Write or stick your student number here

**2026** **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

# Mathematics Extension 2

**General instructions**

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Stick or write your Student Number at the top of this page and on each answer booklet

**Total marks: 100**

**Section I – 10 marks** (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II – 90 marks** (pages 5–13)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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**1** Given $z_1 = 4e^{\frac{i\pi}{3}}$ and $z_2 = 2e^{\frac{i\pi}{6}}$, which of the following is the complex number $\frac{z_1}{z_2}$ expressed in Cartesian form?

A. $2\left(1 + i\sqrt{3}\right)$

B. $2\left(\sqrt{3} + i\right)$

C. $1 + i\sqrt{3}$

D. $\sqrt{3} + i$

**2** Consider the statement:

'If a triangle is equilateral, then it is isosceles'.

Which of the following is the converse of that statement?

A. If a triangle is isosceles, then it is equilateral.

B. A triangle is equilateral if and only if it is isosceles.

C. If a triangle is not equilateral, then it is not isosceles.

D. If a triangle is not isosceles, then it is not equilateral.

**3** The three cube roots of a complex number $w$ are $z_1$, $z_2$ and $z_3$.

If $z_1 = 2i$, what are the other two roots?

A. $1 - i\sqrt{3}$ and $-1 - i\sqrt{3}$

B. $-1 - i\sqrt{3}$ and $-1 + i\sqrt{3}$

C. $\sqrt{3} - i$ and $-\sqrt{3} - i$

D. $-\sqrt{3} - i$ and $-\sqrt{3} + i$

**4** Consider the statement:

$$\forall\, x \in \mathbb{Z}\,,\, \ln|x| > 0$$

Which of the following is the negation of this statement?

A. $\exists\, x \in \mathbb{Z}$ , such that $\ln|x| > 0$

B. $\exists\, x \in \mathbb{Z}$ , such that $\ln|x| \leq 0$

C. $\forall\, x \in \mathbb{Z}\,,\, \ln|x| \leq 0$

D. $\ln|x| > 0 \Rightarrow x \in \mathbb{Z}$

**5** A particle's velocity $v$ is given by $v = 4x^2 + 2$, where $x$ is its displacement from the origin.

Which of these expressions represents the acceleration of the particle?

A. $8x$

B. $32x^3 + 16x$

C. $8x^4 + 8x^2 + 2$

D. $16x^4 + 16x^2 + 4$

**6** Let $P$ and $Q$ be the points with position vectors $\underset{\sim}{p} = 3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{q} = -\underset{\sim}{i} - 8\underset{\sim}{j} + 4\underset{\sim}{k}$ respectively.

What is the Cartesian equation of the sphere which has $PQ$ as a diameter of the sphere?

A. $(x-1)^2 + (y+2)^2 + (z-1)^2 = 49$

B. $(x+2)^2 + (y+6)^2 + (z-3)^2 = 49$

C. $(x-1)^2 + (y+2)^2 + (z-1)^2 = 196$

D. $(x+2)^2 + (y+6)^2 + (z-3)^2 = 196$

**7** Let $\omega$ be a non-real complex fifth root of unity.

What is the value of the expression $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$?

A. 0

B. 1

C. 4

D. 5

**8** The polynomial $P(z)$ has real coefficients. The complex number $\alpha$ is of the form $a+ib$, where $a$ and $b$ are both real, non-zero and distinct.

If $P(a)$, $P'(a)$, $P(b)$, $P'(b)$ and $P(\alpha)$ are all zero, what is the minimum degree of $P(z)$?

A. 4

B. 5

C. 6

D. 7

**9** Which of the following is a primitive of $\int \sin^4 x \cos^5 x \, dx$?

A. $\frac{1}{6}\cos^6 x + C$

B. $\frac{1}{80}\sin^5 2x + C$

C. $\frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$

D. $\frac{1}{6}\cos^6 x - \frac{1}{4}\cos^8 x + \frac{1}{10}\cos^{10} x + C$

**10** A complex number $z$ is defined such that $|z-i| = \left|z-\sqrt{3}+2i\right|$.

What is the value of $\text{Arg}(z)$ when $|z|$ is a minimum?

A. $-\frac{\pi}{3}$

B. $-\frac{\pi}{6}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{3}$

## Section II

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hours and 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use the Question 11 Writing Booklet

a) The location of the complex number $z$ is shown on the diagram on page 1 of the Question 11 Writing Booklet. **2**

On the diagram provided in the writing booklet, indicate the locations of $z^2$ and $-iz$.

b) Consider the vector $\underset{\sim}{v} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}$. **1**

Find a unit vector in the direction of $\underset{\sim}{v}$.

c) Let $z = 1 - i$.

