Fort Street High School

NESA Number:

Name:

2026 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Teacher	Mr Choi	Ms Kaur	Mr Razzaghi
	□	□	□

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/ or calculations
- Marks may be deducted for careless or badly arranged work.

Total marks: 100

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section
- Write your student number on each answer booklet

Question	Mark
Multiple Choice	/10
Question 11	/15
Question 12	/15
Question 13	/15
Question 14	/15
Question 15	/15
Question 16	/15
Total	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

Question 1.

A particles motion can be described by $\ddot{x} = -9x$.

Which equation below could be a representation of the particle's displacement at any time *t*?

A. $3\sin(t)$

B. $3t\sin(t)$

C. $3\sin(3t)$

D. $3t\sin(3t)$

Question 2.

Given $z = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$, what is the value of z^8?

A. 1

B. -1

C. i

D. -i

Question 3.

What is the value of $\arg\bar{z}$ given the complex number $z = -1 - i\sqrt{3}$?

A. $-\frac{2\pi}{3}$

B. $-\frac{\pi}{3}$

C. $\frac{2\pi}{3}$

D. $\frac{\pi}{3}$

Question 4.

Which of the following points lies on the line $\underset{\sim}{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$?

A. $(5,0,11)$

B. $(3,1,5)$

C. $(7,-1,13)$

D. $(5,1,11)$

Question 5.

The velocity, v ms^{-1} , of a particle moving in a straight line is given by $v = 2\sin(3x)$, where x is the position of the particle from the origin.

What is the acceleration of the particle when it is at the origin?

A. 12 ms^{-2}

B. 24 ms^{-2}

C. 6 ms^{-2}

D. 0 ms^{-2}

Question 6.

Consider the complex number z such that $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$.

What is the value of $Arg\left(\dfrac{\overline{z} + z}{\overline{z}z(\overline{z} - z)}\right)$?

A. $\dfrac{\pi}{2}$

B. π

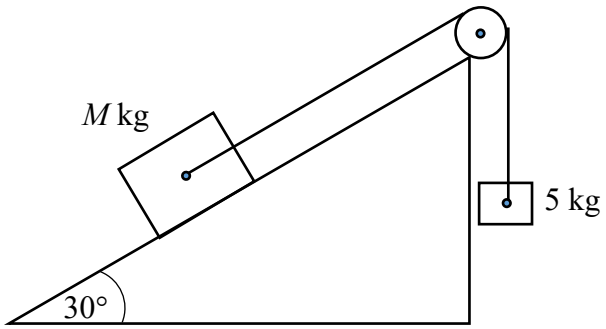
C. $-\dfrac{\pi}{2}$

D. 0

Question 7.

A smooth plane is inclined at an angle of $30°$ to the horizontal. A mass of M kg is attached to a smooth pully using a light inextensible string and moving down the ramp with an acceleration of 2 ms^{-2}.

A 5 kg mass is attached at the other end of the string.



Assuming the acceleration due to gravity is 10 ms^{-2} .

What is the value of M ?

A. 25 kg

B. 20 kg

C. 10 kg

D. $\frac{10(5\sqrt{3}+3)}{11}$ kg

Question 8.

Which expression is equal to $\int \frac{1}{x\sqrt{1-\ln^2 x}}dx$?

A. $\sin^{-1}(\ln x)+c$

B. $\ln\left(\sqrt{1-\ln^2 x}\right)+c$

C. $\tan^{-1}(\ln x)+c$

D. $\ln\left(1-\ln^2 x\right)+c$

Question 9.

Which of the following is an expression for $\int \cos^2 x \sin^7 x\, dx$?

A. $\frac{\cos^3 x}{3} - \frac{3\cos^5 x}{5} + \frac{3\cos^7 x}{7} - \frac{\cos^9 x}{9} + c$

B. $\cos^3 x - 3\cos^5 x + 3\cos^7 x - \cos^9 x + c$

C. $-\frac{\cos^3 x}{3} + \frac{3\cos^5 x}{5} - \frac{3\cos^7 x}{7} + \frac{\cos^9 x}{9} + c$

D. $-\cos^3 x + 3\cos^5 x - 3\cos^7 x + \cos^9 x + c$

Question 10.

Let a, b, c, d be complex numbers such that $a \neq b$ and $c \neq d$.

Which of the following is a necessary and sufficient condition for the line through a and b to be perpendicular to the line through c and d?

A. $\frac{a-b}{c-d} = -\frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}i$

B. $\frac{a-b}{c-d} = -\frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}$

C. $\frac{a-b}{c-d} = \frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}i$

D. $\frac{a-b}{c-d} = \frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}$

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations

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Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) The location of the complex number z is shown on the diagram on page 2 of the Question 11 Writing Booklet.

On the diagram provided in the writing booklet, indicate the location of

(i) $-2zi$ **1**

(ii) $\overline{z} - z$. **1**

(b) The complex numbers z and w are given by $z = 2cis\dfrac{\pi}{6}$ and $w = 3cis\dfrac{2\pi}{3}$. **2**

Find the modulus and argument of $\dfrac{z}{w}$.

(c) (i) Show that $\sqrt{3+4i} = \pm(2+i)$. **1**

(ii) By completing the square solve $z^2 - (4-2i)z - 8i = 0$ where $z \in \mathbb{C}$. **2**

Question 11 continues on page 8

(d) Three forces are acting on an object in such a way that the object is stationary.

(i) Given the first force vector $\boldsymbol{F_1} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$, **1**

show that the magnitude of vector $\boldsymbol{F_1}$ is 3 Newtons.

(ii) The second force vector, $\boldsymbol{F_2}$, has double the magnitude of vector $\boldsymbol{F_1}$ and is **2**

parallel to $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$. Show that vector $\boldsymbol{F_2} = 2\sqrt{3}\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$.

(iii) Hence find the third force vector $\boldsymbol{F_3}$ that is acting on the object. **2**

(e) Evaluate $\int_0^1 \frac{2x^2 - x + 1}{x - 2}\,dx$. **3**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sec x}{\cos x + \sin x}\, dx$. **3**

(b) A particle is moving in a straight line, horizontally away from a fixed point O such that its displacement from the fixed point at time t seconds is x metres. The only force acting on the particle is a resistance force of magnitude mkv^2 newtons, where k is a constant.

Given that the particle starts its motion at O, with a positive initial velocity of u ms^{-1},

(i) Show that the displacement of the particle is given by **2**

$$x = \frac{1}{k}\ln\left(\frac{u}{v}\right).$$

(ii) Show that the velocity of the particle at any time is given by **2**

$$v = \frac{u}{kut + 1}.$$

(iii) Hence find an expression for the displacement of the particle after $\frac{1}{u}$ seconds in terms of k. **2**

(c) Sketch the region of the complex plane defined by $\left|z^2 - \overline{z}^2\right| > 4$. **3**

(d) Find $\int \frac{\left(\sqrt{x}+1\right)^{10}}{\sqrt{x}}\, dx$. **3**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Consider the complex numbers $z = \sqrt{2} + i\sqrt{6}$ and $w = \sqrt{3} - i$. **3**

Find the smallest positive integer value of n for which $\frac{z^n}{w^n}$ is purely real.

(b) The centre of an equilateral triangle is at the origin, and one vertex is at $z_1 = \sqrt{3} + i$. **2**
Using $\text{Arg}(z_1)$, find, in the form of $a + ib$, the complex numbers giving the other two vertices.

(c) A particle is moving in simple harmonic motion with speed v ms^{-1} given by

$$v^2 = -2\pi^2 x^2 - 4\pi^2 x + 16\pi^2,$$

where x is the displacement from the origin, in metres.

(i) Find the exact period of the particle's motion? **2**

(ii) Find the maximum and minimum x-values for which the particle oscillates between. **2**

(d) Find the acute angle between unit vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ such that $\left|\sqrt{3}\underset{\sim}{a} - \underset{\sim}{b}\right| = 1$. **3**

(e) Find $\int \frac{dx}{e^{3x} - 1}$. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Given that $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$,

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$. **2**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx$. **2**

(b) A particle is projected vertically upwards in a resistant medium. The velocity of projection is u ms^{-1}, the mass of the particle is m kg and the resistance force is given by mkv^2 Newtons, where k is a constant.

Given g is acceleration due to gravity,

(i) Show that the maximum height attained is given by $h = \frac{1}{2k} \ln\left(1 + \frac{k}{g} u^2\right)$. **2**

(ii) If the speed of projection is half the terminal velocity, find the ratio of velocity on return to the projection point to the velocity of projection. **4**

(c) Consider the points P and Q with position vectors $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$ and $\underset{\sim}{q} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, respectively.

(i) Find the vector equation of line l passing through P and Q. **1**

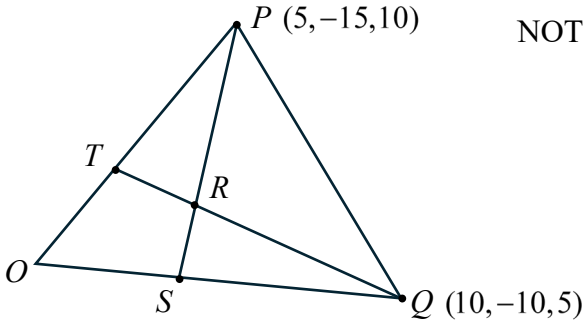
(ii) A sphere has centre $(1,1,4)$ and radius R units. **4**

Find the value of R for which the line l is a tangent to the sphere.

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) The points $O\ (0,0,0)$, $P\ (5,-15,10)$, and $Q\ (10,-10,5)$ are the vertices of a triangle OPQ as shown below. **4**



The point S lies on OQ such that $OS : SQ = 2:3$, and the point T lies on OP such that $OT : TP = 2:3$.

