k^{\frac{3}{2}} \geq (2k-1)\sqrt{k+1}$

$$4k^3 \geq (2k-1)^2(k+1)$$

RHS: $(2k-1)^2(k+1)$

$= 4k^3 - 3k + 1$

so need to prove $-3k+1 \leq 0$

i.e $3k \geq 1$

$k \geq \frac{1}{3}$

but $k \geq 1$ so all the above hold true

$$\Rightarrow \sqrt{1} + \sqrt{2} + \cdots + \sqrt{k+1} \geq \frac{2}{3}(k+1)\sqrt{k+1}$$

b) i). $|\vec{OA}| = |\vec{CB}| = 1$

- $\vec{CP} = 3\vec{PB}$
- $\vec{AQ} = 2\vec{QP}$
- O, Q, R collinear

* $\vec{AR} = k\vec{AB}$
* $\vec{OP} = \frac{3}{4}\vec{CB} = \frac{3}{4}\underset{\sim}{b}$
* $\vec{OQ} = \vec{OA} + \vec{AQ}$

$$= \vec{OA} + \frac{2}{3}\vec{AP}$$

$$= \vec{OA} + \frac{2}{3}(\vec{OP} - \vec{OA})$$

$$= \vec{OA} + \frac{2}{3}\left(\frac{3}{4}\vec{OB} - \vec{OA}\right)$$

$$= \underset{\sim}{a} + \frac{2}{3}\left(\frac{3}{4}\vec{CB} - \vec{OA}\right)$$

$$= \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \frac{2}{3}\underset{\sim}{a}$$

$$= \frac{1}{3}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} \quad \text{---} (1)$$

* O, R, Q collinear so $\vec{OR} = \mu\vec{OQ}$

$$= \mu\left[\frac{1}{3}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}\right]$$

* Also $\vec{OR} = \vec{OA} + \vec{AR} = \vec{OA} + k\vec{AB}$

$= \vec{OA} + k(\vec{OB} - \vec{OA})$

$= \underset{\sim}{a} + k(\underset{\sim}{b} - \underset{\sim}{a})$

$= (1-k)\underset{\sim}{a} + k\underset{\sim}{b}$ — (1)

but $\vec{OR} = \frac{1}{3}\mu\underset{\sim}{a} + \frac{1}{2}\mu\underset{\sim}{b}$

$\Rightarrow \frac{\mu}{3} = 1-k$ — (1) $\qquad \frac{\mu}{2} = k$, $\mu = 2k$ — (2)

sub (2) in (1)

$\frac{2k}{3} = 1-k$

$\frac{5k}{3} = 1$

$k = \frac{3}{5}$ — (1)

ii) $\vec{OQ} \cdot \vec{AP} = 0$

$\left(\frac{1}{3}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}\right) \cdot \left(\frac{3}{4}\underset{\sim}{b} - \underset{\sim}{a}\right) = 0$

$\frac{1}{4}\underset{\sim}{a}\cdot\underset{\sim}{b} - \frac{1}{3}|\underset{\sim}{a}|^2 + \frac{3}{8}|\underset{\sim}{b}|^2 - \frac{1}{2}\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$

$-\frac{1}{4}\underset{\sim}{a}\cdot\underset{\sim}{b} + \frac{1}{24} = 0 \qquad \left(|\underset{\sim}{a}| = |\underset{\sim}{b}| = 1\right)$