(i) Express $z$ in modulus-argument form. **2**

(ii) Find $z^{10}$ and express your answer in Cartesian form. **2**

d) A particle moves such that its position vector at time $t$ is given by **2**
$\underset{\sim}{r}(t) = (t^2)\underset{\sim}{i} + (2t)\underset{\sim}{j} + (\ln t)\underset{\sim}{k}$ for $t > 0$.

Determine the time $t$ at which the particle's velocity vector is perpendicular to its acceleration vector.

**Question 11 continues on page 6**

e) A polynomial $P(z) = z^3 + az^2 + bz + c$ has real coefficients $a, b, c$. **1**

If $z_1 = 2 + i$ and $z_2 = 3$ are roots of $P(z)$, what is the value of the coefficient $b$?

f) Prove by contradiction that $\log_2 5$ is an irrational number. **2**

g) By using a suitable substitution, or otherwise, evaluate $\int_0^1 \sqrt{4 - x^2}\, dx$. **3**

**End of Question 11**

**Question 12** (15 marks) Use the Question 12 Writing Booklet

a) Let $a$, $b$, $c$ be positive real numbers.

(i) Show that $\frac{a+b}{2} \geq \sqrt{ab}$. **1**

(ii) Prove that, if $a + b + c = 1$, then $(1 - a)(1 - b)(1 - c) \geq 8abc$. **2**

b) A particle moves in a straight line with acceleration $a = 6x^2 - 4$, where $x$ is the displacement from the origin. The particle starts from rest at $x = 1$. **3**

Show that the particle is at rest at three different positions.

c) Find $\int xe^{2x}\, dx$. **2**

d) (i) Find the values of $A$, $B$ and $C$ such that: **2**

$$\frac{8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

(ii) Hence, evaluate the definite integral: **2**

$$\int_1^2 \frac{8}{x(x^2+4)}\, dx$$

e) The points $A$, $B$ and $C$ have position vectors $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{j} + 6\underset{\sim}{k}$ respectively. **3**

Find the size of $\angle ABC$ to the nearest degree.

**End of Question 12**

**Question 13** (15 marks) Use the Question 13 Writing Booklet

a) Given that $|z-1|=2$ and $|w+2|=1$, find the maximum possible value of $|z-w|$. Justify your answer. **2**

b) Let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be two non-zero vectors. **2**

Prove that if the vectors $\underset{\sim}{u}+\underset{\sim}{v}$ and $\underset{\sim}{u}-\underset{\sim}{v}$ are perpendicular, then $\left|\underset{\sim}{u}\right|=\left|\underset{\sim}{v}\right|$.

c) A line $L$ has the vector equation $\underset{\sim}{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$. **3**

A sphere $S$ has the equation $(x-4)^2+(y-5)^2+(z-1)^2=9$.

Show that $L$ and $S$ do not have any points of intersection.

d) By considering $e^{i(A+B)}$ and using Euler's formula, prove the trigonometric identity: **2**

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

e) Evaluate $\int_2^5 \frac{x-1}{\sqrt{5+4x-x^2}}\,dx$. **3**

f) Let $z$ be a complex number such that $|z|=1$. **3**

Prove that $\frac{z-1}{z+1}$ is a purely imaginary number, given $z\neq -1$.

**End of Question 13**

**Question 14** (15 marks) Use the Question 14 Writing Booklet

a) (i) Sketch the region in the complex plane where both inequalities $|z-2| \leq 2$ and $0 \leq \arg(z) \leq \frac{\pi}{4}$ are satisfied. **2**

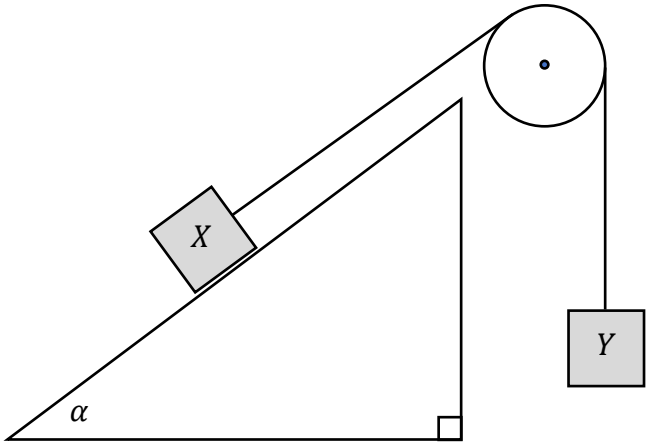
(ii) Find the exact area of this region. **2**

b) Solve $z^2 - (2+i)z + (3+i) = 0$. **2**

c) An object, $X$, of mass 5 kg lies on a smooth plane inclined at an angle $\alpha$ to the horizontal, where $\sin\alpha = \frac{3}{5}$.