By using vector methods, find coordinates of R.

(b) Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$. **4**

(c) (i) Prove that $(1+i\tan\theta)^n + (1-i\tan\theta)^n = 2\sec^n(\theta)\cos(n\theta)$. **3**

(ii) Using part (i) express $\text{Re}\left(\left(1+i\tan\frac{\pi}{8}\right)^8\right)$, in the form $a\sqrt{2}+b$, where a and b are integers. (You must justify your answer.) **4**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

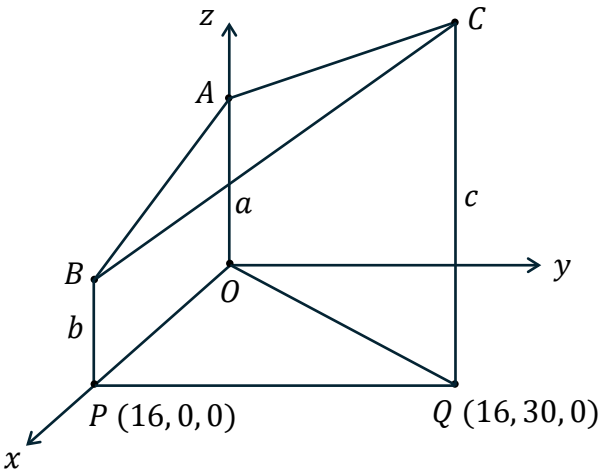
(a) Three spheres S_1, S_2, S_3 are placed on a horizontal surface such that they are touching each other.

Sphere S_1 has centre at A and touches the surface of $O(0, 0, 0)$.

Sphere S_2 has centre at B and touches the surface of $P(16, 0, 0)$.

Sphere S_3 has centre at C and touches the surface of $Q(16, 30, 0)$.

The diagram shows the centres of the spheres and their point of contact with the x-y plane.



By finding the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$, find the coordinates of the centres A, B and C of the three spheres. **4**

(b) Define $f_n(x) = x^{2n} + x^n + 1$, where $n \in \mathbb{Z}^+$, and $\omega = e^{\frac{i2\pi}{3}}$.

(i) Show that $f_1(x) = (x - \omega)(x - \omega^2)$. **2**

(ii) Calculate $f_{3k}(\omega)$. **1**

Question 16 continues on page 14

(c) (i) Using integration by parts, show that **2**

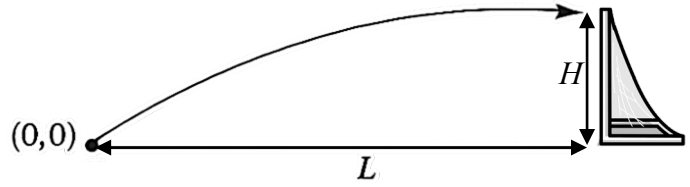
$$\int_x^{x+1} \sin\left(t^2\right) dt = \frac{\cos\left(x^2\right)}{2x} - \frac{\cos\left[\left(x+1\right)^2\right]}{2\left(x+1\right)} - \int_x^{x+1} \frac{\cos\left(t^2\right)}{2t^2} dt .$$

(ii) Hence, given $|a+b+c| \le |a|+|b|+|c|$ and $x > 0$, deduce that **2**

$$\left| \int_x^{x+1} \sin\left(t^2\right) dt \right| \le \frac{1}{x}.$$

(d) A soccer ball is kicked from ground level with initial velocity of u and an angle of projection of θ. The only forces acting on the ball are gravity and air resistance. The air resistance has magnitude mkv Newtons, where k is a constant. **4**

The ball is kicked straight so that it reaches the top corner of the goal, modelled as the point (L,H), Where L is the distance from the initial position of the ball to the goal and H is the height the ball reaches when it goes into the goal. It takes exactly T seconds for the ball to reach the goal.



If $(L,H) = (12,2)$, assuming acceleration due to gravity is equal to 10 ms^{-2}, find the angle of projection of the ball in terms of k if the ball reaches this point after one second.

End of paper

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Mathematics Extension 2

Solutions

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Section I

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Question 1.

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B. $3t\sin(t)$

C. $3\sin(3t)$

D. $3t\sin(3t)$

Question 2.

Given $z = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$, what is the value of z^8?

A. 1

B. -1

C. i

D. -i

$z^8 = (\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})^8$

$= \cos\pi - i\sin\pi$

$= -1$

Question 3.

What is the value of $\arg\bar{z}$ given the complex number $z = -1 - i\sqrt{3}$?

A. $-\frac{2\pi}{3}$

B. $-\frac{\pi}{3}$

C. $\frac{2\pi}{3}$

D. $\frac{\pi}{3}$

Question 4.

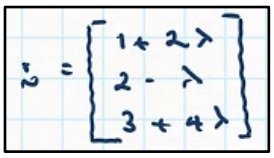
Which of the following points lies on the line $\underset{\sim}{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$?

A. $(5,0,11)$

B. $(3,1,5)$

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D. $(5,1,11)$



Question 5.

The velocity, v ms^{-1} , of a particle moving in a straight line is given by $v = 2\sin(3x)$, where x is the position of the particle from the origin.

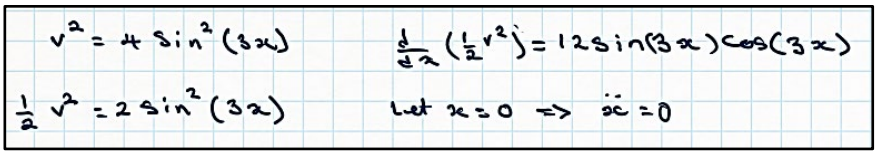
What is the acceleration of the particle when it is at the origin?

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D. 0 ms^{-2}



Question 6.

Consider the complex number z such that $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$.

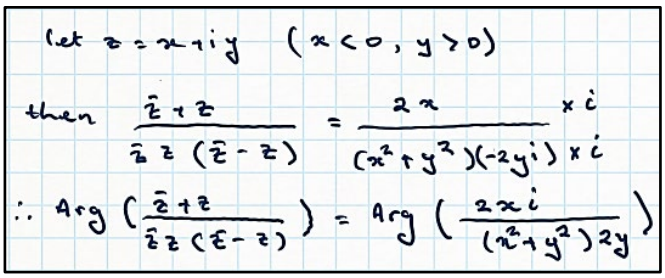
What is the value of $Arg\left(\dfrac{\overline{z}+z}{\overline{z}z(\overline{z}-z)}\right)$?

A. $\dfrac{\pi}{2}$

B. π

C. $-\dfrac{\pi}{2}$

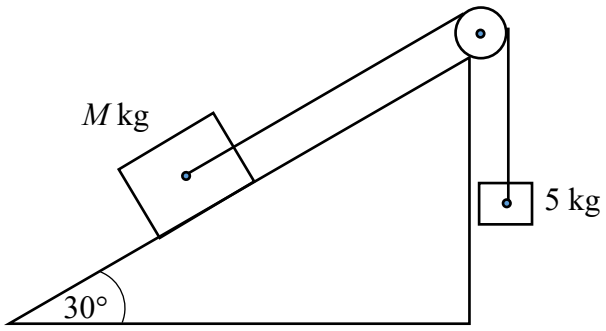
D. 0



Question 7.

A smooth plane is inclined at an angle of $30°$ to the horizontal. A mass of M kg, is attached to a smooth pully using a light inextensible string and moving down the ramp with an acceleration of 2 ms^{-2}.

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Assuming the acceleration due to gravity is 10 ms^{-2} .

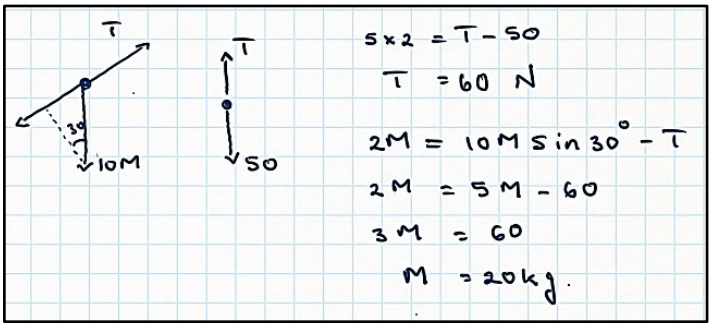
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Question 8.

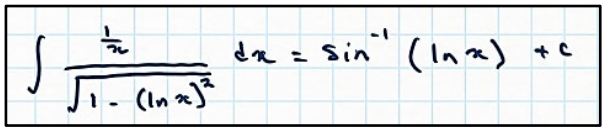
Which expression is equal to $\int \frac{1}{x\sqrt{1-\ln^2 x}}dx$?

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D. $\ln\left(1-\ln^2 x\right)+c$



Question 9.

Which of the following is an expression for $\int \cos^2 x \sin^7 x\, dx$?

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B. $\cos^3 x - 3\cos^5 x + 3\cos^7 x - \cos^9 x + c$

C. $-\dfrac{\cos^3 x}{3} + \dfrac{3\cos^5 x}{5} - \dfrac{3\cos^7 x}{7} + \dfrac{\cos^9 x}{9} + c$

D. $-\cos^3 x + 3\cos^5 x - 3\cos^7 x + \cos^9 x + c$

Question 10.

Let a, b, c, d be complex numbers such that $a \neq b$ and $c \neq d$.

Which of the following is a necessary and sufficient condition for the line through a and b to be perpendicular to the line through c and d?