$\frac{1}{4}\underset{\sim}{a}\cdot\underset{\sim}{b} = \frac{1}{24}$

$\underset{\sim}{a}\cdot\underset{\sim}{b} = \frac{1}{6}$ — (1)

$$\text{Cos}\angle AOB = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$$

$$\text{Cos}\angle AOB = \frac{1}{6}$$

$$\angle AOB \simeq 80°24' \quad \text{—— (1)}$$

c)

$$a_{up} = -g - kv \qquad a_{down} = g - kv$$

$$\frac{dv}{dt} = -g - kv \qquad \frac{dv}{dt} = g - kv$$

$$\frac{dt}{dv} = \frac{1}{-g-kv} \qquad \frac{dt}{dv} = \frac{1}{g-kv}$$

$$\int_0^{t_{up}} dt = \int_u^0 \frac{dv}{-g-kv} \qquad \int_0^{t_{down}} dt = \int_0^w \frac{dv}{g-kv}$$

$$t_u = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right) \qquad t_D = \frac{1}{k}\ln\left(\frac{g}{g-kw}\right)$$

$$T_{total} = \frac{1}{k}\ln\left(\frac{g+kw}{g-kw}\right) \quad \text{—— (1)}$$

$$* \quad -\frac{v\,dv}{dx} = -g - kv \qquad v\frac{dv}{dx} = g - kv$$

$$\frac{dx}{dv} = \frac{-v}{g+kv} \qquad \frac{dx}{dv} = \frac{v}{g-kv}$$

$$x_{up} = \int_u^0 \frac{-v}{g+kv}\,dv \qquad x_{down} = \int_0^w \frac{v}{g-kv}\,dv$$

$$x_{up} = -\int_{w}^{0} \frac{v}{g+kv}dv \qquad x_D = -\int_{0}^{w} \frac{-v}{g-kv}dv$$

$$x_u = \frac{-1}{k}\int_{w}^{0} \frac{kv+g-g}{g+kv}dv \qquad x_D = \frac{-1}{k}\int_{0}^{w} \frac{-kv+g-g}{g-kv}dv$$

$$x_w = -\frac{1}{k}\left[\int_{w}^{0} 1 - \frac{g}{g+kv}\right]dv \qquad x_D = -\frac{1}{k}\left[\int_{0}^{w} 1 - \frac{g}{g-kv}\right]dv$$

$$x_w = \frac{w}{k} - \frac{g}{k^2}\ln\left(\frac{g+kw}{g}\right) \qquad x_D = -\frac{w}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g-kw}\right)$$

— (1)

$$x_{up} = x_D$$

$$\frac{w}{k} - \frac{g}{k^2}\ln\left(\frac{g+kw}{g}\right) = -\frac{w}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g-kw}\right)$$

$$\frac{kw+w}{k} = \frac{g}{k^2}\ln\left(\frac{g+kw}{g-kw}\right)$$

$$\text{so} \quad T = \frac{1}{k}\ln\left(\frac{g+kw}{g-kw}\right)$$

$$\frac{g}{k}T = \frac{w+w}{k}$$

$$\boxed{T = \frac{w+w}{g}}$$

— (1)

d) Since $x \ln x$ is strictly increasing, the right-endpoint rectangle of width 1 will overestimate the area under curve

$$\sum_{k=1}^{n} k \ln k > \int_1^n x \ln x \, dx \quad -(1)$$

$$\int_1^n x \ln x \, dx = \left[\frac{x^2}{2} \ln x\right]_1^n - \int_1^n \frac{x}{2} dx$$

$$= \frac{n^2}{2} \ln n - \left[\frac{x^2}{4}\right]_1^n$$

$$= \frac{n^2}{2} \ln n - \frac{n^2}{4} + \frac{1}{4}$$

$$= \frac{n^2}{4}(2 \ln n - 1) + \frac{1}{4} \quad -(1)$$

So $$\sum_{k=1}^{n} k \ln k > \int_1^n x \ln x \, dx$$

$$\ln(1 \cdot 2^2 \times 3^3 \times \cdots n^n) > \frac{n^2}{4}(2 \ln n - 1) + \frac{1}{4}$$

$$\ln a_n > \frac{n^2}{4}(2 \ln n - 1) \quad -(1)$$

e both sides $\quad a_n > e^{\frac{n^2}{4}(2 \ln n - 1)} \quad -(1)$