Object $X$ is connected by a light inextensible string, which passes over a smooth pulley at the top of the plane, to a second object, $Y$, of mass 3 kg that hangs freely.

The system is released from rest.




(i) Show that both objects are in equilibrium. **2**

(ii) Find the tension in the string. Use $g = 9.8$ ms$^{-2}$. **1**

**Question 14 continues on page 10**

d) The points $A$, $B$ and $C$ in the complex plane represent the complex numbers $z_1$, $z_2$ and $z_3$ respectively. **3**

If $\frac{z_2-z_1}{z_3-z_1} = i$, show that triangle $ABC$ is a right-angled isosceles triangle.

e) Prove by mathematical induction that, for all non-negative integers $n$, **3**

$$(x+a)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} a^r$$

**End of Question 14**

**Question 15** (15 marks) Use the Question 15 Writing Booklet

a) Let $z$ be a root of the equation $z^5 = -1$.

(i) Show that the roots of this equation are given by $z_k = e^{i\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)}$, for $k = 0, 1, 2, 3, 4$. **2**

(ii) Hence, prove that $\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{3\pi}{5}\right) = -\frac{1}{4}$. **3**

b) The polynomial $P(z) = z^4 - 4z^3 + 14z^2 - 36z + 45$ has a purely imaginary root. Find all the roots of $P(z) = 0$. **3**

c) Let $a$, $b$, $c$ be the side lengths of a triangle. **3**

Prove that the following inequality must hold:

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3$$

d) (i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

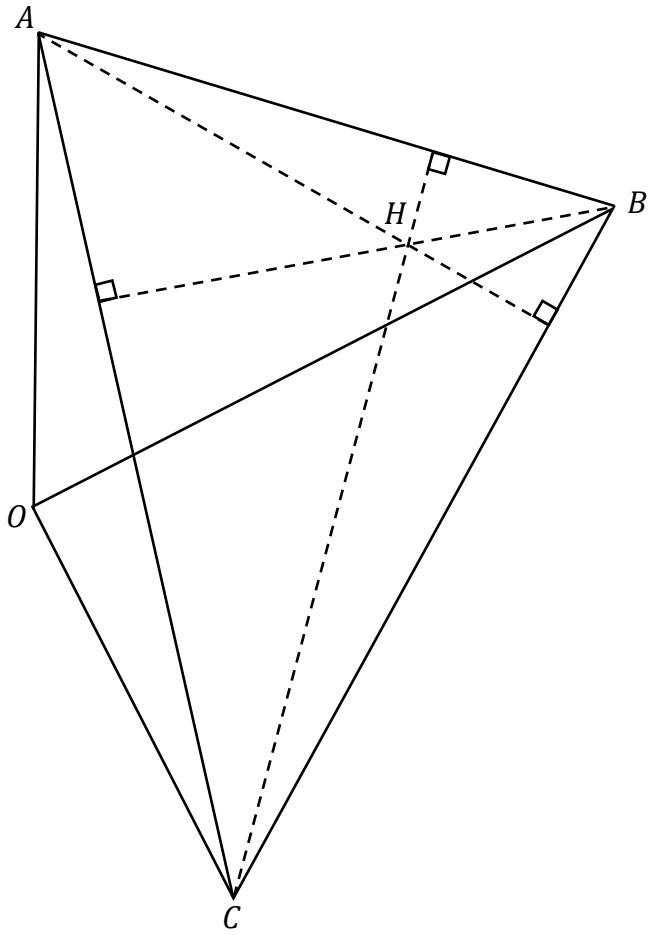
(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan(x)}\,dx$. **2**

**End of Question 15**

**Question 16** (15 marks) Use the Question 16 Writing Booklet

a) In the tetrahedron $OABC$, the edges $OA$, $OB$ and $OC$ are mutually perpendicular. **4**
Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$.

The altitude of a triangle is the perpendicular line segment drawn from any vertex to the opposite side. Every triangle has three altitudes, which intersect at a single point called the orthocentre. Let $H$ be the orthocentre of triangle $ABC$.




Similarly, the altitude of a pyramid is the perpendicular line segment drawn from any vertex to the opposite face. Consider the altitude of pyramid $OABC$, drawn from $O$ and perpendicular to triangle $ABC$. Use vector methods to show that this altitude passes through $H$.

**Question 16 continues on page 13**

b) Find $\int \frac{1}{(x^2+1)^2}\,dx$. **3**

c) Prove that, for any real numbers $a$ and $b$, $a^2 + b^2 - ab - a - b + 1 \geq 0$. **2**

d) Consider the integral $I_n = \int_0^1 (1-x^2)^n dx$ for integers $n \geq 0$.