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B. $\dfrac{a-b}{c-d} = -\dfrac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}$

C. $\dfrac{a-b}{c-d} = \dfrac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}i$

D. $\dfrac{a-b}{c-d} = \dfrac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}$

$a-b \perp c-d \Rightarrow a-b = ki(c-d) \quad (k \in \mathbb{R})$

$\Rightarrow \dfrac{a-b}{c-d} = ki$

since $\dfrac{a-b}{c-d}$ is purely imaginary then

$\dfrac{a-b}{c-d} + \overline{\left(\dfrac{a-b}{c-d}\right)} = 0 \Rightarrow \dfrac{a-b}{c-d} = -\dfrac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}$ (B)

Section II

90 marks

Attempt Questions 11–16

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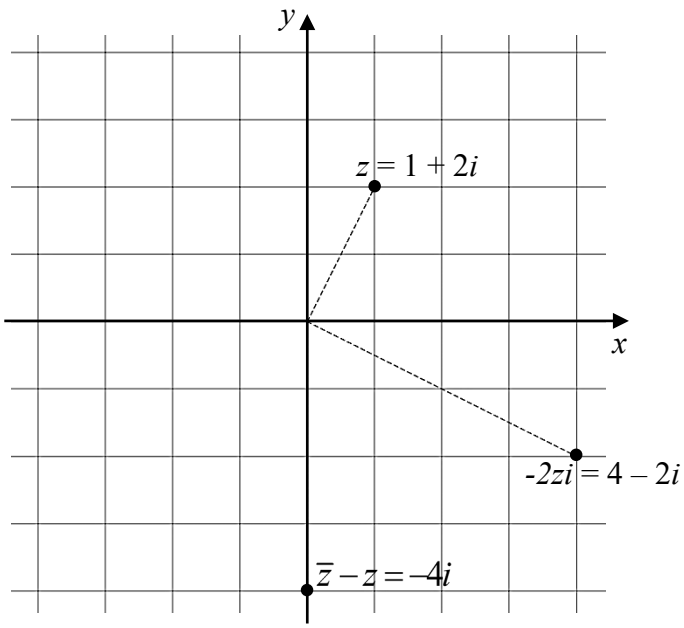
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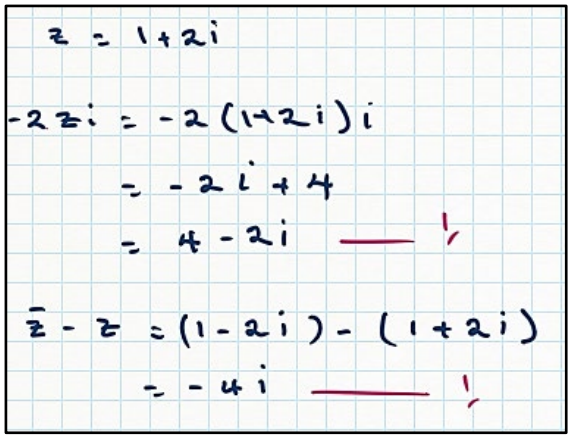
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On the diagram provided in the writing booklet, indicate the location of

(i) $-2zi$ **1**

(ii) $\overline{z} - z$. **1**





(b) The complex numbers z and w are given by $z = 2cis\frac{\pi}{6}$ and $w = 3cis\frac{2\pi}{3}$. **2**

Find the modulus and argument of $\frac{z}{w}$.

$\frac{z}{w} = \frac{2\,cis\,\pi/6}{3\,cis\,\frac{2\pi}{3}}$

$= \frac{2}{3}\,cis\left(\frac{\pi}{6} - \frac{2\pi}{3}\right)$

$= \frac{2}{3}\,cis\left(-\frac{\pi}{2}\right)$ ——— 1

$\therefore \left|\frac{z}{w}\right| = \frac{2}{3}$; $Arg\left(\frac{z}{w}\right) = -\frac{\pi}{2}$ — 1

Marking guidelines:

As on the solution.

(c) (i) Show that $\sqrt{3+4i} = \pm(2+i)$. **1**

Method 1

Let $(x+iy)^2 = 3+4i$

$x^2 - y^2 = 3$

$2xy = 4$

$y = \frac{2}{x}$

$x^2 - \frac{4}{x^2} = 3$

$x^4 - 3x^2 - 4 = 0$

$(x^2 - 4)(x+1) = 0$

$x = 2,\ y = 1$

$x = -2,\ y = -1$

$\therefore \sqrt{3+4i} = \pm(2+i)$

Method 2

$[\pm(2+i)]^2 = (2+i)^2$

$= 4 + 4i - 1$

$= 3 + 4i$

$\therefore \sqrt{3+4i} = \pm(2+i)$

(ii) By completing the square solve $z^2-(4-2i)z-8i=0$ where $z \in \mathbb{C}$. **2**

$z^2-(4-2i)z+(2-i)^2-8i=(2-i)^2$ —1

$(z-(2-i))^2=4-1-4i+8i$

$=3+4i$

$z=(2-i)\pm\sqrt{3+4i}$

$=(2-i)\pm(2+i)$

$=4,-2i$ —1

Marking guidelines:

As on the solution.

(d) Three forces are acting on an object in such a way that the object is stationary.

(i) Given the first force vector $\boldsymbol{F_1}=\underset{\sim}{i}+2\underset{\sim}{j}-2\underset{\sim}{k}$, **1**

show that the magnitude of vector F_1 is 3 Newtons.

$|F_1|=\sqrt{1^2+2^2+2^2}$

$=\sqrt{1+4+4}$

$=\sqrt{9}$

$=3$

(ii) The second force vector, $\boldsymbol{F_2}$, has double the magnitude of vector $\boldsymbol{F_1}$ and is **2**

parallel to $\underset{\sim}{r}=\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$. Show that vector $\boldsymbol{F_2}=2\sqrt{3}\left(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right)$.

$\hat{r}=\frac{\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}}{\sqrt{1+1+1}}$

$=\frac{1}{\sqrt{3}}(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k})$ —1

$\underset{\sim}{F_2}=6\hat{r}$

$=\frac{6}{\sqrt{3}}(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k})$

$=2\sqrt{3}(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k})$ —1

Marking guidelines:

As on the solution.

(iii) Hence find the third force vector $\boldsymbol{F_3}$ that is acting on the object. **2**

$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = 0$

$\underset{\sim}{F}_3 = -\underset{\sim}{F}_1 - \underset{\sim}{F}_2$ — 1

$= -(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}) - 2\sqrt{3}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

$= (-1-2\sqrt{3})\underset{\sim}{i} + (-2-2\sqrt{3})\underset{\sim}{j} + (2-2\sqrt{3})\underset{\sim}{k}$ — 1

Marking guidelines:

As on the solution.

(e) Evaluate $\int_0^1 \frac{2x^2 - x + 1}{x - 2} dx$. **3**

$\int_0^1 \frac{2x^2 - x + 1}{x-2} dx = \int_0^1 \left(2x + 3 + \frac{7}{x-2}\right) dx$ — 1

$= \left[x^2 + 3x + 7\ln|x-2|\right]_0^1$ — 1

$= (1 + 3 + 7\ln|-1|) - (0 + 0 + 7\ln|-2|)$

$= 4 - 0 - 7\ln 2$

$= 4 - 7\ln 2$ — 1

Marking guidelines:

As on the solution.

let $u = x - 2$ $\quad x = 0 \to u = -2$ $\quad x = 1 \to u = -1$

$x = u + 2$

$dx = du$

$I = \int_{-2}^{-1} \frac{2(u+2)^2 - (u+2) + 1}{u} du$

$= \int_{-2}^{-1} \frac{2u^2 + 8u + 8 - u - 2 + 1}{u} du$

$= \int_{-2}^{-1} \left(2u + 7 + \frac{7}{u}\right) du$

$= \left[u^2 + 7u + 7\ln|u|\right]_{-2}^{-1}$

$= (1 - 7 + 7\ln|-1|) - (4 - 14 + 7\ln|-2|)$

$= 4 - 7\ln 2$.

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sec x}{\cos x + \sin x}\,dx$. **3**

Method 1

$$\int_0^{\frac{\pi}{4}} \frac{\sec x}{\cos x + \sin x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1 + \tan x}\,dx \quad -1$$

$$= \left[\ln(1+\tan x)\right]_0^{\frac{\pi}{4}} \quad -1$$

$$= \ln\left(1+\tan\tfrac{\pi}{4}\right) - \ln(1+\tan 0)$$

$$= \ln 2 - \ln 1$$

$$= \ln 2 \quad -1$$

Method 2

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos x(\cos x + \sin x)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} + \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

$$= \left[-\ln\cos x + \ln(\cos x + \sin x)\right]_0^{\frac{\pi}{4}}$$

$$= \left(-\ln\frac{1}{\sqrt{2}} + \ln\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right) -$$

$$(-\ln(1) + \ln(1+0))$$

$$= \ln\left(\sqrt{2} \times \frac{2}{\sqrt{2}}\right)$$

$$= \ln 2$$

Marking guidelines:

As on the solution.

For method 3, only one mark was awarded for t substitution and expressing the integral in terms of t only and correct limit change.

