Girraween High School

2026

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time: 10 minutes
- Working time: 3 Hours
- Write using a black pen
- NESA approved calculators may be used
- Laminated reference sheets are provided
- Answer multiple-choice questions by completely colouring in the appropriate circle on your multiple-choice answer sheet.
- Answer questions in the writing booklet provided and show all relevant mathematical reasoning and/or calculations.

Total Marks: 100

Section 1 (Pages 1 – 6) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

Section 2 (Pages 7-15) **90 marks**

- Attempt Q11 - Q16
- Allow about 2 hours and 45 minutes for this section

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Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1 – 10

Question 1

The angle between the vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$ is

A) 15^o

B) 50^o

C) 75^o

D) 105^o

Question 2

The straight lines

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ -8 \\ 2 \end{pmatrix}$ are

A) Parallel

B) Perpendicular

C) Neither parallel or perpendicular

D) The same line

Examination continues next page…

Question 3

Consider the statement: $\exists\ x \in Z$ such that $\frac{1}{x} > x$.

The negation of this statement is

A) $\forall\, x \in Z, \frac{1}{x} < x$

B) $\exists\ x \in Z$ such that $\frac{1}{x} < x$

C) $\forall\, x \in Z, \frac{1}{x} > x$

D) $\exists\ x \in Z$ such that $\frac{1}{x} = x$

Question 4

The contrapositive of the statement

"If a number is irrational, its square root will be irrational" is

A) "If a number is irrational, its square will be irrational"

B) "If a number is rational its square root will be rational"

C) "If a number is irrational its square root will be rational"

D) "If a number is rational its square will be rational"

Examination continues next page…

Question 5

For two complex numbers z_1 and z_2, $|z_1| = |z_2| = 1$ and $|z_1 + z_2| = \frac{6}{5}$.

$|z_1 - z_2| =$

A) $\frac{3}{5}$

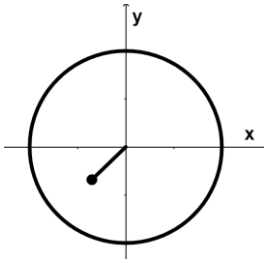
B) $\frac{4}{5}$

C) 1

D) $\frac{8}{5}$

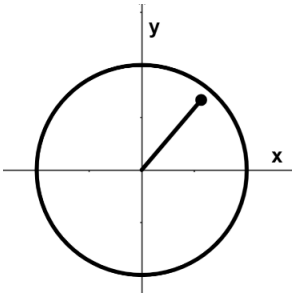
Question 6

Below is a diagram of the complex number z, where $|z| < 1$ as shown.

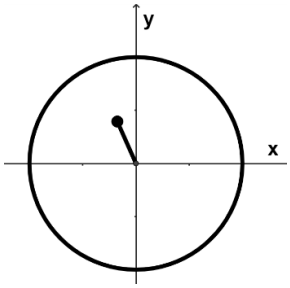


One of the square roots of z could be

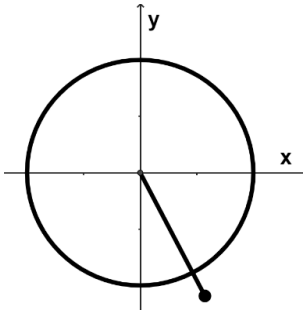
A)



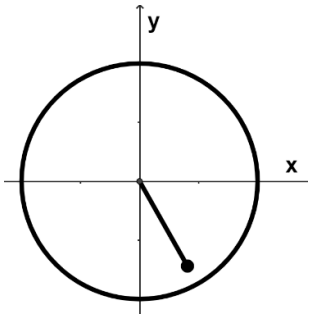
B)



C)



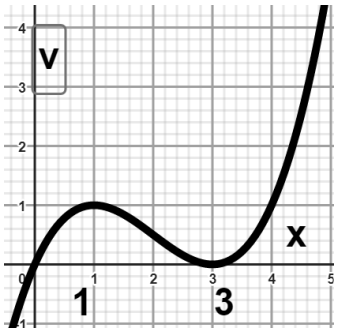
D)



Examination continues next page…

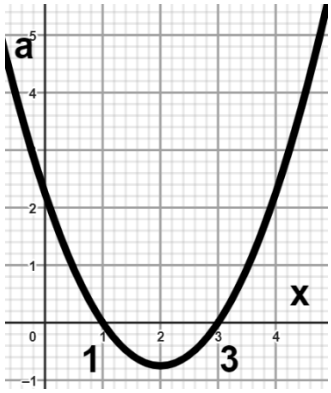
Question 7

The following diagram is a graph of the velocity (v) of a particle with respect to its position (x).

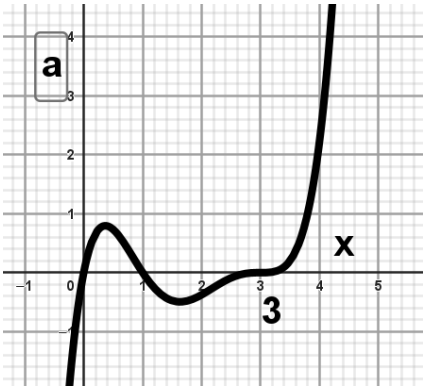


The graph of the acceleration (a) in terms of x would look like

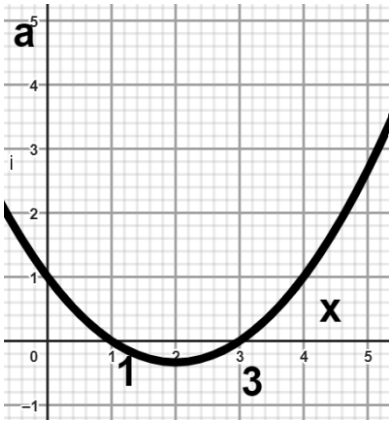
A)



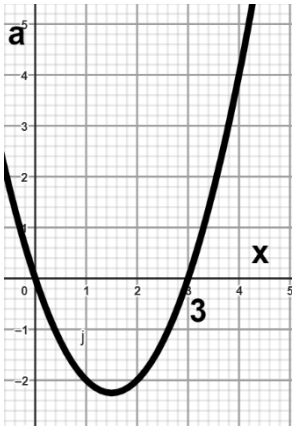
B)



C)



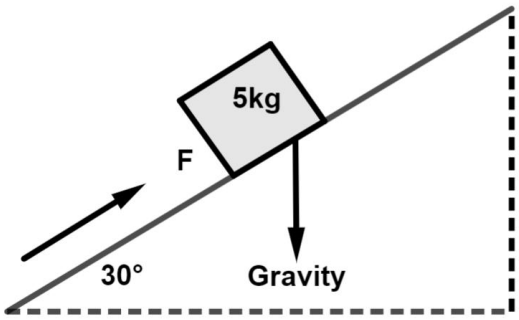
D)



Examination continues next page…

Question 8

A $5kg$ weight is slipping down a plane inclined at an angle of 30^o to the horizontal with an acceleration of $2\ m/s^2$. It is slipping due to gravity, which has an acceleration of $10m/s^2$. The slipping is being resisted by friction *(see diagram).*



The force due to friction is

A) 3 Newtons

B) 10 Newtons

C) 15 Newtons

D) 40 Newtons

Question 9

In the complex plane, $\overrightarrow{OA} = a, \overrightarrow{OB} = b$ and $\overrightarrow{OC} = a + b$, where $\frac{a+b}{a-b}$ is entirely imaginary and $|a + b| \neq |a - b|$.

$OABC$ is a

A) rhombus

B) rectangle

C) square

D) trapezium

Examination continues next page…

Question 10

One of these curves is exactly the same as the spiral $\begin{pmatrix}\cos t\\ \sin t\\ t\end{pmatrix}$.

The equivalent curve is

A) $\begin{pmatrix}\cos\left(\frac{\pi}{2}-2t\right)\\ \sin\left(\frac{\pi}{2}-2t\right)\\ 2t\end{pmatrix}$.

B) $\begin{pmatrix}\cos(\pi-2t)\\ \sin(\pi-2t)\\ 2t\end{pmatrix}$.

C) $\begin{pmatrix}-\sin\left(\frac{3\pi}{2}-2t\right)\\ -\cos\left(\frac{3\pi}{2}-2t\right)\\ 2t\end{pmatrix}$.

D) $\begin{pmatrix}\cos(2\pi-2t)\\ \sin(2\pi-2t)\\ 2t\end{pmatrix}$.