(i) Show that for $n \geq 1$, the recurrence relation $I_n = \frac{2n}{2n+1} I_{n-1}$ holds. **3**

(ii) Hence, show that $I_n = \frac{4^n (n!)^2}{(2n+1)!}$. **3**

**End of paper**

**SOLUTIONS**

**2026** **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

# Mathematics Extension 2

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**Total marks: 100**

**Section I – 10 marks** (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II – 90 marks** (pages 5–13)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

## Section I

### Multiple-choice answer key

| | |
|---|---|
| 1 | D |
| 2 | A |
| 3 | C |
| 4 | B |
| 5 | B |
| 6 | A |
| 7 | D |
| 8 | C |
| 9 | C |
| 10 | A |

**1** Given $z_1 = 4e^{\frac{i\pi}{3}}$ and $z_2 = 2e^{\frac{i\pi}{6}}$, which of the following is the complex number $\frac{z_1}{z_2}$ expressed in Cartesian form?

A. $2\left(1 + i\sqrt{3}\right)$

B. $2\left(\sqrt{3} + i\right)$

C. $1 + i\sqrt{3}$

D. $\sqrt{3} + i$

***Sample solution:***

$$\frac{z_1}{z_2} = \frac{4}{2}e^{i\left(\frac{\pi}{3} - \frac{\pi}{6}\right)}$$

$$= 2e^{\frac{i\pi}{6}}$$

$$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= \sqrt{3} + i$$

**2** Consider the statement:

'If a triangle is equilateral, then it is isosceles'.

Which of the following is the converse of that statement?

A. If a triangle is isosceles, then it is equilateral.

B. A triangle is equilateral if and only if it is isosceles.

C. If a triangle is not equilateral, then it is not isosceles.

D. If a triangle is not isosceles, then it is not equilateral.

***Sample solution:***

Converse of $P \Longrightarrow Q$ is $Q \Longrightarrow P$

Hence, converse is 'if a triangle is isosceles, then it is equilateral'.

**3** The three cube roots of a complex number $w$ are $z_1$, $z_2$ and $z_3$.

If $z_1 = 2i$, what are the other two roots?

A. $1 - i\sqrt{3}$ and $-1 - i\sqrt{3}$

B. $-1 - i\sqrt{3}$ and $-1 + i\sqrt{3}$

C. $\sqrt{3} - i$ and $-\sqrt{3} - i$

D. $-\sqrt{3} - i$ and $-\sqrt{3} + i$

***Sample solution:***

$2i = 2\,\text{cis}\,\frac{\pi}{2}$

Other two roots will be $2\,\text{cis}\left(\frac{\pi}{2} \pm \frac{2\pi}{3}\right)$

$$2\,\text{cis}\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) = 2\,\text{cis}\,\frac{7\pi}{6} \qquad 2\,\text{cis}\left(\frac{\pi}{2} - \frac{2\pi}{3}\right) = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$$

$$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \qquad = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= -\sqrt{3} - i \qquad = \sqrt{3} - i$$

**4** Consider the statement:

$$\forall\, x \in \mathbb{Z}\,,\, \ln|x| > 0$$

Which of the following is the negation of this statement?

A. $\exists\, x \in \mathbb{Z}$, such that $\ln|x| > 0$

B. $\exists\, x \in \mathbb{Z}$, such that $\ln|x| \leq 0$

C. $\forall\, x \in \mathbb{Z}\,,\, \ln|x| \leq 0$

D. $\ln|x| > 0 \Rightarrow x \in \mathbb{Z}$

***Sample solution:***

Original statement: For all integers $x$, $\ln|x| > 0$

Negation of a 'for all' statement is a 'there exists' statement.

Negation: There exists an integer $x$ such that $\ln|x| \leq 0$

**5** A particle's velocity $v$ is given by $v = 4x^2 + 2$, where $x$ is its displacement from the origin.

Which of these expressions represents the acceleration of the particle?

A. $8x$

B. $32x^3 + 16x$

C. $8x^4 + 8x^2 + 2$

D. $16x^4 + 16x^2 + 4$

***Sample solution:***

$$\frac{dv}{dx} = 8x$$

$$a = v\frac{dv}{dx}$$
$$= (4x^2 + 2)(8x)$$
$$= 32x^3 + 16x$$

**6** Let $P$ and $Q$ be the points with position vectors $\underset{\sim}{p} = 3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{q} = -\underset{\sim}{i} - 8\underset{\sim}{j} + 4\underset{\sim}{k}$ respectively.