12 (a) Method 3

let $t = \tan\frac{x}{2}$ $\quad x = \frac{\pi}{4} \to t = \tan\frac{\pi}{8}$, $\quad x = 0 \to t = 0$

$$dx = \frac{2\,dt}{1+t^2}$$

$$\int_0^{\frac{\pi}{4}} \frac{\sec x}{\cos x + \sin x}\,dx = \int_0^{\tan\frac{\pi}{8}} \frac{2(1+t^2)}{(1-t^2)(1+2t-t^2)}\,dt$$

$$= \int_0^{\tan\frac{\pi}{8}} \frac{1}{1-t} - \frac{1}{1+t} + \frac{1}{t-1+\sqrt{2}} + \frac{1}{t-1-\sqrt{2}}\,dt$$

$$= \int_0^{\tan\frac{\pi}{8}} \frac{2t}{1-t^2} + \frac{2t-2}{t^2-2t-1}\,dt$$

$$= \left[\ln\left|\frac{t^2-2t-1}{1-t^2}\right|\right]_0^{\tan\frac{\pi}{8}}$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

12 (a) Method 4

$$\int_0^{\frac{\pi}{4}} \frac{\sec x}{\cos x + \sin x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + \cos x \sin x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2}{1+\cos 2x + \sin 2x}\,dx$$

Let $t = \tan x$

$$= \int_0^1 \frac{2}{1 + \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \cdot \frac{dt}{1+t^2}$$

$$= \int_0^1 \frac{2}{1+t^2+1-t^2+2t}\,dt$$

$$= \int_0^1 \frac{2}{2+2t}\,dt$$

$$= \int_0^1 \frac{1}{1+t}\,dt$$

$$= \left[\ln|1+t|\right]_0^1$$

$$= \ln 2 - \ln 1$$

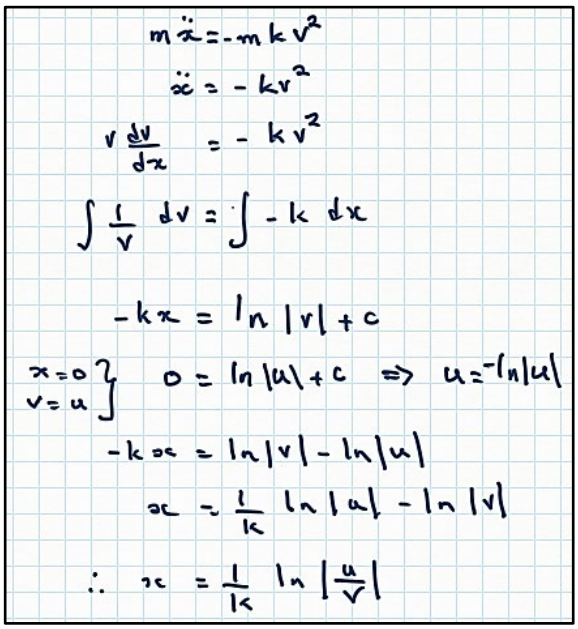
$$= \ln 2$$

(b) A particle is moving in a straight line, horizontally away from a fixed point O such that its displacement from the fixed point at time t seconds is x metres. The only force acting on the particle is a resistance force of magnitude mkv^2 newtons, where k is a constant.

Given that the particle starts its motion at O, with a positive initial velocity of u ms^{-1} ,

(i) Show that the displacement of the particle is given by **2**

$$x = \frac{1}{k}\ln\left(\frac{u}{v}\right).$$

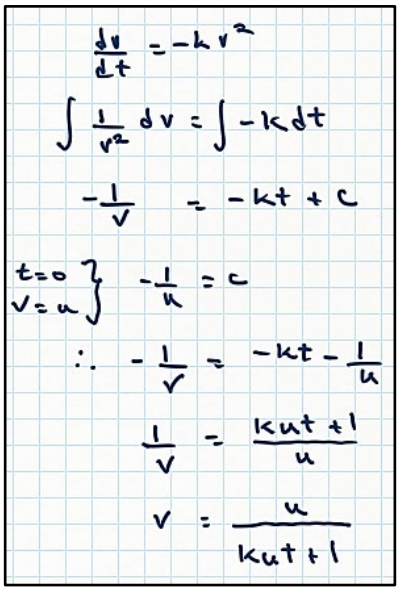


Marking guidelines:

1 – correctly setting up integration or integrate 1/v correctly

2 – complete solution, including finding the

(ii) Show that the velocity of the particle at any time is given by $v = \dfrac{u}{kut+1}$. **2**

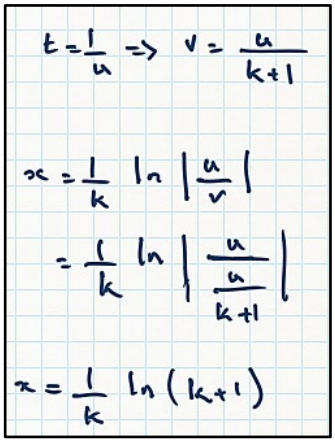


Marking guidelines:

1 – correctly setting up integration or integrate $1/v^2$ correctly.

2 – complete solution, including finding the

(iii) Hence find an expression for the displacement of the particle after $\frac{1}{u}$ seconds in terms of k. **2**

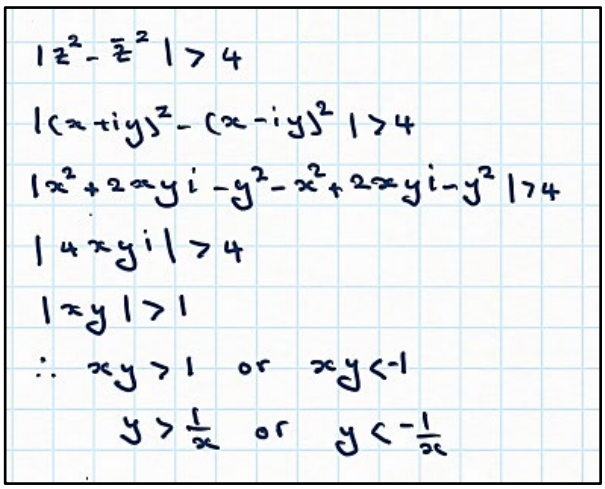


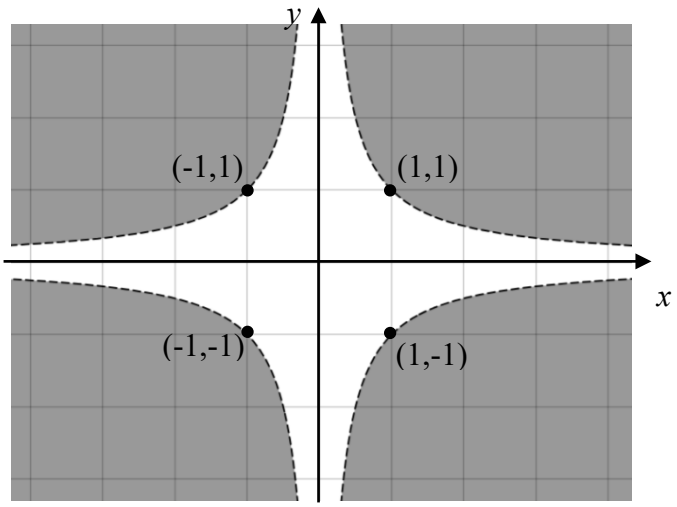
Marking guidelines:

1 – reasonable steps leading to answer or attempt to integrate u/(kut+1) in respect to t.

2 – complete solution

(c) Sketch the region of the complex plane defined by $\left|z^2-\bar{z}^2\right|>4$. **3**





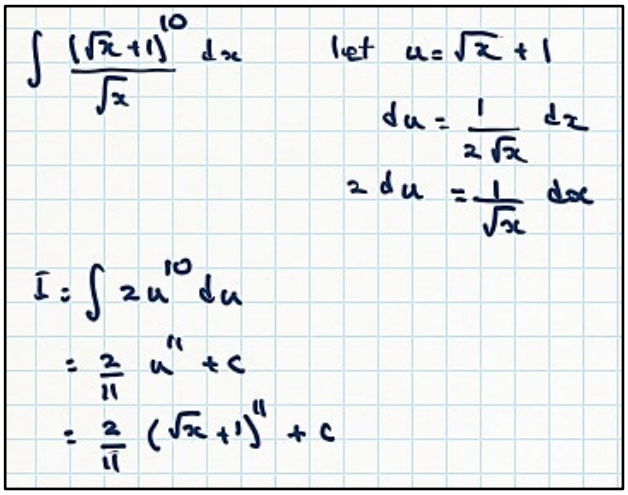
Marking guidelines:

1 – Deriving the equation $|xy|>1$ OR correctly shaded the region for $xy>1$.

2 – Deriving the equation $|xy|>1$ and correctly shaded the region for $xy>1$ OR correctly shaded the region for $|xy|>1$ using solid line.

3 – the equation as well as all regions with correct points on the hyperbola labelled.

(d) Find $\int \frac{\left(\sqrt{x}+1\right)^{10}}{\sqrt{x}}dx$. **3**



Marking guidelines:

1 – setting up u substitution equations

2 – above and rewriting the integral

3 – full solution

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Consider the complex numbers $z = \sqrt{2} + i\sqrt{6}$ and $w = \sqrt{3} - i$. **3**

Find the smallest positive integer value of n for which $\frac{z^n}{w^n}$ is purely real.

Method 1

$$\frac{z}{w} = \frac{\sqrt{2} + i\sqrt{6}}{\sqrt{3} - i} \times \frac{\sqrt{3} + i}{\sqrt{3} + i}$$

$$= \frac{\sqrt{6} + \sqrt{2}i + 3\sqrt{2}i - \sqrt{6}}{3 + 1}$$

$$= \frac{4\sqrt{2}}{4} i$$

$$= \sqrt{2}\, i$$

$$\left(\frac{z}{w}\right)^n = (\sqrt{2}i)^n$$

which is purely real when n is even, with smallest value $n = 2$.

Method 2

$|z| = 2\sqrt{2}$, $\text{Arg}(z) = \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$

$|w| = 2$, $\text{Arg}(w) = \tan^{-1}\frac{-1}{\sqrt{3}}$

$= \frac{-\pi}{6}$

$$\frac{z}{w} = \frac{2\sqrt{2}}{2} \frac{\text{cis}\frac{\pi}{3}}{\text{cis}\frac{-\pi}{6}}$$

$$= \sqrt{2}\ \text{cis}\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$$

$$= \sqrt{2}\ \text{cis}\left(\frac{\pi}{2}\right)$$

$$\left(\frac{z}{w}\right)^n = \sqrt{2}^n \left(\text{cis}\frac{\pi}{2}\right)^n$$

$$= \sqrt{2}^n \left(\text{cis}\frac{n\pi}{2}\right) \text{ (By DeMoivier)}$$

when $n = 2$ $\left(\frac{z}{w}\right)^2 = 2[\cos\pi + i\sin\pi]$

$= -2$

$\therefore n = 2$.