Examination continues next page…

Section II

90 marks
Attempt Questions 11 - 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Start a new writing page**

a) i) Find $\frac{-\sqrt{3}+i}{-1-i}$ in Cartesian form. 2

ii) Express $-\sqrt{3}+i$ in exponential form. 2

iii) $-1-i = \sqrt{2}\, e^{-\frac{3\pi i}{4}}$. Use this information and your answers to (i) and (ii) to find the exact value of $cos\, \frac{19\pi}{12}$ 2

b) i) Find the two square roots of $55+48i$. 2

ii) Hence solve the equation $z^2-(2+i)z-(13+11i)=0$ 2

c) Sketch on an Argand diagram the region where $|z+2i| \leq 2$ and $-\frac{\pi}{2} \leq Arg\, z < -\frac{\pi}{4}$ 3

d) Find the seven seventh roots of $-128i$. Leave your answer in modulus argument form. 2

Examination continues next page…

Question 12 (15 marks) **Start a new writing page**

a) Find $\int \frac{1}{\sin x+\cos x+1}\ dx$ 2

b) (i) Find A, B and C so that $\frac{-7x-43}{(x+4)^2(x-1)} = \frac{A}{(x+4)^2} + \frac{B}{x+4} + \frac{C}{x-1}$ 3

(ii) Hence find $\int \frac{-7x-43}{(x+4)^2(x-1)}\ dx$ 2

c) Find $\int cosec^3 x\ dx$ 3

d) Find $\int \frac{x^2}{\sqrt{1-x^2}}\ dx$ 2

e) If $I_N = \int_{\frac{\pi}{4}} tan^N x\ dx$

i) Show that $I_N = \frac{1}{N-1} - I_{N-2}$ 2

ii) Find I_4 1

Examination continues next page…

Question 13 (15 marks) **Start a new writing page**

a) A particle moves with simple harmonic motion about $x = 0$. It starts at $x = 0$ and takes 2 seconds to reach its first stopping point at $x = 3$.

(i) Write a formula for the particle's motion in terms of time (t). 3

(ii) If the particle weighs $4kg$, find the force which produces the maximum acceleration of the particle. 2

b) A particle is moving with simple harmonic motion about $x = 1$. Its speed when at $x = 1$ is $1m/s$ and its speed at $x = 2$ is $\frac{\sqrt{3}}{2}\, m/s$. Find the period and amplitude of the particle's motion. 3
(You may use $v^2 = n^2(a^2 - (x - C)^2)$)

c) Prove by contradiction that $\log_5 11$ is irrational. 2

d) Prove by contradiction that $\sqrt{5} + \sqrt{6} \geq \sqrt{21}$ 2

e) (i) If a, b, c are real numbers, prove $a^2 + b^2 + c^2 \geq ab + ac + bc$ 1
(You may use $a^2 + b^2 \geq 2ab$)

(ii) If a, b, c are the lengths of the sides of a triangle, prove that $c^2 \leq 2a^2 + 2b^2$ 1

(iii) Hence or otherwise prove $3a^2 + 3b^2 \geq ab + ac + bc$ if a, b, c are the sides of a triangle. 1

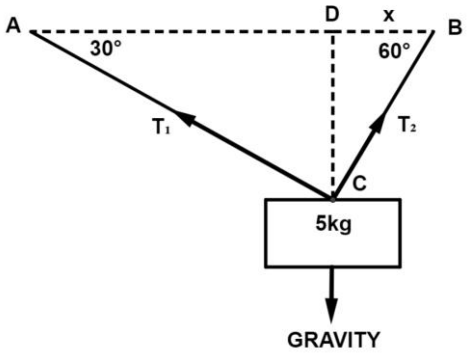
Examination continues next page…

Question 14 (15 marks) **Start a new writing page**

a) (i) Solve $4k^2 - 2k - 1 > 0$ where $k > 0$. 1

(ii) Prove by induction that $5^N > 6N^2$ for all integers $N \geq 2$. *(You may use your answer to Part (i))* 3

b) A $5kg$ weight is suspended from two taut non elastic ropes at angles of $30^o$and $60^o$to the horizontal respectively. The acceleration due to gravity is $10m/s^2$. *(see diagram).*

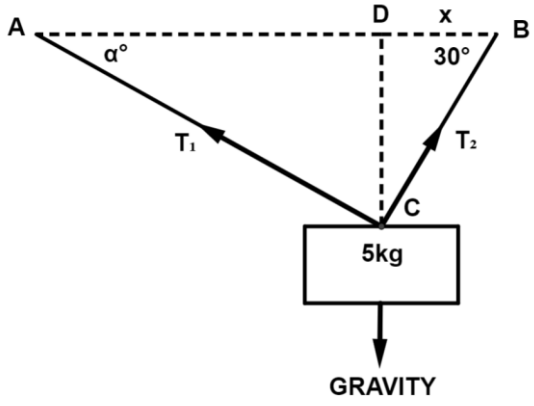


(i) If the weight is stationary and the only forces acting on it are gravity and the tensions in the two ropes, find the tensions in the ropes. 2

(ii) If the distance $DB = x$ metres, show that the distance $AD = 3x$ metres 1

(iii) The weight is now winched up so that $\angle DBC = 30^o$. *(see diagram below).*

Find the new angle α $(\angle DAC)$. 2



Question 14 continues on the following page

Question 14 (continued)

c) A bullet with a mass of 20 grams is fired horizontally at $400m/s$. It encounters air resistance of $0.0048v$ Newtons in a horizontal direction. Ignoring gravity:

(i) Show that $x = \frac{10\,000-25v}{6}$ and show that the bullet has travelled 1500 metres when its velocity has dropped to $40m/s$. 3

(ii) Show that the time of flight in seconds is given by $t = -\frac{25}{6}\ ln\left(\frac{v}{400}\right)$ and find the time taken for the bullet's velocity to drop to $40m/s$. 3

Examination continues next page…

Question 15 (15 marks) **Start a new writing page**

a) (i) State the centre and radius of the sphere 1

$(x-2)^2+(y+3)^2+(z-1)^2=169$

(ii) Show that the distance d of any point on the line $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\-1\\29\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\-2\end{pmatrix}$ 1

from the centre of the sphere is given by the expression

$d^2=29\lambda^2-116\lambda+792$

(iii) Hence or otherwise find the minimum distance from the line 2

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\-1\\29\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\-2\end{pmatrix}$ to the centre of the sphere.

(iv) Show that the line $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-3\\1\end{pmatrix}+\mu\begin{pmatrix}0\\3\\-1\end{pmatrix}$ is skew to the line 3

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\-1\\29\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\-2\end{pmatrix}$

(v) Find the equation of the line that is perpendicular to both 3

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\-1\\29\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\-2\end{pmatrix}$ and $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-3\\1\end{pmatrix}+\mu\begin{pmatrix}0\\3\\-1\end{pmatrix}$ that passes through

the centre of the sphere.

Question 15 continues next page…

Question 15 (continued)

b) A projectile with mass m is fired vertically upwards with initial velocity Um/s.
It experiences a force due to gravity of mg Newtons and air resistance of mkv^2 Newtons.

(i) Taking up as positive, state the acceleration in terms of the velocity and show that the vertical displacement x is given by $x = \frac{1}{2k} \ln\left(\frac{g+kU^2}{g+kv^2}\right)$ 3

(ii) Find the greatest height the projectile reaches. Express your answer in terms of g, k and U. 2

Examination continues next page…

Question 16 (10 marks) **Start a new writing page**

a) (i) Prove for any complex number $z, z\bar{z} = |z|^2$ 1

(ii) The roots of the equation $9z^4 - 18z^3 + 26z^2 - 18z + 9 = 0$ 3
are $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ where $|\alpha| = |\beta| = 1$.
Show that the quadratic equation with roots $\alpha + \bar{\alpha}$ and $\beta + \bar{\beta}$ is
$9z^2 - 18z + 8 = 0$.

(iii) By solving this equation, find $\alpha + \bar{\alpha}$ and $\beta + \bar{\beta}$. 1

(iv) Show that if $|\alpha| = 1$ and $\arg \alpha = \theta, \alpha + \bar{\alpha} = 2\cos\theta$ 1

(v) Hence or otherwise solve $9z^4 - 18z^3 + 26z^2 - 18z + 9 = 0$ 2

Question 16 continues next page

Question 16 (continued)

b) (i) Prove that for a complex number z, if $|z| = 1$, and $Arg\ z = \theta, 0 < \theta < \pi$ 2

then $Arg(z-1) = \frac{\theta+\pi}{2}$ and $Arg\ (z+1) = \frac{\theta}{2}$.