What is the Cartesian equation of the sphere which has $PQ$ as a diameter of the sphere?

A. $(x-1)^2 + (y+2)^2 + (z-1)^2 = 49$

B. $(x+2)^2 + (y+6)^2 + (z-3)^2 = 49$

C. $(x-1)^2 + (y+2)^2 + (z-1)^2 = 196$

D. $(x+2)^2 + (y+6)^2 + (z-3)^2 = 196$

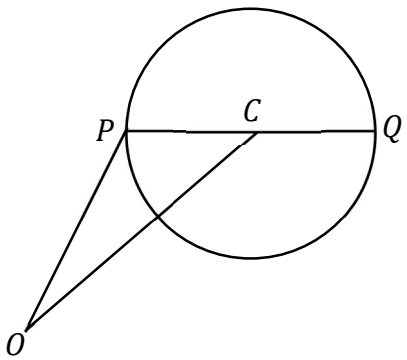
***Sample solution:***

$$\overrightarrow{PQ} = \begin{pmatrix} -4 \\ -12 \\ 6 \end{pmatrix}$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{16 + 144 + 36}$$
$$= 14$$
$$r = 7$$

Centre: $\overrightarrow{OC} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$

$$= \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$




Hence equation is $(x-1)^2 + (y+2)^2 + (z-1)^2 = 49$

**7** Let $\omega$ be a non-real complex fifth root of unity.

What is the value of the expression $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$?

A. 0

B. 1

C. 4

D. 5

***Sample solution:***

$P(z) = z^5 - 1$, roots are 1, $\omega$, $\omega^2$, $\omega^3$, $\omega^4$

Consider two different factorisations of $P(z)$:

$(z-1)(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4) = (z-1)(z^4+z^3+z^2+z+1)$

Divide through by $(z-1)$:

$$(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4) = z^4+z^3+z^2+z+1$$

Let $z = 1$:

$$(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) = 1+1+1+1+1$$
$$= 5$$

**8** The polynomial $P(z)$ has real coefficients. The complex number $\alpha$ is of the form $a+ib$, where $a$ and $b$ are both real, non-zero and distinct.

If $P(a)$, $P'(a)$, $P(b)$, $P'(b)$ and $P(\alpha)$ are all zero, what is the minimum degree of $P(z)$?

A. 4

B. 5

C. 6

D. 7

***Sample solution:***

$P(a)$, $P'(a)$ are zero $\Longrightarrow$ $a$ is a double root

$P(b)$, $P'(b)$ are zero $\Longrightarrow$ $b$ is a double root

$P(\alpha) = 0 \Longrightarrow P(\bar{\alpha}) = 0$, so both $\alpha$ and $\bar{\alpha}$ are roots

Hence $P(z)$ has degree at least 6

**9** Which of the following is a primitive of $\int \sin^4 x \cos^5 x \, dx$?

A. $\frac{1}{6}\cos^6 x + C$

B. $\frac{1}{80}\sin^5 2x + C$

C. $\frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$

D. $\frac{1}{6}\cos^6 x - \frac{1}{4}\cos^8 x + \frac{1}{10}\cos^{10} x + C$

***Sample solution:***

$$\begin{aligned}\int \sin^4 x \cos^5 x \, dx &= \int \cos x \, (1 - \sin^2 x)^2 \sin^4 x \, dx \\ &= \int \cos x \, (1 - 2\sin^2 x + \sin^4 x) \sin^4 x \, dx \\ &= \int \cos x \, (\sin^4 x - 2\sin^6 x + \sin^8 x) dx \\ &= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + c\end{aligned}$$

**10** A complex number $z$ is defined such that $|z - i| = |z - \sqrt{3} + 2i|$.

What is the value of $\text{Arg}(z)$ when $|z|$ is a minimum?

A. $-\frac{\pi}{3}$

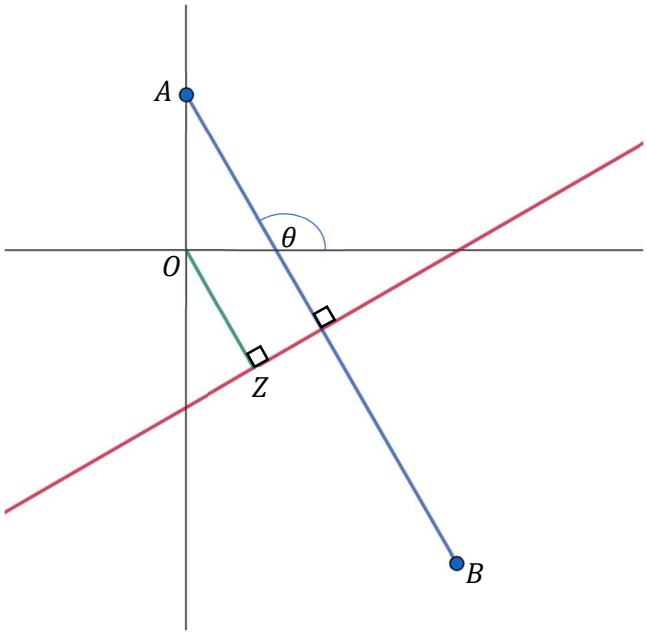
B. $-\frac{\pi}{6}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{3}$