Marking guidelines:

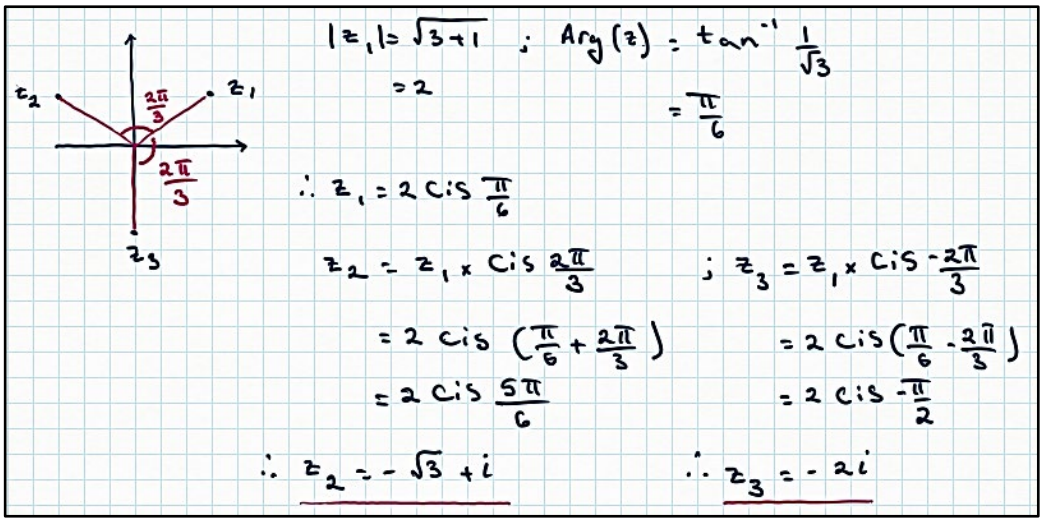
1 – finding $\frac{z}{w}$

2 – demonstrating the impact of index n.

3 – full solution

(b) The centre of an equilateral triangle is at the origin, and one vertex is at $z_1 = \sqrt{3} + i$. **2**

Using $\text{Arg}(z_1)$, find the complex numbers giving the other two vertices, giving your answer in $a + ib$ form.



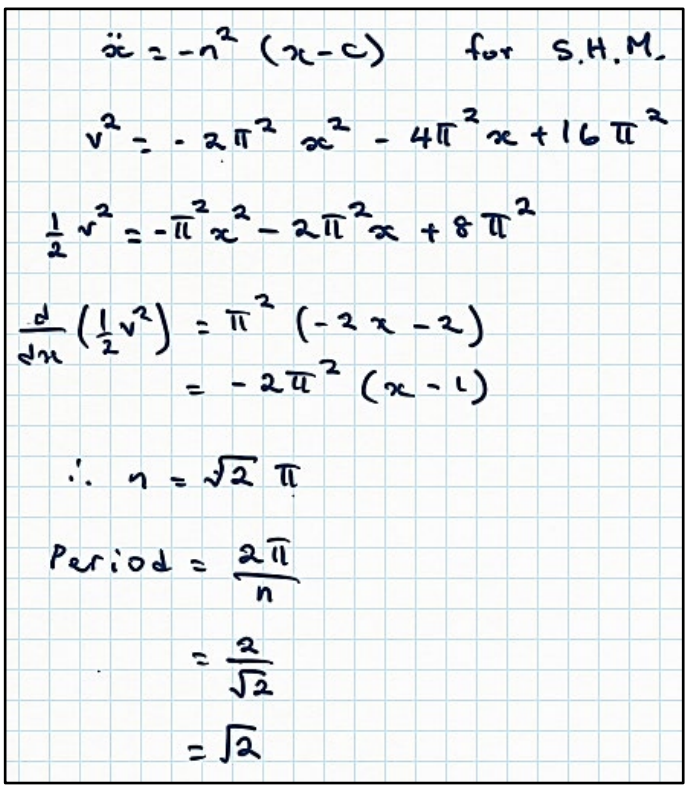
Marking guidelines:

1 – Rotating anti-clockwise correctly

2 – full solution

(c) A particle is moving in simple harmonic motion with speed v ms^{-1} given by $v^2 = -2\pi^2 x^2 - 4\pi^2 x + 16\pi^2$, where x is the displacement from the origin, in metres.

(i) Find the exact period of the particle's motion? **2**



Marking guidelines:

1 – Finding acceleration in terms of x

2 – Full solution

(ii) Find the maximum and minimum x-values for which the particle oscillates between. **2**

Let $v = 0$

$\Rightarrow -2\pi^2 x^2 - 4\pi^2 x + 16\pi^2 = 0$

$\therefore x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

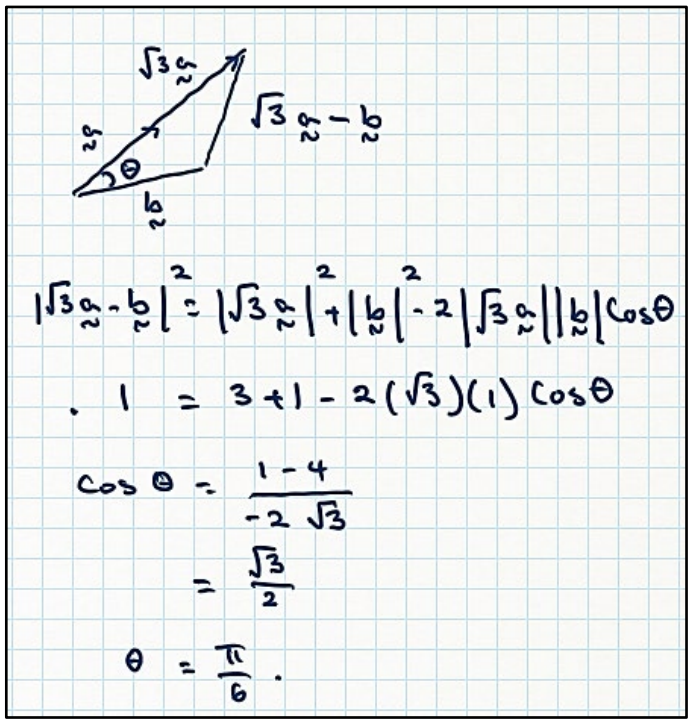
$\therefore x = -4\,m, \quad x = 2\,m$

Marking guidelines:

1 – reasonable correct steps that leads to the solution

2 – Full solution

(d) Find the acute angle between unit vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ such that $\left|\sqrt{3}\underset{\sim}{a} - \underset{\sim}{b}\right| = 1$. **3**



$\left|\sqrt{3}\underset{\sim}{a} - \underset{\sim}{b}\right|^2 = \left|\sqrt{3}\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 - 2\left|\sqrt{3}\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos\theta$

$1 = 3 + 1 - 2(\sqrt{3})(1)\cos\theta$

$\cos\theta = \dfrac{1-4}{-2\sqrt{3}}$

$= \dfrac{\sqrt{3}}{2}$

$\theta = \dfrac{\pi}{6}$.

Marking guidelines:

1 – Establish an appropriate equation

2 – Steps towards the solution

3 – Full solution

(e) Find $\int \frac{dx}{e^{3x}-1}$. **3**

$\int \frac{dx}{e^{3x}-1}$ let $u = e^{3x}$

$\frac{du}{dx} = 3e^{3x}$

$= \int \frac{1}{(u-1)3u}\, du$ $\frac{du}{dx} = 3u \Rightarrow dx = \frac{1}{3u}\, du$

$= \frac{1}{3}\int \left(\frac{A}{u-1} + \frac{B}{u}\right) du$

Now $Au + B(u-1) = 1$

Let $u = 0 \Rightarrow B = -1$

$u = 1 \Rightarrow A = 1$

$\Rightarrow I = \frac{1}{3}\int \left(\frac{1}{u-1} - \frac{1}{u}\right) du$

$= \frac{1}{3}\left[\ln|u-1| - \ln|u|\right] + C$

$= \frac{1}{3}\ln\left|\frac{u-1}{u}\right| + C$

$= \frac{1}{3}\ln\left|\frac{e^{3x}-1}{e^{3x}}\right| + C$

Marking guidelines:

1 – setting up u substitutions.

2 – Rewriting the integral in terms of u.

3 – Full solution

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Given that $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$,

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$. **2**

$$I_n = \int_0^{\pi/2} \sin^n x\,dx$$

$$= \int_0^{\pi/2} \sin x \sin^{n-1} x\,dx$$

let $u = \sin^{n-1} x$; $v' = \sin x$

$u' = (n-1)\cos x \sin^{n-2} x$; $v = -\cos x$

$$I_n = \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x\,dx$$

$$= (n-1)\int_0^{\pi/2} (1-\sin^2 x)(\sin^{n-2} x)\,dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x - (n-1)\int_0^{\pi/2} \sin^n x\,dx$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$n I_n = (n-1) I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2}.$$

Marking guidelines:

1 – setting up integration by parts.