(ii) State the roots of $z^n - 1 = 0$. 1

In the diagram below, $\overrightarrow{OA} = 1, \overrightarrow{AB} = w$, $BC = w^2$, $CD = w^3$ etc where w is the root of $z^n - 1 = 0$ with the smallest positive argument.

$CD = w^3$

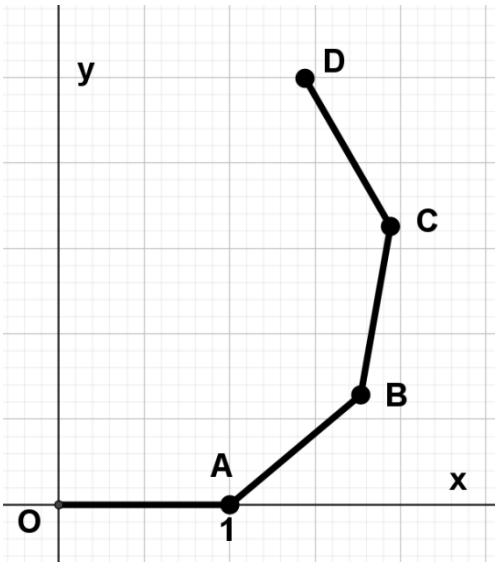


DIAGRAM NOT TO SCALE

(iii) State $Arg\ (1+w)$ and show that $|1+w| = \frac{\sin\left(\frac{2\pi}{n}\right)}{sin\left(\frac{\pi}{n}\right)}$ 2

(iv) Show that $Arg\ \frac{\left(1+w+w^2+w^3+\cdots+w^k\right)}{\left(1+w+w^2+w^3+\cdots+w^{k-1}\right)} = \frac{\pi}{n}$ and 2

$|(1+w+w^2+w^3+\cdots+w^k)| = \frac{\sin\left(\frac{(k+1)\pi}{n}\right)}{sin\left(\frac{\pi}{n}\right)}$

End of Examination

2026 Year 12 Trial

Mathematics Extension 2

Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

1.	A	○	B	○	C	●	D	○
2.	A	○	B	○	C	○	D	●
3.	A	●	B	○	C	○	D	○
4.	A	○	B	○	C	○	D	●
5.	A	○	B	○	C	○	D	●
6.	A	○	B	○	C	○	D	●
7.	A	○	B	●	C	○	D	○
8.	A	○	B	○	C	●	D	○
9.	A	●	B	○	C	○	D	○
10.	A	○	B	○	C	●	D	○

Mathematics Assessment Solutions

Girraween High School

☐ Year 11	☐ Extension 1	Year 2026
☑ Year 12	☑ Extension 2	Task Number 4 TRIAL

Suggested solutions	Marker's Feedback

(1) $\cos\theta = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$= \dfrac{\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\4\\6\end{pmatrix}}{\sqrt{6}\times\sqrt{61}}$

$= \dfrac{5}{\sqrt{366}}$

$\theta = 75^\circ$ (C)

(2) $\begin{pmatrix}-6\\8\\-2\end{pmatrix} = -2\begin{pmatrix}3\\4\\-1\end{pmatrix}$ so they are going in the same direction. (D)

Also $\begin{pmatrix}1\\7\\0\end{pmatrix} - 2\begin{pmatrix}3\\4\\-1\end{pmatrix} = \begin{pmatrix}-5\\-1\\2\end{pmatrix}$ so they are THE SAME LINE.

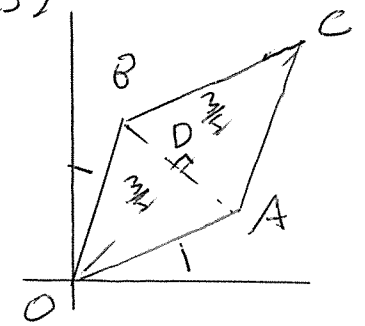
(3) Original statement:
There exists an integer x such that $\frac{1}{x} > x$

Negation:
For all integers x, $\frac{1}{x} < x$ (A)

$\forall\ x \in z,\ \frac{1}{x} < x$

(4) Contrapositive: (D)
The square of a rational number is rational.

(5)



$\vec{OA} = z_1$, $\vec{OB} = z_2$

$\vec{OC} = z_1 + z_2$.

OACB is a rhombus.
Diagonals $\perp$ at O
& bisect each other.

$OD = \frac{1}{2}OC = \frac{1}{2}\times\frac{6}{5} = \frac{3}{5}$.

$\therefore OA = \frac{4}{5}$ [Pythagoras].

$\therefore BA = |z_1 - z_2| = \frac{8}{5}$ (D)

Suggested solutions	Marker's Feedback

Q.(6) $-\pi < \text{Arg } z < -\frac{\pi}{2}$

& $0 < |z| < 1$.

$\therefore -\frac{\pi}{2} < \text{Arg}(\sqrt{z}) < 0$ [or $\frac{\pi}{2} < \text{Arg}(\sqrt{z}) < \pi$]

& $|\sqrt{z}| > |z|$.

A → wrong quadrant.
B → modulus too small.
C → modulus > 1.

(D)

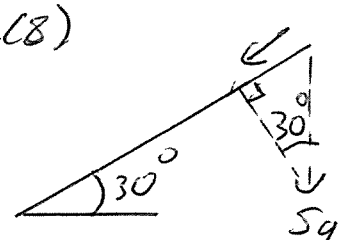
Q.(7) $a = v \cdot \frac{dv}{dx}$

$a = 0$ where $v = 0$ (x intercepts) or

$\frac{dv}{dx} = 0$ (turning points).

$a = 0$ where $x = 0, 1$ & 3. (B) only.

Q.(8)



Gravity DOWN plane
$= 50 \sin 30^\circ$
$= 25$ N.

$Sg = 50$N

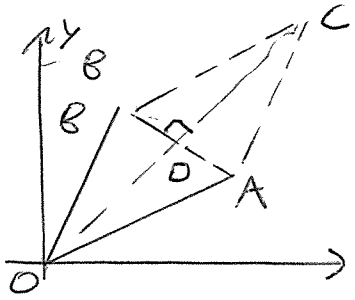
Gravity ~~Acc~~ FORCE of acceleration down plane
$= 2 \times 5 = 10$N.

$25 - F = 10$

$F = 15$ Newtons.

(C)

(9)



OABC is a parallelogram.

D is midpoint OC & AB.

As $\frac{a+b}{a-b}$ is imaginary $\angle ODA = 90^\circ$

→ Diagonals ⊥ but not equal.

OACB is a RHOMBUS

(A)

Suggested solutions	Marker's Feedback

Q.(10) A not good. $\cos\left(\frac{\pi}{2}-2t\right)=\sin 2t$ ×

B $\cos(\pi-2t)=-\cos 2t$ ×

C $-\sin\left(\frac{3\pi}{2}-2t\right)$

$=-(-\cos 2t)=\cos 2t.$

$-\cos\left(\frac{3\pi}{2}-2t\right)=\sin 2t.$

$\therefore C=\begin{pmatrix}\cos 2t\\ \sin 2t\\ 2t\end{pmatrix}$ & if $2t=u$,

$C=\begin{pmatrix}\cos u\\ \sin u\\ u\end{pmatrix}$ (C)

Note: D $\cos(2\pi-2t)=\cos 2t$

But $\sin(2\pi-2t)=-\sin 2t$ ×

Suggested solutions	Marker's Feedback

Q.(11)(a) (i) $\frac{-\sqrt{3}+i}{-1-i} \times \frac{(-1+i)}{(-1+i)}$

$= \frac{(\sqrt{3}-1)}{2} - i\frac{(\sqrt{3}+1)}{2}$.

(ii) $-\sqrt{3}+i = 2e^{\frac{5\pi}{6}i}$ $\quad$ $\text{fm } |z| = \sqrt{(\sqrt{3})^2+1^2}$

$= 2$.

$\text{Arg } z = \frac{5\pi}{6}$.