***Sample solution:***

$z$ lies on the perpendicular bisector of the interval joining $A(0, 1)$ and $B(\sqrt{3}, -2)$.




When $|z|$ is a minimum, the interval $OZ$ is perpendicular to the perpendicular bisector of the interval joining $A(0, 1)$ and $B(\sqrt{3}, -2)$. That is, $OZ$ is parallel to $AB$.

$$m_{OZ} = m_{AB} = -\sqrt{3}$$

$$\tan\theta = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3}$$

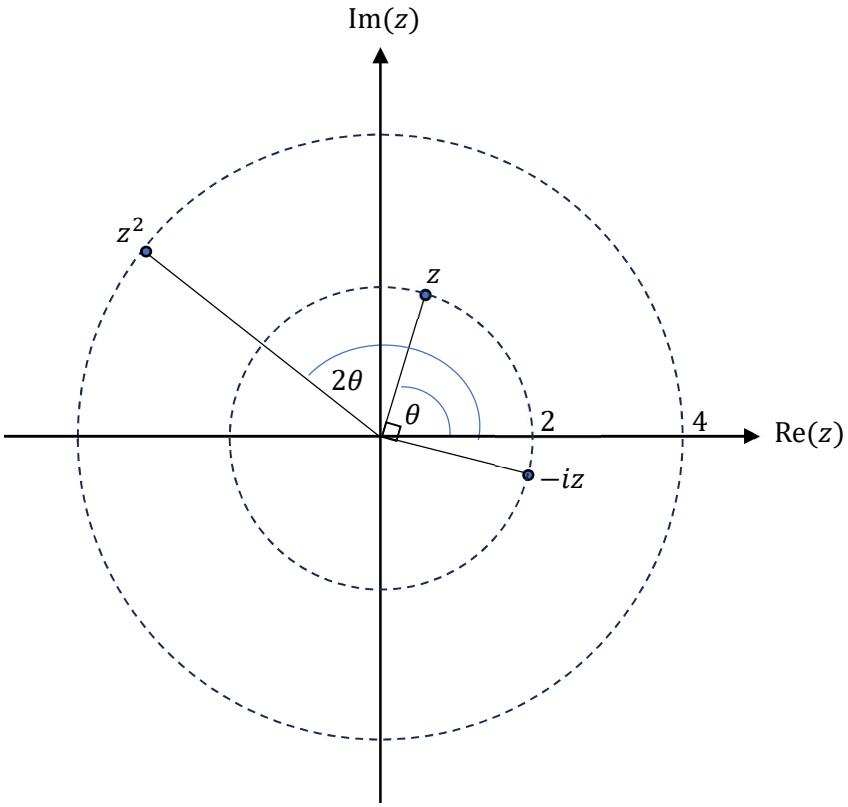
$$\text{Arg}(z) = -\frac{\pi}{3}$$

**Question 11** (15 marks)

a) The location of the complex number $z$ is shown on the diagram on page 1 of the Question 11 Writing Booklet. **2**

On the diagram provided in the writing booklet, indicate the locations of $z^2$ and $-iz$.

***Sample solution:***




| Marks | Criteria |
|---|---|
| **2** | Both points correct, with necessary details labelled |
| **1** | One point correct with details labelled OR both points in correct vicinity, but necessary details not labelled |

b) Consider the vector $\underset{\sim}{v} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}$. **1**

Find a unit vector in the direction of $\underset{\sim}{v}$.

***Sample solution:***

$$\left|\underset{\sim}{v}\right| = \sqrt{3^2 + (-4)^2 + 12^2}$$

$$= 13$$

$$\hat{\underset{\sim}{v}} = \frac{1}{13}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)$$

| Marks | Criteria |
|---|---|
| 1 | Correct answer |

c) Let $z = 1 - i$.