2 – Full solution

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx$. **2**

$$I_7 = \frac{6}{7} I_5$$

$$= \frac{6}{7} \times \frac{4}{5} I_3$$

$$= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} I_1$$

$$= \frac{16}{35} \int_0^{\pi/2} \sin x\,dx$$

$$= \frac{16}{35} \left[-\cos x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{16}{35} (-0+1)$$

$$= \frac{16}{35}.$$

Marking guidelines:

1 – reasonable correct steps that leads to the solution

2 – Full solution

(b) A particle is projected vertically upwards in a resistant medium. The velocity of projection is u ms^{-1}, the mass of the particle is m kg and the resistance force is given by mkv^2 Newtons, where k is a constant. Given g is acceleration due to gravity,

(i) Show that the maximum height attained is given by $h = \frac{1}{2k}\ln\left(1+\frac{k}{g}u^2\right)$. **2**

$m\ddot{x} = -mg - mkv^2$

$\ddot{x} = -g - kv^2$

+ ↑ $x = h$, $v = 0$; $x = 0$, $v = u$, $t = 0$

$v\frac{dv}{dx} = -(g+kv^2)$

$\frac{1}{2k}\int \frac{2kv}{g+kv^2}\,dv = -\int dx$

$-x = \frac{1}{2k}\ln(g+kv^2) + C$

$x = 0$, $v = u$: $0 = \frac{1}{2k}\ln(g+ku^2) + C$

$\Rightarrow C = \frac{-1}{2k}\ln(g+ku^2)$

$\therefore x = \frac{1}{2k}\ln(g+ku^2) - \frac{1}{2k}\ln(g+kv^2)$

$= \frac{1}{2k}\ln\left(\frac{g+ku^2}{g+kv^2}\right)$

At max height $v = 0$

$\therefore h = \frac{1}{2k}\ln\left(\frac{g}{g}+\frac{k}{g}u^2\right)$

$= \frac{1}{2k}\ln\left(1+\frac{k}{g}u^2\right)$

Marking guidelines:

1 – setting up the integration.

2 – Full solution

(ii) If the speed of projection is half the terminal velocity, find the ratio of velocity on return to the projection point to the velocity of projection. **4**

Terminal velocity: $mg = mkv^2$

$v^2 = \frac{g}{k} \Rightarrow v = \sqrt{\frac{g}{k}}$

$\therefore u = \frac{1}{2}\sqrt{\frac{g}{k}} \Rightarrow u^2 = \frac{1}{4}\frac{g}{k}$



$\ddot{x} = g - kv^2$

$v\frac{dv}{dx} = g - kv^2$

$\int \frac{v}{g-kv^2}\,dv = \int dx$

$-\frac{1}{2k}\int \frac{-2kv}{g-kv^2}\,dv = \int dx$

$\Rightarrow \quad x = -\frac{1}{2k}\ln|g-kv^2| + c$

$\left.\begin{matrix} x=0 \\ v=0 \end{matrix}\right\} \Rightarrow c = \frac{1}{2k}\ln g$

$\therefore x = \frac{1}{2k}\ln\left|\frac{g}{g-kv^2}\right|$

Let $x = \frac{1}{2k}\ln\left(1+\frac{k}{g}u^2\right)$ and $u^2 = \frac{1}{4}\frac{g}{k}$:

$\Rightarrow \frac{1}{2k}\ln\left(1+\frac{k}{g}\times\frac{1}{4}\times\frac{g}{k}\right) = \frac{1}{2k}\ln\left|\frac{g}{g-kv^2}\right|$

$\ln\frac{5}{4} = \ln\left|\frac{g}{g-kv^2}\right|$

$5(g-kv^2) = 4g$

$5kv^2 = g$

$v^2 = \frac{g}{5k}$

Ratio: $\frac{v^2}{u^2} = \frac{g}{5k} \div \frac{g}{4k}$

$= \frac{4}{5} \Rightarrow \frac{v}{u} = \frac{2}{\sqrt{5}}$ (or $\frac{2\sqrt{5}}{5}$)

Marking guidelines:

1 – Terminal Velocity.

2 – Finding displacement in terms of v^2

3 – finding v^2

4 – complete solution

(c) Consider the points P and Q with position vectors $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$ and $\underset{\sim}{q} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, respectively.

(i) Find the vector equation of line l passing through P and Q. **1**

$l: \quad \underset{\sim}{r} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -2-4 \\ 3+2 \\ 1-2 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}$

(ii) A sphere has centre $(1,1,4)$ and radius R units. **4**

Find the value of R for which the line l is a tangent to the sphere.

Eqn of sphere: $(x-1)^2 + (y-1)^2 + (z-4)^2 = R^2$

To find pts of intersection, solve simultaneously:

$(4-6\lambda-1)^2 + (-2+5\lambda-1)^2 + (2-\lambda-4)^2 = R^2$

$(3-6\lambda)^2 + (-3+5\lambda)^2 + (-2-\lambda)^2 = R^2$

$(36+25+1)\lambda^2 + (-36-30+4)\lambda + 9 + 9 + 4 - R^2 = 0$

$62\lambda^2 - 62\lambda + 22 - R^2 = 0$

For one point of contact $\Delta = 0$

$b^2 - 4ac = 0$

$62^2 - 4(62)(22-R^2) = 0$

$62^2 - 4(62)(22) + 4(62)R^2 = 0$

$R^2 = \dfrac{4(62)(22) - 62^2}{(4)(62)}$

$= \dfrac{13}{2}$

$\therefore R = \dfrac{\sqrt{26}}{2}$ units $(R > 0)$.

Marking guidelines:

1 – equation of the sphere.

2 – solving simultaneously

3 – recognising $\Delta = 0$

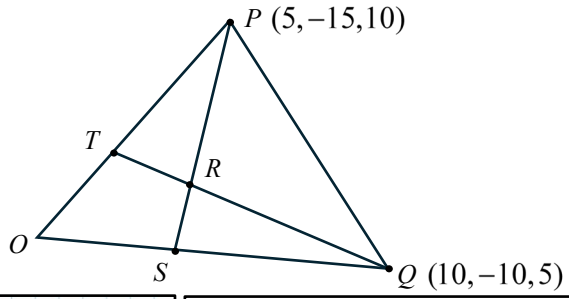
4 – complete solution

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) The points $O\ (0,0,0)$, $P\ (5,-15,10)$, and $Q\ (10,-10,5)$ are the vertices of a triangle OPQ as shown below. **4**

The point S lies on OQ such that $OS:SQ=2:3$, and the point T lies on OP such that $OT:TP=2:3$. By using vector methods, find coordinates of R.



Location of S: $\left(\frac{3\times0+2\times10}{3+2}, \frac{3\times0+2\times-10}{3+2}, \frac{3\times0+2\times5}{3+2}\right)$

$: (4,-4,2)$

Location of T: $\left(\frac{3\times0+2\times5}{3+2}, \frac{3\times0+2\times-15}{3+2}, \frac{3\times0+2\times10}{3+2}\right)$

$: (2,-6,4)$

$\overrightarrow{SP} = \begin{bmatrix}4\\-4\\2\end{bmatrix} + \lambda\begin{bmatrix}5-4\\-15+4\\10-2\end{bmatrix}$; $\overrightarrow{TQ} = \begin{bmatrix}2\\-6\\4\end{bmatrix} + \mu\begin{bmatrix}10-2\\-10+6\\5-4\end{bmatrix}$

$= \begin{bmatrix}4\\-4\\2\end{bmatrix} + \lambda\begin{bmatrix}1\\-11\\8\end{bmatrix}$ $= \begin{bmatrix}2\\-6\\4\end{bmatrix} + \mu\begin{bmatrix}8\\-4\\1\end{bmatrix}$

To find R, solve simultaneously:

$4+\lambda = 2+8\mu \Rightarrow \lambda = 8\mu-2$ — (*)

$-4-11\lambda = -6-4\mu$ — (1)

$2+8\lambda = 4+\mu$ — (2)

sub (*) into (1) $\Rightarrow -4-11(8\mu-2) = -6-4\mu$

$84\mu = 24$

$\mu = \frac{2}{7} \Rightarrow \lambda = \frac{2}{7}$

check with (3) $\Rightarrow$ LHS $= 2+8\times\frac{2}{7}$ RHS $= 4+\frac{2}{7}$

$= 4\frac{2}{7}$ $= 4\frac{2}{7}$

R: $\begin{bmatrix}4\\-4\\2\end{bmatrix} + \frac{2}{7}\begin{bmatrix}1\\-11\\8\end{bmatrix} \Rightarrow R: \left(4\frac{2}{7}, -7\frac{1}{7}, 4\frac{2}{7}\right)$

Marking guidelines:

1 – Find the location of S and T OR correct equation of one of the line or equivalent merit.

2 – write the vector equations for SP and TQ or equivalent merit.

3 – Steps to correctly find the scalar values

4 – Full solution

See next page for alternative solution

15(a) ~~15~~ Alternative solution 1.

$\overrightarrow{SR} = \lambda \overrightarrow{SP}$ $\quad \overrightarrow{TR} = \mu \overrightarrow{TQ}$

$\overrightarrow{OR} = \lambda(\overrightarrow{OP} - \frac{2}{5}\overrightarrow{OQ}) + \frac{2}{5}\overrightarrow{OQ} = \mu(\overrightarrow{OQ} - \frac{2}{5}\overrightarrow{OP}) + \frac{2}{5}\overrightarrow{OP}$

equate coefficient of $\overrightarrow{OP}$ & $\overrightarrow{OQ}$

$\lambda = \frac{2}{5}(1-\mu)$

$\mu = \frac{2}{5}(1-\lambda)$

$\lambda = \mu = \frac{2}{7}$

$\therefore \overrightarrow{OR} = \frac{2}{7}\left[\begin{pmatrix}5\\-15\\10\end{pmatrix} - \frac{2}{5}\begin{pmatrix}10\\-10\\5\end{pmatrix}\right] + \frac{2}{5}\begin{pmatrix}10\\-10\\5\end{pmatrix}$

$= \left(\frac{30}{7}, -\frac{50}{7}, \frac{30}{7}\right)$

(NB) No student expressed in mixed fraction.

other methods include writing.