(iii) $\frac{-\sqrt{3}+i}{-1-i}$

$= \frac{2e^{\frac{5\pi}{6}i}}{\sqrt{2}e^{-\frac{3\pi}{4}i}}$

$= \sqrt{2}e^{\left[\left(\frac{5\pi}{6}i - \left(-\frac{3\pi}{4}i\right)\right)\right]}$

$= \sqrt{2}e^{\frac{19\pi i}{12}}$

$= \sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi i}{12}\right)$.

Equating real parts of (i) & (iii)

$\sqrt{2}\cos\frac{19\pi}{12} = \frac{\sqrt{3}-1}{2}$

$\therefore \cos\frac{19\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Suggested solutions	Marker's Feedback

Q.(11)(b)(i) Let square roots of $55+48i$
$= x+iy$, $x, y \in \mathbb{R}$.

$\therefore (x+iy)^2 = 55+48i$.

$(x^2-y^2)+2ixy = 55+48i$.

Equating reals, $x^2-y^2 = 55$ (1)

Equating imaginaries, $2xy = 48$

$y = \frac{24}{x}$ (2)

Either

Sub. (2) in (1):

$x^2 - \left(\frac{24}{x}\right)^2 = 55$

$x^4 - 55x^2 - 576 = 0$

$(x^2+9)(x^2-64) = 0$

As x is real, $x = \pm 8$.

$y = \frac{24}{x} = \pm 3$.

$\sqrt{55+48i} = \pm(8+3i)$.

OR

$|x+iy|^2 = x^2+y^2 = |55+48i| = \sqrt{5329} = 73$

$\therefore x^2+y^2 = 73$ (3)

$x^2-y^2 = 55$ (1) +

$2x^2 = 128$

$x = \pm 8$.

Noting $55+48i$ is in Q_1, one root of $55+48i$ must be in $Q_1 \Rightarrow y$ has the same sign as x. From (3)−(1) $2y^2 = 18$, $y = \pm 3$.

$\therefore \sqrt{55+48i} = \pm(8+3i)$.

Suggested solutions	Marker's Feedback

Q.(11)(b)(ii) $z^2-(2+i)z-(13+11i)=0$

$\Delta = b^2-4ac$

$=(2+i)^2-4\times1\times-(13+11i)$

$= 55+48i.$

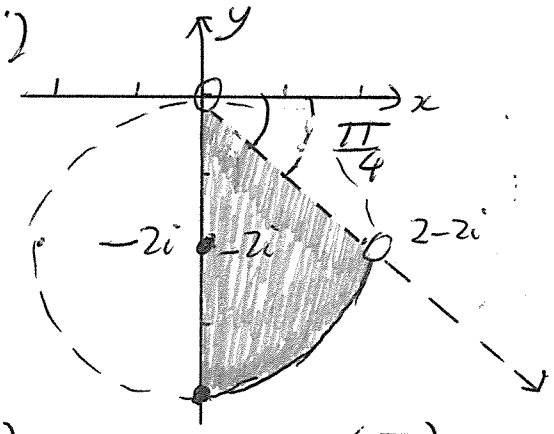
$z = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$= \dfrac{(2+i)\pm\sqrt{55+48i}}{2\times1}$

$= \dfrac{(2+i)+(8+3i)}{2}$ or $\dfrac{(2+i)-(8+3i)}{2}$

$z = 5+2i$ or $z=-3-i.$

(c)



(d) $-128i = 128\text{cis}\left(\frac{-\pi}{2}\right)$

$\sqrt[7]{-128i} = 2\text{cis}\left[\frac{-\pi}{14}+\frac{2k\pi}{7}, k=0,1,2,3,4,5,6\right].$

$= 2\text{cis}\left(\frac{-\pi}{14}\right), 2\text{cis}\frac{3\pi}{14}, 2\text{cis}\frac{\pi}{2}, 2\text{cis}\frac{11\pi}{14},$

$2\text{cis}\frac{15\pi}{14}, 2\text{cis}\frac{19\pi}{14}, 2\text{cis}\frac{23\pi}{14}.$

Suggested solutions	Marker's Feedback

Q.(12)(a) $\int \frac{1}{\sin x + \cos x + 1}\,dx$ $\left| \begin{array}{l} t = \tan\left(\frac{x}{2}\right) \\ \frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) \\ \quad = \frac{1+t^2}{2} \\ dx = \frac{dx}{dt}\,dt \\ \quad = \frac{2}{1+t^2}\,dt \end{array} \right.$

$= \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1} \cdot \frac{2}{1+t^2}\,dt$

$= \int \frac{1}{t+1}\,dt$

$= \ln[t+1] + C$

$= \ln\left[1 + \tan\left(\frac{x}{2}\right)\right] + C$

(b)(i) $\frac{A}{(x+4)^2} + \frac{B}{x+4} + \frac{C}{x-1} = \frac{-7x-43}{(x+4)^2(x-1)}$

$\times (x+4)^2(x-1)$

$A(x-1) + B(x+4)(x-1) + C(x+4)^2 = -7x - 43 \quad (1)$

Sub. $x=1$ in (1): $C(1+4)^2 = -7\times 1 - 43$

$C = -2.$

Sub. $x=-4$ in (1): $A(-4-1) = -7\times -4 - 43$

$-5A = -15 \Rightarrow A = 3.$

Sub. $x=0, A=3, C=-2$ in (1):

$3\times -1 + B\times 4\times -1 - 2\times 4^2 = -43 \Rightarrow B = 2.$

Could also expand & equate co-efficients of x^2, x, constant.

(1) expanded is

$(B+C)x^2 + (A+3B+8C)x + (16C - 4B - A) = 0x^2 - 7x - 43.$

Equations are $B+C=0$, $A+3B+8C=-7$, $16C-4B-A=-43$.

(ii) Hence $\int \frac{-7x-43}{(x+4)^2(x-1)}\,dx$

$= \int \frac{3}{(x+4)^2} + \frac{2}{x+4} - \frac{2}{x-1}\,dx$

$= \frac{-3}{(x+4)} + 2\ln(x+4) - 2\ln(x-1) + C.$

Suggested solutions	Marker's Feedback

Q.(12)(c) $\int \operatorname{cosec}^3 x \, dx$ $\quad u = \operatorname{cosec} x \,/\, v = -\cot x$

$u' = -\operatorname{cosec} x \cot x \,/\, v' = \operatorname{cosec}^2 x$

By $\int uv' \, dx = uv - \int vu' \, dx$

$$\int \operatorname{cosec}^3 x \, dx = -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x \, dx$$

$$\int \operatorname{cosec}^3 x \, dx = -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$\int \operatorname{cosec}^3 x \, dx = -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x \, dx + \int \operatorname{cosec} x \, dx$$

$$2\int \operatorname{cosec}^3 x \, dx = -\operatorname{cosec} x \cot x + \int \operatorname{cosec} x \, dx$$

$$= -\operatorname{cosec} x \cot x - \int \frac{-\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{\operatorname{cosec} x + \cot x} \, dx$$

$$= -\operatorname{cosec} x \cot x - \int \frac{-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x} \, dx$$

$$\int \operatorname{cosec}^3 x \, dx = -\frac{1}{2} \operatorname{cosec} x \cot x - \frac{1}{2} \ln \left| \operatorname{cosec} x + \cot x \right| + C \quad \left[\text{by } \int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)|\right]$$

Notes: (1) $\frac{d}{dx}(\operatorname{cosec} x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{-\cos x}{\sin^2 x}$

$$= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cot x.$$

(2) As a result,

$$\frac{d}{dx}(\operatorname{cosec} x + \cot x) = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$$

Suggested solutions	Marker's Feedback

Q.(12)(d) $\int \frac{x^2}{\sqrt{1-x^2}} \cdot dx$

Let $x = \sin\theta$. $dx = \cos\theta \cdot d\theta$

$= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \cdot d\theta$

$= \int \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta \cdot d\theta$

$= \int \sin^2\theta \cdot d\theta$

$= \frac{1}{2}\int 1 - \cos 2\theta \cdot d\theta$

$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$

$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C$

$\left(\frac{1}{2} \times 2\sin\theta\cos\theta\right)$.

$= \frac{1}{2}\sin^{-1}(x) - \frac{1}{2} x\sqrt{1-x^2} + C.$

Suggested solutions	Marker's Feedback

Q.12 (b) (i) $I_n = \int_0^{\frac{\pi}{4}} \tan^N x \, dx$

$I_n = \int_0^{\frac{\pi}{4}} \tan^{N-2} x \cdot \tan^2 x \, dx$

$I_n = \int_0^{\frac{\pi}{4}} \tan^{N-2} x \cdot (\sec^2 x - 1) \, dx$

$I_n = \int_0^{\frac{\pi}{4}} \tan^{N-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{N-2} x \, dx$

$I_n = \left[\frac{1}{N-1} \tan^{N-1} x\right]_0^{\frac{\pi}{4}} - I_{N-2}$

$I_n = \frac{1}{N-1} - I_{N-2}$

(ii) $I_0 = \int_0^{\frac{\pi}{4}} 1 \, dx = [x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$.