(i) Express $z$ in modulus-argument form. **2**

***Sample solution:***

$$r = \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

$$\tan\theta = -1$$

$$\theta = -\frac{\pi}{4} \left(\text{or}\frac{7\pi}{4}\right)$$

$$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) \text{ or } \sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)$$

| Marks | Criteria |
|---|---|
| 2 | Provides correct solution (principal argument form not necessary) |
| 1 | Correct modulus or argument |

(ii) Find $z^{10}$ and express your answer in Cartesian form. **2**

***Sample solution:***

$$z^{10} = \left(\sqrt{2}\right)^{10}\left(\cos\left(10 \times -\frac{\pi}{4}\right) + i\sin\left(10 \times -\frac{\pi}{4}\right)\right)$$

$$= 32\left(\cos\left(-\frac{5\pi}{2}\right) + i\sin\left(-\frac{5\pi}{2}\right)\right)$$

$$= 32(0 - i)$$

$$= -32i$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution |
| **1** | Correctly finds $z^{10}$ in modulus-argument form OR finds correct modulus or argument and then correctly converts their $z^{10}$ into Cartesian form |

d) A particle moves such that its position vector at time $t$ is given by $\underset{\sim}{r}(t) = (t^2)\underset{\sim}{i} + (2t)\underset{\sim}{j} + (\ln t)\underset{\sim}{k}$ for $t > 0$. **2**

Determine the time $t$ at which the particle's velocity vector is perpendicular to its acceleration vector.

***Sample solution:***

$$\underset{\sim}{v}(t) = (2t)\underset{\sim}{i} + (2)\underset{\sim}{j} + \left(\frac{1}{t}\right)\underset{\sim}{k}$$

$$\underset{\sim}{a}(t) = (2)\underset{\sim}{i} + (0)\underset{\sim}{j} - \left(\frac{1}{t^2}\right)\underset{\sim}{k}$$

Perpendicular when $\underset{\sim}{v} \cdot \underset{\sim}{a} = 0$

$$2t(2) + 2(0) + \frac{1}{t}\left(-\frac{1}{t^2}\right) = 0$$

$$4t - \frac{1}{t^3} = 0$$

$$4t^4 - 1 = 0$$

$$4t^4 = 1$$

$$t^4 = \frac{1}{4}$$

$$t = \pm\frac{1}{\sqrt{2}}$$

But $t \geq 0$, so $t = \frac{1}{\sqrt{2}}$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution (accept $t = \pm\frac{1}{\sqrt{2}}$) |
| **1** | Finds correct acceleration vector and states $\underset{\sim}{v} \cdot \underset{\sim}{a} = 0$, or equivalent merit |

e) A polynomial $P(z) = z^3 + az^2 + bz + c$ has real coefficients $a, b, c$. **1**

If $z_1 = 2 + i$ and $z_2 = 3$ are roots of $P(z)$, what is the value of the coefficient $b$?

***Sample solution:***

Real coefficients $\Rightarrow 2 - i$ is also a root

Sum of roots 2 at a time $= b$

$b = (2 + i)(2 - i) + 3(2 + i) + 3(2 - i)$

$= 4 + 1 + 6 + 3i + 6 - 3i$

$= 17$

| Marks | Criteria |
|---|---|
| 1 | Correct answer |

f) Prove by contradiction that $\log_2 5$ is an irrational number. **2**

***Sample solution:***

Assume $\log_2 5$ is a rational number

i.e. $\log_2 5 = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$ and $p, q$ are coprime.

$$\log_2 5 = \frac{p}{q}$$

$$2^{\frac{p}{q}} = 5$$

$$2^p = 5^q$$

But since 2 and 5 are both prime numbers, this cannot be true.

Hence the assumption is false, so $\log_2 5$ is irrational.

| Marks | Criteria |
|---|---|
| 2 | Provides complete and logical proof |
| 1 | Significant progress, but minor errors made or minor details missing |

g) By using a suitable substitution, or otherwise, evaluate $\int_0^1 \sqrt{4-x^2}\,dx$. **3**

***Sample solution:***

Let $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\,d\theta$

When $x = 0$, $\theta = 0$

When $x = 1$, $\theta = \frac{\pi}{6}$

$$\int_0^1 \sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{4\cos^2\theta}\cdot 2\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} 4\cos^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} 1+\cos 2\theta\,d\theta$$

$$= 2\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= 2\left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}-\left(0+\frac{1}{2}\sin 0\right)\right)$$

$$= 2\left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}-0\right)$$

$$= \frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Obtains correct primitive OR correct definite integral in terms of $\theta$, or equivalent merit |
| **1** | Attempts to use a suitable substitution or equivalent merit |

**Question 12** (15 marks)

a) Let $a$, $b$, $c$ be positive real numbers.