$\overrightarrow{OR} = \underline{q} + \lambda(-\underline{q} + \frac{2}{5}\underline{p})$ where $\underline{q} = \overrightarrow{OQ}$, $\underline{p} = \overrightarrow{OP}$

(b) Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$. **4**

$$I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} dx \quad (1)$$

$$\sin(\pi - x) = \sin x$$

$$\cos^2(\pi - x) = (-\cos x)^2 = \cos x \quad (1)$$

$$\therefore I = \int_0^{\pi} (\pi - x) \frac{\sin x}{1+\cos^2 x} dx$$

$$2I = \int_0^{\pi} \pi \frac{\sin x}{1+\cos^2 x} dx$$

$$= -\pi \left[\tan^{-1}(\cos x)\right]_0^{\pi} \quad (1)$$

$$= -\pi \left(\tan^{-1}(-1) - \tan^{-1}(1)\right)$$

$$= -\pi \left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$2I = \frac{\pi^2}{2}.$$

$$\therefore I = \frac{\pi^2}{4} \quad (1)$$

Marking guidelines:

1 – Rearrange the integral.

2 – Rewrite the integral

3 – Perform the integration

4 – Full solution

See next page for alternative solutions

15. Method 2

(b)

$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$ $u' = \frac{\sin x}{1+\cos^2 x}$ $v' = 1$ / 1 mark

$u = -\tan^{-1}(\cos x)$ $v = x$

$= \left[-x\tan^{-1}(\cos x)\right]_0^{\pi} - \int_0^{\pi} -\tan^{-1}(\cos x)\,dx$

$= \left[-\pi\tan^{-1}(\cos\pi) + 0\tan^{-1}(\cos 0)\right] + \int_0^{\pi} \tan^{-1}(\cos x)\,dx$

$= \frac{\pi^2}{4} + \int_0^{\pi} \tan^{-1}(\cos x)\,dx$ [2 marks]

$= \frac{\pi^2}{4} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^{-1}(\cos(u+\frac{\pi}{2}))\,du$ $u = x - \frac{\pi}{2}$

~~$= \frac{\pi^2}{4} + 0$ as $\tan^{-1}(\cos x)$~~

~~$\frac{\pi^2}{4}$~~

$= \frac{\pi^2}{4} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^{-1}(\cos(u-\frac{\pi}{2}))\,du$

$= \frac{\pi^2}{4} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^{-1}(\cos(\frac{\pi}{2}-u))\,du$ $\cos\theta$ is even

$= \frac{\pi^2}{4} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^{-1}(\sin u)\,du$ $\tan^{-1}(\sin u)$ is an odd function

$= \frac{\pi^2}{4} + 0$

$= \frac{\pi^2}{4}$

15(b) Method 3

$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$ Let $u = \cos x$ $\begin{cases} \pi \to -1 \\ 0 \to 1 \end{cases}$ $du = -\sin x\,dx$

$= \int_1^{-1} \frac{\cos^{-1}(u)}{1+u^2}\,du$ (1 marks]

$= \int_{-1}^{1} \cos^{-1}u \left(\frac{1}{1+u^2}\right)du$ $[\,]' = \frac{1}{1+u^2}$ $v' = -\frac{1}{\sqrt{1-u^2}}$

$[\,] = \tan^{-1}(u)$ $v = \cos^{-1}u$

$= \left[\tan^{-1}(u)\cos^{-1}u\right]_{-1}^{1} - \int_{-1}^{1} -\frac{1}{\sqrt{1-u^2}}\tan^{-1}u\,du$ [2 marks]

$= \frac{\pi}{4}\times 0 - \left(-\frac{\pi}{4}\times\pi\right) + \int_{-1}^{1} \frac{\tan^{-1}u}{\sqrt{1-u^2}}\,du$

$= \frac{\pi^2}{4} + \int_{-1}^{1} \frac{\tan^{-1}u}{\sqrt{1-u^2}}\,du$ [3 marks]

$= \frac{\pi^2}{4} + 0$ $\frac{\tan^{-1}u}{\sqrt{1-u^2}}$ is an odd function.

$= \frac{\pi^2}{4}$

(c) (i) Prove that $(1+i\tan\theta)^n + (1-i\tan\theta)^n = 2\sec^n(\theta)\cos(n\theta)$. **3**

LHS $= (1+i\tan\theta)^n + (1-i\tan\theta)^n$

$= \left(1+i\frac{\sin\theta}{\cos\theta}\right)^n + \left(1-i\frac{\sin\theta}{\cos\theta}\right)^n$

$= \frac{(\cos\theta+i\sin\theta)^n}{\cos^n\theta} + \frac{(\cos\theta-i\sin\theta)^n}{\cos^n\theta}$

$= \frac{1}{\cos^n\theta}\left(\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\right)$

$= \sec^n\theta\,(2\cos n\theta)$

$= 2\sec^n\theta\cos(n\theta)$

$=$ RHS

Marking guidelines:

1 – correct attempt that will lead to proof.

2 – Significant steps that lead to the proof

3 – Complete proof.

(ii) Using part (i) express $\text{Re}\left(\left(1+i\tan\frac{\pi}{8}\right)^8\right)$, in the form $a\sqrt{2}+b$, where a and b **4**

are integers. (You must justify your answer.)

$z + \bar{z} = 2\,\text{Re}(z)$

$\therefore \text{Re}(z) = \frac{1}{2}(z+\bar{z})$

$\Rightarrow \text{Re}\left(\left(1+i\tan\frac{\pi}{8}\right)^8\right) = \frac{1}{2}\left[\left(1+i\tan\frac{\pi}{8}\right)^8 + \left(1-i\tan\frac{\pi}{8}\right)^8\right]$

$= \frac{1}{2} \times 2 \times \sec^8\frac{\pi}{8} \times \cos\frac{8\pi}{8}$

$= -\sec^8\frac{\pi}{8}$

Now: $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\Rightarrow \cos^2\frac{\pi}{8} = \frac{1}{2}\left(\cos\frac{\pi}{4}+1\right)$

$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}+1\right)$

$= \frac{1+\sqrt{2}}{2\sqrt{2}}$

$\therefore \sec^2\frac{\pi}{8} = \frac{2\sqrt{2}}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}}$

$= \frac{2\sqrt{2}-4}{1-2}$

$\therefore -\sec^8\left(\frac{\pi}{8}\right) = (-4+2\sqrt{2})^4$

$= 256 - 512\sqrt{2} + 768 - 256\sqrt{2} + 64$

$= 1088 - 768\sqrt{2}$

Marking guidelines:

1 – use part (i) and substituted θ = π/8.

2 – found the exact value in terms of sec π/8 only.

3 – Find a surd expression for $\cos^2\frac{\pi}{8}$

4 – Complete solution

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

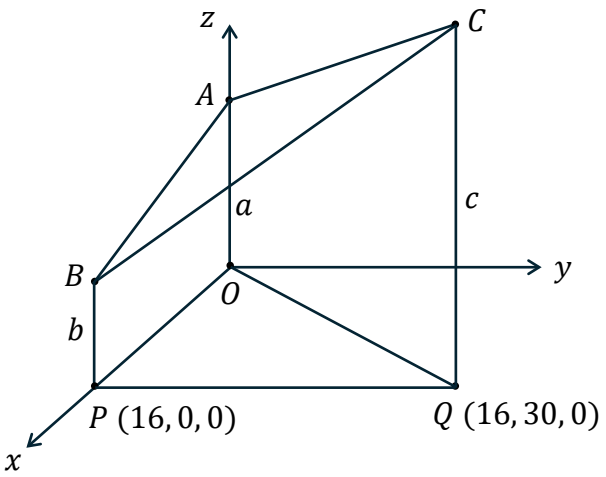
(a) Three spheres S_1, S_2, S_3 are placed on a horizontal surface such that they are touching each other.

Sphere S_1 has centre at A and touches the surface of $O(0, 0, 0)$.

Sphere S_2 has centre at B and touches the surface of $P(16, 0, 0)$.

Sphere S_3 has centre at C and touches the surface of $Q(16, 30, 0)$.

The diagram shows the centres of the spheres and their point of contact with the *x*-*y* plane.



By finding the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$, **4**

find the coordinates of the centres A, B and C of the three spheres.

Solution on next page

a, b, and c are the radii of the three spheres.