$I_2 = \frac{1}{1-1} - I_0$

$= 1 - \frac{\pi}{4}$.

$I_4 = \frac{1}{4-1} - I_2$

$= \frac{1}{3} - \left(1 - \frac{\pi}{4}\right)$

$= \frac{\pi}{4} - 1 + \frac{1}{3}$.

$I_4 = \frac{\pi}{4} - \frac{2}{3}$.

Suggested solutions	Marker's Feedback

Q.(13)(a) Starts at centre of motion: sin.

(i) Amplitude = 3.

Period = $2 \times 4 = 8$ seconds.

$\frac{2\pi}{n} = 8 \Rightarrow n = \frac{\pi}{4}$.

$x = 3\sin\frac{\pi}{4}t$.

(ii) $\ddot{x} = -n^2 x$ [centre $x = 0$].

$= -\frac{\pi^2}{16} x$.

Max. acceleration is at $x = -3$ (LH end of motion).

$= -\frac{\pi^2}{16} \times -3$

$= \frac{3\pi}{16}^2 \text{ m/s}^2$.

Force: $F = ma = 4 \times \frac{3\pi}{16}^2 = \frac{3\pi}{4}^2$ Newtons.

Note: Can also do max $\ddot{x}$ is when $x = -3$

$3\sin\frac{\pi}{4}t = -3$

$\sin\frac{\pi}{4}t = -1$.

$\frac{\pi}{4}t = \frac{3\pi}{2}$

$t = 6$.

$\dot{x} = \frac{3\pi}{4}\cos\frac{\pi}{4}t$.

$\ddot{x} = -\frac{3\pi}{16}^2 \sin\frac{\pi}{4}t$.

$t = 6 \quad \ddot{x} = -\frac{3\pi}{16}^2 \times \sin\frac{3\pi}{2} = \frac{3\pi}{16}^2 \text{ m/s}^2$.

$F = ma = 4 \times \frac{3\pi}{16}^2 = \frac{3\pi}{4}^2$ Newtons.

Suggested solutions	Marker's Feedback

Q.(13)(b) $v^2 = n^2[a^2-(x-1)^2]$.

(as centre of motion is $x=1$).

When $x=1$, speed $=1$. $v^2 = 1^2 = 1$.

$1 = n^2[a^2-(1-1)^2]$.

$1 = n^2a^2$ (1)

When $x=2$, $v^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$.

$\frac{3}{4} = n^2[a^2-(2-1)^2]$.

$\frac{3}{4} = n^2a^2 - n^2$ (2)

(1) − (2): $n^2 = \frac{1}{4} \Rightarrow n = \frac{1}{2}$.

Period $= \frac{2\pi}{(\frac{1}{2})} = 4\pi$ seconds.

Sub. $n=\frac{1}{2}$ in : $1 = \frac{1}{4}a^2 \Rightarrow a = 2$.

Amplitude = 2m.

∴ Period $= 4\pi$ seconds.

Amplitude = 2m.

(c) Assume $\log_5 11$ is rational

i.e. $\log_5 11 = \frac{p}{q}$, $p, q \in \mathbb{Z}^+$.

$\therefore 5^{\frac{p}{q}} = 11$

$5^p = 11^q$

But these means that there is a number which = both 5^p & 11^q, which contradicts the fundamental theorem of arithmetic as 5 & 11 are both prime.

There is a contradiction.

∴ $\log_5 11$ is irrational.

Suggested solutions	Marker's Feedback
Q.(13)(d) Assume $\sqrt{5}+\sqrt{6} < \sqrt{21}$ This means $(\sqrt{21})^2 > (\sqrt{5}+\sqrt{6})^2$ $21 > 5 + 2\sqrt{30} + 6$ $21 > 11 + 2\sqrt{30}$ $10 > 2\sqrt{30}$ $5 > \sqrt{30}$ But as $30 > 25 (5^2)$, $\sqrt{30} > 5$. There is a contradiction $\sqrt{5}+\sqrt{6}$ must be $\geq \sqrt{21}$. [Note: There can be variations on this).	

Suggested solutions	Marker's Feedback

Q.(13)(e) (i) $\forall\ a, b, \in \mathbb{R}$,

$a^2+b^2 \geq 2ab$ (1)

$a^2+c^2 \geq 2ac$ (2) +

$b^2+c^2 \geq 2bc$ (3)

$2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$ ÷2

$a^2+b^2+c^2 \geq ab+ac+bc.$

(ii) If a, b, c are $\triangle$ sides.

$c < a+b$

$c^2 < (a+b)^2$

$= a^2+2ab+b^2$

$\leq a^2+(a^2+b^2)+b^2$

[as $a^2+b^2 \geq 2ab$].

$c^2 \leq 2a^2+2b^2.$

(iii) $3a^2+3b^2$

$= a^2+b^2+2a^2+2b^2$

$> a^2+b^2+c^2$ [using (ii)]

$\geq ab+ac+bc$ [using (i)].

$\therefore 3a^2+3b^2 \geq ab+ac+bc.$

if a, b, c $\triangle$ sides.

Suggested solutions	Marker's Feedback

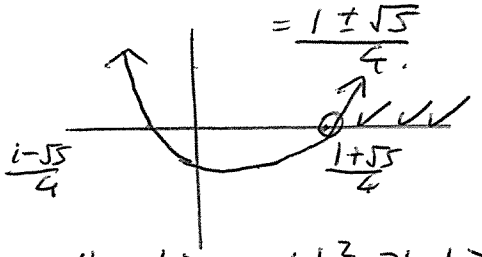
Q.(14)(a)(i) $4k^2-2k-1>0$

For $4k^2-2k-1=0$

$$k=\frac{2\pm\sqrt{(-2)^2-4\times4\times-1}}{2\times4}$$

$$=\frac{2\pm2\sqrt{5}}{8}$$

$$=\frac{1\pm\sqrt{5}}{4}.$$



Where $k>0$, $4k^2-2k-1>0$ if $k>\frac{1+\sqrt{5}}{4}$.

(ii) Step 1: Show true for $n=2$

LHS $=5^2=25$.

RHS $=6\times2^2=24$.

LHS > RHS. True for $n=2$.

Step 2: Assume true for $n=k$

i.e. $5^k>6k^2$, $k\geq2$.

Step 3: Prove true for $n=k+1$

i.e. RTP $5^{k+1}>6(k+1)^2$, $k\geq2$.

LHS

$=5^{k+1}$

$=5\times5^k$

$>5\times6k^2$ [by assumption]

$=30k^2$

$>6(k+1)$

$\therefore 5^{k+1}>30k^2>6(k+1)^2$

$5^{k+1}>6(k+1)^2$.

∴ If it is true for $n=k$ it will be true for $n=k+1$.

Hence it will be true $\forall n\in\mathbb{Z}^+\geq2$ by the principle of mathematical induction.

Note:

$30k^2-6(k+1)^2$

$=24k^2-12k-6$

$=6(4k^2-2k-1)$

>0 if $k>2$ as $4k^2-2k-1>0$ if $k>2$ from (i)

[as $\frac{1+\sqrt{5}}{4}<\frac{1+3}{4}=2$]

NOTE: CAN ALSO DO: $5^{k+1}-6(k+1)^2$

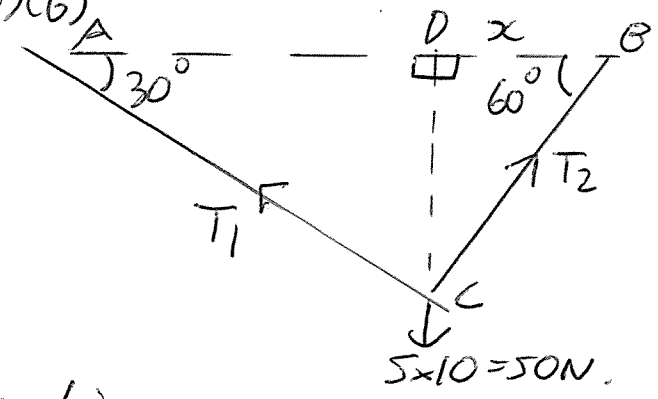
$>30k^2-6k^2-12k-6$

$=6(4k^2-2k-1)$

>0 from (i)

Suggested solutions	Marker's Feedback

(14)(b)



Resolving

horizontally: $T_1 \cos 30^\circ = T_2 \cos 60^\circ$ (1)

Vertically: $T_1 \sin 30^\circ + T_2 \sin 60^\circ = 50$ (2)

Using (1): $T_1 \dfrac{\cos 30^\circ}{\cos 60^\circ} = T_2$

$T_1 \sqrt{3} = T_2$ (1)*

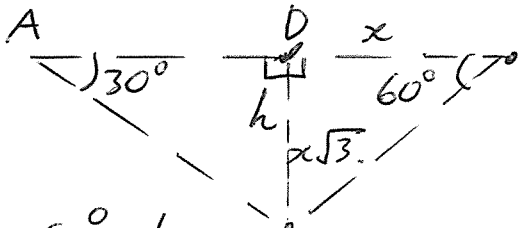
Sub (1)* in (2): $T_1 \sin 30^\circ + T_2 \sin 60^\circ = 50$

$\dfrac{T_1}{2} + T_1\sqrt{3} \times \dfrac{\sqrt{3}}{2} = 50$

$2T_1 = 50 \Rightarrow T_1 = 25N$

Sub. $T_1 = 25N$ in (1)* $T_2 = 25\sqrt{3}$ N.

(ii)



$\tan 60^\circ = \dfrac{h}{x} = \sqrt{3}$.

$\therefore h = x\sqrt{3}$.

$\tan 30^\circ = \dfrac{x\sqrt{3}}{AO} = \dfrac{1}{\sqrt{3}}$

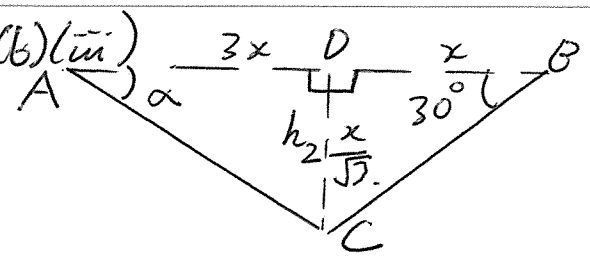
$\times AO\sqrt{3}$.

$3x = AO$.

$AO = 3x$ m.

Suggested solutions	Marker's Feedback

Q. (iv) (b) (iii)



Using $\triangle DBC$: $\tan 30^\circ = \frac{h_2}{x}$

$$\frac{x}{\sqrt{3}} = h_2$$

Using $\triangle AOC$: $\tan \alpha = \frac{h_2}{AO}$

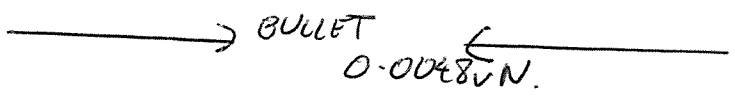
$$= \frac{\left(\frac{x}{\sqrt{3}}\right)}{3x}$$

$$= \frac{1}{3\sqrt{3}}$$

$\angle DAC = \alpha = 10^\circ 54'$

Suggested solutions	Marker's Feedback

Q.(14)(c)(i)



$F = 0.02a = -0.0048v.$

$a = -0.24v.$

$v.\frac{dv}{dx} = -0.24v.$

$\frac{dv}{dx} = -0.24.$

$\frac{dx}{dv} = \frac{25}{6}.$

$x = \int \frac{-25}{6}.dv$

$= \frac{-25v}{6} + C$

As $v = 400$ when $x = 0$

$0 = -\frac{25 \times 400}{6} + C$

$\frac{10000}{6} = C.$

$x = \frac{10\,000 - 25v}{6}.$

Note: we can also use a definite integral:

$x = \int_{400}^{v} -\frac{25}{6}.dv$

When $v = 40$,

$x = \frac{10\,000 - 25 \times 40}{6}$

$= 1500$ metres.

Suggested solutions	Marker's Feedback

Q.(14)(i)(ii) $a = \frac{dv}{dt} = -\frac{6v}{25}$ $(-0.24v)$.

$$\frac{dt}{dv} = -\frac{25}{6v}$$

$$t = -\frac{25}{6}\int \frac{1}{v}\,.\,dv$$

$$t = -\frac{25}{6}\ln|v| + C$$

As $v = 400$ when $t = 0$,

$$0 = -\frac{25}{6}\ln 400 + C.$$

$$C = \frac{25}{6}\ln 400$$

$$t = -\frac{25}{6}\ln|v| + \frac{25}{6}\ln 400$$

Note: As $v > 0$ at all times, just $\ln(v)$ goal.

$$t = -\frac{25}{6}\ln\left(\frac{v}{400}\right).$$

When $v = 40$, $t = -\frac{25}{6}\ln\left(\frac{40}{400}\right) = \frac{25}{6}\ln 10$

$= 9.59$ seconds.

Note: A definite integral can also be used here:

$$t = -\frac{25}{6}\int_{400}^{v} \frac{1}{v}\,.\,dv$$

Suggested solutions	Marker's Feedback

Q.(15)(a)(i) Centre $= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ $r = 13$.

(ii) Point on line $= \begin{pmatrix} 4\lambda \\ 3\lambda - 1 \\ 29 - 2\lambda \end{pmatrix}$.

$\therefore$ Distance:

$d^2 = (4\lambda - 2)^2 + (3\lambda - 1 + 3)^2 + (29 - 2\lambda - 1)^2$

$= (4\lambda - 2)^2 + (3\lambda + 2)^2 + (28 - 2\lambda)^2$

$= \underline{29\lambda^2 - 116\lambda + 792}$.

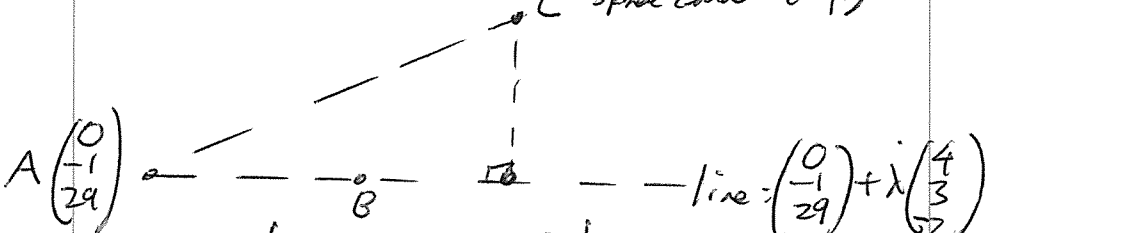
(iii) Hence MINIMUM distance from line

$\lambda = \frac{-b}{2a} = \frac{116}{2 \times 29} = 2$.

$d^2 = 29 \times 2^2 - 116 \times 2 + 792 = 676$.

Minimum distance $= \sqrt{676} = \underline{26 \text{ units}}$.

OR otherwise



Distance $= \left| \vec{AC} - \text{Proj}_{\vec{AB}} \vec{AC} \right|$

$= \left| \begin{pmatrix} 2 \\ -2 \\ -28 \end{pmatrix} - \begin{pmatrix} 8 \\ 6 \\ -4 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} -6 \\ -8 \\ -24 \end{pmatrix} \right|$

$= \underline{26 \text{ units}}$.

$\text{Proj}_{\vec{AB}} \vec{AC} = \dfrac{\begin{pmatrix} 2 \\ -2 \\ -28 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}}{\left| \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \right|^2} \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$

$= \frac{58}{29} \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 6 \\ -4 \end{pmatrix}$

Suggested solutions	Marker's Feedback

Q.(15)(a)(iv)

$$\begin{pmatrix}2\\-3\\1\end{pmatrix}+\mu\begin{pmatrix}0\\3\\-1\end{pmatrix}=\begin{pmatrix}0\\-1\\29\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\-2\end{pmatrix}$$

$x: 2=4\lambda \quad (1)$

$y: -3+3\mu=-1+3\lambda \quad (2)$

$z: 1-\mu=29-2\lambda \quad (3)$

Using $\lambda=\frac{1}{2}$.