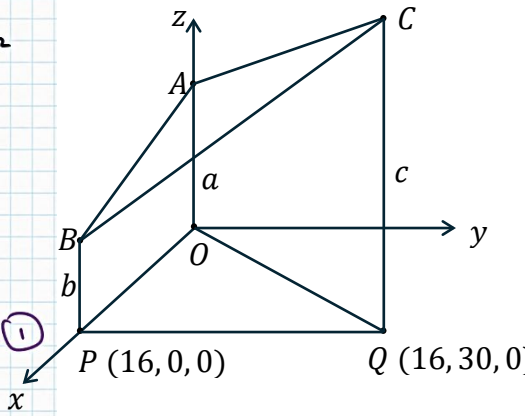
$\overrightarrow{AB} = \begin{bmatrix} 16-0 \\ 0-0 \\ b-a \end{bmatrix} \Rightarrow |\overrightarrow{AB}|^2 = (16-0)^2 + (0-0)^2 + (b-a)^2$

But $|\overrightarrow{AB}| = a+b$ (sum of the radii)

$\therefore 16^2 + (b-a)^2 = (a+b)^2 \Rightarrow 16^2 - 2ab = 2ab$

$ab = \frac{16^2}{4}$

$\therefore ab = 64$ — (1)



Similarly

$\overrightarrow{AC} = \begin{bmatrix} 16-0 \\ 30-0 \\ c-a \end{bmatrix} \Rightarrow |\overrightarrow{AC}|^2 = 16^2 + 30^2 + (c-a)^2$

But $|\overrightarrow{AC}| = a+c$ (sum of the radii)

$\therefore 16^2 + 30^2 + (c-a)^2 = (a+c)^2$

$16^2 + 30^2 - 2ac = 2ac$

$ac = \frac{16^2 + 30^2}{4}$

$ac = 289$ — (2)

similarly

$\overrightarrow{BC} = \begin{bmatrix} 16-16 \\ 30-0 \\ c-b \end{bmatrix} \Rightarrow |\overrightarrow{BC}|^2 = 0^2 + 30^2 + (c-b)^2$

But $|\overrightarrow{BC}| = c+b$ (sum of the radii)

$\therefore 30^2 + (c-b)^2 = (c+b)^2$

$bc = 225$ — (3)

Marking guidelines:

1 – Finding Vectors AB, AC and BC

2 – Equating the length of the vectors with the distance between the centres

3 – Reasonable steps in solving simultaneously

4 – Full solution

Rearranging (1) $\Rightarrow a = \frac{64}{b}$

Rearranging (3) $\Rightarrow c = \frac{225}{b}$

Sub into (2): $\frac{64 \times 225}{b^2} = 289$

$b = \frac{120}{17}$ $(b>0)$

$b = 7\frac{1}{17}$

Sub $b = 7\frac{1}{17}$ into (1) $\Rightarrow a = 9\frac{1}{15}$

into (3) $\Rightarrow c = 31\frac{7}{8}$

$\therefore$ centre of S_1: $(0, 0, 9\frac{1}{15})$

centre of S_2: $(16, 0, 7\frac{1}{17})$

centre of S_3: $(16, 30, 31\frac{7}{8})$

(b) Define $f_n(x) = x^{2n} + x^n + 1$, where $n \in \mathbb{Z}^+$, and $\omega = e^{\frac{i2\pi}{3}}$.

(i) Show that $f_1(x) = (x-\omega)(x-\omega^2)$. **2**

$$(x-\omega)(x-\omega^2) = x^2 - (\omega+\omega^2)x + \omega^3$$
$$= x^2 - \left(e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}}\right)x + \omega^{i2\pi}$$
$$= x^2 - \left(e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}}\right)x + 1$$
$$= x^2 - 2\,\mathrm{Re}\left(e^{i\frac{2\pi}{3}}\right)x + 1$$
$$= x^2 - 2\left(-\tfrac{1}{2}\right)x + 1$$
$$= x^2 + x + 1$$
$$= f_1(x)$$

Marking guidelines:

1 – sufficient steps leading to the correct answer

2 – Complete solution

(ii) Calculate $f_{3k}(\omega)$. **1**

$$f_{3k}(x) = x^6 + x^3 + 1$$
$$f_{3k}(\omega) = e^{i4\pi} + e^{i2\pi} + 1$$
$$= 1 + 1 + 1$$
$$= 3.$$

(c) (i) Using integration by parts, show that **2**

$$\int_x^{x+1} \sin\left(t^2\right) dt = \frac{\cos\left(x^2\right)}{2x} - \frac{\cos\left[(x+1)^2\right]}{2(x+1)} - \int_x^{x+1} \frac{\cos\left(t^2\right)}{2t^2}\,dt.$$

Let $I = \int_x^{x+1} \sin(t^2)\,dt$

let $u = \frac{1}{2t}$ $\quad v' = 2t\sin(t^2)$

$u' = -\frac{1}{2t^2}$ $\quad v = -\cos(t^2)$

$\therefore I = \left[-\frac{\cos t^2}{2t}\right]_x^{x+1} - \int_x^{x+1} \frac{\cos t^2}{2t^2}\,dt$

$$= -\frac{\cos(x+1)^2}{2(x+1)} + \frac{\cos x^2}{2x} - \int_x^{x+1} \frac{\cos t^2}{2t^2}\,dt$$

Marking guidelines:

1 – sufficient steps leading to the correct answer

2 – Complete solution

(ii) Hence, given $|a+b+c| \le |a|+|b|+|c|$ and $x > 0$, deduce that $\left| \int_x^{x+1} \sin\left(t^2\right) dt \right| \le \frac{1}{x}$. **2**

$$\left| \int_x^{x+1} \sin(t^2)\,dt \right| = \left| \frac{\cos x^2}{2x} + \frac{\cos(x+1)^2}{2(x+1)} - \int_x^{x+1} \frac{\cos t^2}{2t^2}\,dt \right| \quad \text{(from (i))}$$

$$\le \left| \frac{\cos x^2}{2x} \right| + \left| \frac{\cos(x+1)^2}{2(x+1)} \right| + \left| -\int_x^{x+1} \frac{\cos t^2}{2t^2}\,dt \right|$$

(since $\cos\theta \le 1$)

$$\le \frac{1}{2x} + \frac{1}{2(x+1)} + \int_x^{x+1} \frac{1}{2t^2}\,dt$$

$$\le \frac{1}{2x} + \frac{1}{2(x+1)} + \left[\frac{-1}{2t} \right]_x^{x+1}$$

$$\le \frac{1}{2x} + \frac{1}{2(x+1)} - \frac{1}{2(x+1)} + \frac{1}{2x}$$

$$\le \frac{1}{x}$$

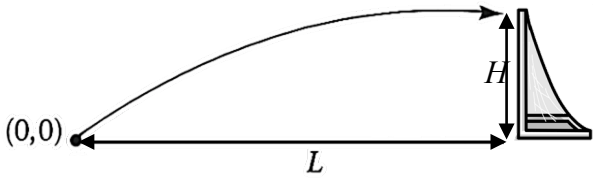
Marking guidelines:

1 – sufficient steps leading to the correct answer

2 – Complete solution

(d) A soccer ball is kicked from ground level with initial velocity of u and an angle of projection of θ. The only forces acting on the ball are gravity and air resistance. The air resistance has magnitude mkv Newtons, where k is a constant. The ball is kicked straight so that it reaches the top corner of the goal, modelled as the point (L,H), Where L is the distance from the initial position of the ball to the goal and H is the height the ball reaches when it goes into the goal. It takes exactly T seconds for the ball to reach the goal. **4**

If $(L,H)=(12,2)$, assuming acceleration due to gravity is equal to 10 ms^{-2}, find the angle of projection of the ball in terms of k if the ball reaches this point after one second.



Horizontal Motion

$\ddot{x} = -k\dot{x}$

$\frac{d\dot{x}}{dt} = -k\dot{x}$

$\int \frac{1}{\dot{x}}\, d\dot{x} = -k\int dt$

$\ln \dot{x} = -kt + c$

$\dot{x} = Ae^{-kt} \quad (A = e^c)$

$t=0,\ \dot{x} = u\cos\theta \Rightarrow A = u\cos\theta$

$\therefore \dot{x} = u\cos\theta\, e^{-kt}$

$x = -\frac{u}{k}\cos\theta\, e^{-kt} + c$

$t=0,\ x=0 \Rightarrow c = \frac{u}{k}\cos\theta$

$\therefore x = \frac{u}{k}\cos\theta\left(1 - e^{-kt}\right)$

Vertical Motion

$\ddot{y} = -(10 + k\dot{y})$

$\frac{d\dot{y}}{dt} = -(10 + k\dot{y})$

$\int \frac{1}{10+k\dot{y}}\, d\dot{y} = -\int dt$

$\frac{1}{k}\ln(10 + k\dot{y}) = -t + c$

$10 + k\dot{y} = Ae^{-kt} \quad (A = e^c)$

$\dot{y} = \frac{1}{k}\left(Ae^{-kt} - 10\right)$

$t=0,\ \dot{y} = u\sin\theta \Rightarrow A = uk\sin\theta + 10$

$\Rightarrow \dot{y} = \frac{1}{k}\left((uk\sin\theta + 10)e^{-kt} - 10\right)$

$= u\sin\theta\, e^{-kt} + \frac{10}{k}e^{-kt} - \frac{10}{k}$

$y = -\frac{u\sin\theta}{k}e^{-kt} - \frac{10}{k^2}e^{-kt} - \frac{10t}{k} + c$

$t=0,\ y=0 \Rightarrow c = \frac{uk\sin\theta + 10}{k^2}$

$y = \frac{uk\sin\theta + 10}{k^2}\left(1 - e^{-kt}\right) - \frac{10}{k}t$

Let $x = 12$, $y = 2$, $t = 1$.

Solution continues on the next page

$$12 = \frac{u}{k}\cos\theta\left(1-e^{-k}\right) \quad \text{——} \quad (1)$$

$$2 = \frac{uk\sin\theta + 10}{k^2}\left(1-e^{-k}\right) - \frac{10}{k} \quad \text{—} \quad (2)$$

From (1) $u\cos\theta = \dfrac{12k}{1-e^{-k}}$

From (2) $\dfrac{uk\sin\theta + 10}{k^2} = \dfrac{2k+10}{k(1-e^{-k})}$

$$uk\sin\theta + 10 = \frac{k(2k+10)}{(1-e^{-k})}$$

$$u\sin\theta = \frac{2k+10}{(1-e^{-k})} - \frac{10}{k}$$

$$\Rightarrow \frac{u\sin\theta}{u\cos\theta} = \frac{k(2k+10)-10(1-e^{-k})}{k(1-e^{-k})} \div \frac{12k}{1-e^{-k}}$$

$$\tan\theta = \frac{k(2k+10)-10(1-e^{-k})}{12k^2}$$

$$\therefore \theta = \tan^{-1}\left[\frac{k^2+5k-5(1-e^{-k})}{6k^2}\right]$$

Marking guidelines:

1 – establish the horizontal equations of motion

2 – above and establish the vertical equations of motion

3 – rearrange the equations for displacement

4 – full solution

End of paper