Sub. (1) in (2): $-3+3\mu=-1+\frac{3}{2}$

$3\mu=\frac{7}{2}$

$\mu=\frac{7}{6}$

Sub. (1) & (2) in (3)

LHS $=1-\frac{7}{6}=-\frac{1}{6}$

RHS $=29-28\times\frac{1}{2}=28$

LHS $\neq$ RHS. Lines don't intersect.

Also, test if lines are //.

If //, $\begin{pmatrix}4\\3\\2\end{pmatrix}=k\begin{pmatrix}0\\3\\-1\end{pmatrix}$ Using $y: 3=3k\Rightarrow k=1$.

$z: 2=-k: k=-2$.

Not possible.

Lines don't intersect & aren't // $\Rightarrow$ they are skew.

(v) $\perp$ to both lines:

$$\begin{pmatrix}0\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=0\Rightarrow 3y_1-z_1=0 \quad (1)$$

$$\begin{pmatrix}4\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=0\Rightarrow 4x_1+3y_1-2z_1=0 \quad (2)$$

Using (1) in (2): $z_1=3y_1\Rightarrow 4x_1+3y_1-6y_1=0$

$4x_1-3y_1=0$

Choose $x_1=3\Rightarrow 12-3y_1=0\Rightarrow y_1=4$.

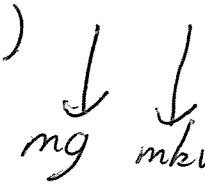
$z_1=3y_1=3\times 4=12$.

Direction vector $=\begin{pmatrix}3\\4\\12\end{pmatrix}$

Line is $\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\12\end{pmatrix}$ → Can also use direction vector $=\begin{pmatrix}\frac{1}{4}\\\frac{1}{3}\\1\end{pmatrix}$ etc.

Suggested solutions	Marker's Feedback

Q.(15)(b)



$$F = ma = mv\cdot\frac{dv}{dx} = -mg - mkv^2$$

$$v\cdot\frac{dv}{dx} = -g - kv^2$$

$$\frac{dv}{dx} = \frac{-g-kv^2}{v}$$

$$\frac{dx}{dv} = \frac{-v}{g+kv^2}$$

$$x = -\frac{1}{2k}\int\frac{2kv}{g+kv^2}\cdot dv$$

$$x = -\frac{1}{2k}\ln[g+kv^2]+C$$

As $v=U$ when $x=0$,

$$0 = -\frac{1}{2k}\ln(g+kU^2)+C$$

$$\frac{1}{2k}\ln(g+kU^2) = C$$

$$x = -\frac{1}{2k}\ln\left[g+kv^2\right]+\frac{1}{2k}\ln\left[g+kU^2\right]$$

$$x = \frac{1}{2k}\ln\left[\frac{g+kU^2}{g+kv^2}\right] \text{ as required.}$$

(ii) Max. height: $v=0$

$$x = \frac{1}{2k}\ln\left[\frac{g+kU^2}{g}\right].$$

Suggested solutions	Marker's Feedback

Q.(16)(a)(i) If $z = x + iy$, $\bar{z} = x - iy$

$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$.

(ii) Quadratic equation:

Sum of roots $= (\alpha + \bar{\alpha}) + (\beta + \bar{\beta})$ = same as sum of roots of original quatic

$= \frac{18}{9}$

$-\frac{b}{1} = 2$. Co-efficient of $z = \underline{-2}$.

∴ Product of roots of quadratic

$= (\alpha + \bar{\alpha})(\beta + \bar{\beta})$

$= \alpha\beta + \alpha\bar{\beta} + \bar{\alpha}\beta + \bar{\alpha}\bar{\beta}$

$= (\alpha\beta + \alpha\bar{\beta} + \bar{\alpha}\beta + \bar{\alpha}\bar{\beta} + \alpha\bar{\alpha} + \beta\bar{\beta}) - (\alpha\bar{\alpha} + \beta\bar{\beta})$

$=$ Sum of original quatic roots 2 at a time $-(1+1)$

as $\alpha\bar{\alpha} = |\alpha|^2 = 1$ & $\beta\bar{\beta} = |\beta|^2 = 1$.

$= \frac{26}{9} - 2$

Constant $= \frac{8}{9}$.

Equation is $z^2 - 2z + \frac{8}{9} = 0$

or $\underline{9z^2 - 18z + 8 = 0.}$

(iii) $9z^2 - 18z + 8 = 0$

$(3z-2)(3z-4) = 0$

$\therefore \alpha + \bar{\alpha} = \frac{2}{3}$, $\beta + \bar{\beta} = \frac{4}{3}$.

Suggested solutions	Marker's Feedback

Q.(16)(a)(iv) If $|\alpha| = 1$,

$\alpha = \cos\theta + i\sin\theta$.

$\bar{\alpha} = \cos\theta - i\sin\theta$

$\alpha + \bar{\alpha} = 2\cos\theta$.

(v) Using (iii) & (iv)

$\alpha + \bar{\alpha} = 2\cos\theta = \frac{2}{3}$.

$\cos\theta = \frac{1}{3}$.

$\sin\theta = \pm\sqrt{1-\left(\frac{1}{3}\right)^2}$

$= \frac{2\sqrt{2}}{3}$.

$\alpha \& \bar{\alpha} = \frac{1 \pm 2i\sqrt{2}}{3}$.

$\beta + \bar{\beta} = 2\cos\theta_2$.

$2\cos\theta_2 = \frac{4}{3}$.

$\cos\theta_2 = \frac{2}{3}$.

$\sin\theta_2 = \pm\sqrt{1-\left(\frac{2}{3}\right)^2}$

$= \frac{\sqrt{5}}{3}$.

$\beta \& \bar{\beta} = \frac{2 \pm i\sqrt{5}}{3}$.

Solutions to $9z^4 - 18z^3 + 26z^2 - 18z + 9 = 0$

are $\frac{1+2i\sqrt{2}}{3}$, $\frac{1-2i\sqrt{2}}{3}$, $\frac{2+i\sqrt{5}}{2}$, $\frac{2-i\sqrt{5}}{2}$.

(16)(a)(v) Suggested solutions | Marker's Feedback

Or otherwise method:

$9z^4 - 18z^3 + 26z^2 - 18z + 9 = 0$

$\div z^2 \quad \div z^2$

(as we know $z \neq 0$).

$9z^2 - 18z + 26 - \frac{18}{z} + \frac{9}{z^2} = 0$

$9\left(z^2 + \frac{1}{z^2}\right) - 18\left(z + \frac{1}{z}\right) + 26 = 0$

Noting $z^2 + \frac{1}{z^2} = \left(z + \frac{1}{z}\right)^2 - 2$,

$9\left(z + \frac{1}{z}\right)^2 - \quad - 18\left(z + \frac{1}{z}\right) + 26 = 0$

Letting $u = z + \frac{1}{z}$,

$9u^2 - 18u + 8 = 0$

$u = \frac{2}{3}$ or $\frac{4}{3}$ [from (iii)].

$z + \frac{1}{z} = \frac{2}{3}$ or $z + \frac{1}{z} = \frac{4}{3}$

$\times 3z$

$3z^2 - 2z + 3 = 0$

$z = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 3 \times 3}}{2 \times 3}$

$= \frac{2 \pm 4i\sqrt{2}}{6}$

$= \frac{1 \pm 2i\sqrt{2}}{3}$.

$3z^2 - 4z + 3 = 0$

$z = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 3 \times 3}}{2 \times 3}$

$= \frac{4 \pm 2i\sqrt{5}}{6}$

$= \frac{2 \pm i\sqrt{5}}{3}$.

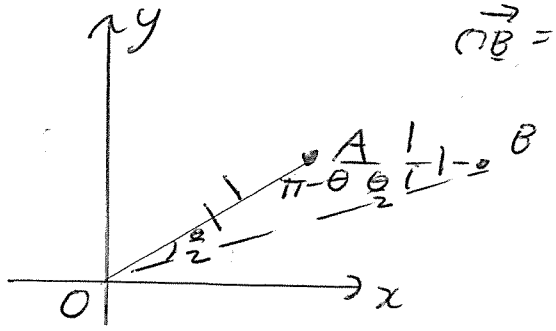
Note: This works because if $|\alpha| = 1$, $|\bar{\alpha}| = \frac{1}{\alpha}$.

Suggested solutions	Marker's Feedback

Q.(16)(b)(i)

$\overrightarrow{OA} = z$.

$\overrightarrow{OB} = z+1$.



If $\text{Arg } z = \angle AOx = \theta$,

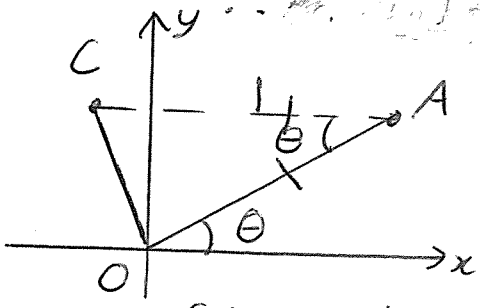
$\angle OAB = \pi - \theta$ [co-interior $\angle$'s, $Ox // AB$].

$\angle ABO = \angle AOB = \left[\frac{1}{2}(\pi - (\pi - \theta)\right] = \frac{\theta}{2}$

[$\angle$'s opposite = sides in isosceles $\triangle$].

$\therefore \angle BOx = \frac{\theta}{2}$ [alternate $\angle$'s, $Ox // AB$ or = $\theta - \frac{\theta}{2}$ [adjacent $\angle$'s].

$\therefore \text{Arg}(z+1) = \frac{\theta}{2}$



$OA = z$

$\overrightarrow{OC} = z - 1$.

$\angle CAO = \theta$ [alternate $\angle$'s, $CA // Ox$].

$\angle COA = \angle OCA = \frac{1}{2}(\pi - \theta) = \frac{\pi}{2} - \frac{\theta}{2}$

[$\angle$'s opposite = sides in isosceles $\triangle$].

$\therefore \text{Arg}(z-1) = \theta + \frac{\pi}{2} - \frac{\theta}{2}$

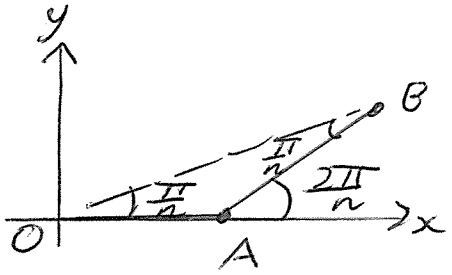
$= \frac{\theta + \pi}{2}$.

Suggested solutions	Marker's Feedback

Q.(16)(b)(ii) Roots of $z^n - 1 = 0$

$z = \text{cis}\frac{2k\pi}{n}$, $k = 0, 1, \ldots (n-1)$.

(iii)



$\overrightarrow{OB} = w + 1 + w$.

$\text{Arg}(1+w) = \frac{1}{2}\text{Arg}\,w = \frac{\pi}{n}$ [from (i)].

Using the sine rule on $\triangle OAB$:

$$\frac{AB}{\sin\frac{\pi}{n}} = \frac{OB}{\sin\left(\pi - \frac{2\pi}{n}\right)}$$

$$\frac{1}{\sin\frac{\pi}{n}} = \frac{OB}{\sin\frac{2\pi}{n}} \quad [\text{as } \sin(\pi-\theta) = \sin\theta]$$

$$\frac{\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}} = OB.$$

$$\therefore |w+1| = \frac{\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}}.$$

& $\text{Arg}(w+1) = \frac{\pi}{n}$.

Suggested solutions	Marker's Feedback

Q.(16)(b)(iv)

$$\frac{1+w+\ldots+w^k}{1+w+\ldots+w^{k-1}}$$

$$= \frac{w^{k+1}-1}{w-1} \div \frac{w^{k-1}-1}{w-1}$$

$$= \frac{w^{k+1}-1}{w-1} \times \frac{w-1}{w^k-1}$$

$$= \frac{w^{k+1}-1}{w^k-1}.$$

$\therefore$ Noting $\text{Arg}(z-1) = \frac{\theta+\pi}{2}$

where $\text{Arg}\ z = \theta$ if $|z|=1$, ↑

$$\text{Arg}\,(w^{k+1}-1) = \frac{\frac{2(k+1)\pi}{n} + \pi}{2} \quad \text{(using (i))}.$$

$$= \frac{2k\pi + 2\pi + n\pi}{2n}.$$

$$\text{Arg}\,(w^k-1) = \frac{\frac{2k\pi}{n} + \pi}{2}$$

$$= \frac{2k\pi + n\pi}{2n}.$$

$$\therefore \text{Arg}\left(\frac{w^{k+1}-1}{w^k-1}\right) = \frac{2k\pi + 2\pi + n\pi - 2k\pi - n\pi}{2n}$$

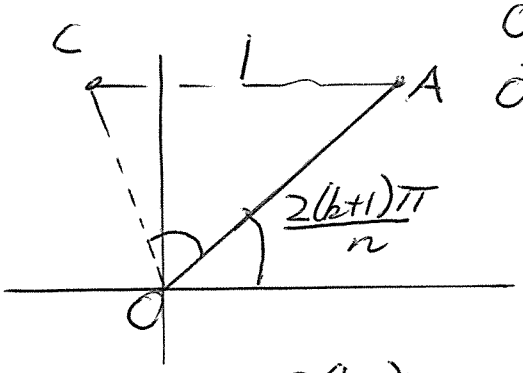
$$= \frac{\pi}{n}.$$

(i)(ii)(iii)(iv) cont'd. Suggested solutions	Marker's Feedback

Noting $1+w+\dots+w^k = \dfrac{w^{k+1}-1}{w-1}$.

$\therefore \left|1+w+\dots+w^k\right| = \dfrac{\left|w^{k+1}-1\right|}{\left|w-1\right|}$

& $|w-1| = \dfrac{\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}}$ from (iii)



$\overrightarrow{OA} = w^{k+1}$

$\overrightarrow{OC} = w^{k+1} - 1.$

$\angle COA = \dfrac{\pi - \frac{2(k+1)\pi}{n}}{2}$ (from (i))

as $|w^{k+1}| = 1.$

$= \dfrac{\pi}{2} - \dfrac{(k+1)\pi}{n}.$

Using the sine rule on $\triangle OAC$

$\dfrac{AC}{\sin\left(\frac{\pi}{2} - \frac{(k+1)\pi}{n}\right)} = \dfrac{OC}{\sin\frac{2(k+1)\pi}{n}}$

$\dfrac{1}{\cos\left[\frac{(k+1)\pi}{n}\right]} = \dfrac{OC}{\sin\frac{2(k+1)\pi}{n}}$ (as $\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$)

$OC = \dfrac{\sin\left[\frac{2(k+1)\pi}{n}\right]}{\cos\frac{(k+1)\pi}{n}}$

$= \dfrac{2\sin\frac{(k+1)\pi}{n}\cos\frac{(k+1)\pi}{n}}{\cos\frac{(k+1)\pi}{n}}$

$\left|w^{k+1} - 1\right| = 2\sin\dfrac{(k+1)\pi}{n}.$

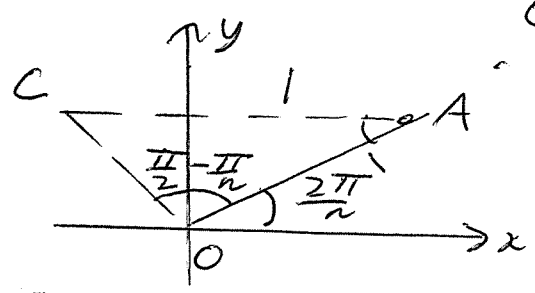
16(b)(iv)	Suggested solutions continued.	Marker's Feedback

Also $w^k - 1$:

Using $\triangle OAC$,

$\overrightarrow{OA} = w$

$\overrightarrow{OC} = w - 1$.



$$\frac{OC}{\sin\frac{2\pi}{n}} = \frac{AC}{\sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)}$$

$$\frac{OC}{\sin\frac{2\pi}{n}} = \frac{AC = 1}{\cos\frac{\pi}{n}}$$

$$OC = \frac{\sin\frac{2\pi}{n}}{\cos\frac{\pi}{n}}$$

$$= \frac{2\sin\frac{\pi}{n}\cos\frac{\pi}{n}}{\cos\frac{\pi}{n}}$$

$$= 2\sin\frac{\pi}{n}.$$

$$\therefore \frac{|w^{k+1} - 1|}{|w - 1|} = |1 + w + \dots + w^k|$$

$$= \frac{2\sin\frac{(k+1)\pi}{n}}{2\sin\frac{\pi}{n}}$$

$$= \frac{\sin\left[\frac{(k+1)\pi}{n}\right]}{\sin\left(\frac{\pi}{n}\right)}$$

as required.

END OF SOLUTIONS