(i) Show that $\frac{a+b}{2} \geq \sqrt{ab}$. **1**

***Sample solution:***

$$\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$$

$$a - 2\sqrt{ab} + b \geq 0$$

$$a + b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

| Marks | Criteria |
|---|---|
| 1 | Provides correct proof |

(ii) Prove that, if $a + b + c = 1$, then $(1 - a)(1 - b)(1 - c) \geq 8abc$. **2**

***Sample solution:***

$1 - a = b + c$

$1 - b = a + c$

$1 - c = a + b$

$$(1 - a)(1 - b)(1 - c) = (b + c)(a + c)(a + b)$$
$$\geq 2\sqrt{bc} \times 2\sqrt{ac} \times 2\sqrt{ab} \quad \text{(using part i)}$$
$$\geq 8\sqrt{a^2b^2c^2}$$
$$\geq 8abc$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct proof |
| **1** | Attempts to use appropriate expressions for $(1 - a)$, $(1 - b)$ and $(1 - c)$ OR successfully incorporates result from part (i), or equivalent merit |

b) A particle moves in a straight line with acceleration $a = 6x^2 - 4$, where $x$ is the displacement from the origin. The particle starts from rest at $x = 1$. **3**

Show that the particle is at rest at three different positions.

***Sample solution:***

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^2 - 4$$

$$\frac{1}{2}v^2 = \int 6x^2 - 4 \; dx$$

$$\frac{1}{2}v^2 = 2x^3 - 4x + c$$

When $x = 1$, $v = 0$

$$0 = 2(1) - 4(1) + c$$

$$c = 2$$

$$\frac{1}{2}v^2 = 2x^3 - 4x + 2$$

$$v^2 = 4x^3 - 8x + 4$$

At rest when $v = 0$

$$0 = 4x^3 - 8x + 4$$

$$0 = x^3 - 2x + 1$$

At rest when $x = 1$, so $(x - 1)$ is a factor

$$\begin{array}{r} x^2+x-1 \\ x-1\overline{)\,x^3+0x^2-2x+1} \\ \underline{x^3 - x^2} \\ x^2-2x+1 \\ \underline{x^2 - x} \\ -x+1 \\ \underline{-x+1} \\ 0 \end{array}$$

$$0 = (x - 1)(x^2 + x - 1)$$

$$x = 1, \frac{-1 \pm \sqrt{5}}{2}$$

i.e. the particle is at rest in 3 different positions

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Forms a cubic equation and attempts to use polynomial division or sums/products of roots to find the factors, or equivalent merit |
| **1** | Finds a correct expression for $v^2$, including finding constant of integration |

c) Find $\int xe^{2x}\,dx$. **2**

***Sample solution:***

Let $u = x$ $\quad v' = e^{2x}$

$u' = 1$ $\quad v = \frac{1}{2}e^{2x}$

$$\int xe^{2x}\,dx = \left[\frac{1}{2}xe^{2x}\right] - \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x} + c\right)$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution (no penalty for omitting $+c$) |
| **1** | Attempts to use integration by parts |

d) (i) Find the values of $A$, $B$ and $C$ such that: **2**

$$\frac{8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

***Sample solution:***

$8 = A(x^2+4) + x(Bx+C)$

$8 = Ax^2 + 4A + Bx^2 + Cx$

$4A = 8 \Rightarrow A = 2$

$A + B = 0 \Rightarrow B = -2$

$C = 0$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution |
| **1** | Evaluates one of the three coefficients |

(ii) Hence, evaluate the definite integral: **2**

$$\int_1^2 \frac{8}{x(x^2+4)}\,dx$$

***Sample solution:***

$$\int_1^2 \frac{8}{x(x^2+4)}\,dx = \int_1^2 \frac{2}{x} - \frac{2x}{x^2+4}\,dx$$

$$= [2\ln x - \ln(x^2+4)]_1^2$$

$$= 2\ln 2 - \ln 8 - (2\ln 1 - \ln 5)$$

$$= \ln 4 - \ln 8 + \ln 5$$

$$= \ln\frac{5}{2}$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution (accept answer as a sum/difference of logs) |
| **1** | Finds a correct primitive, or equivalent merit |

e) The points $A$, $B$ and $C$ have position vectors $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{j} + 6\underset{\sim}{k}$ respectively. **3**

Find the size of $\angle ABC$ to the nearest degree.

***Sample solution:***

$$\overrightarrow{BA} = -2\underset{\sim}{i} + 6\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$\overrightarrow{BC} = -\underset{\sim}{i} + 5\underset{\sim}{j} + \underset{\sim}{k}$$

$$\left|\overrightarrow{BA}\right| = \sqrt{4 + 36 + 4} = \sqrt{44}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1 + 25 + 1} = \sqrt{27}$$

$$\cos(\angle ABC) = \frac{-2(-1) + 6(5) - 2(1)}{\sqrt{44} \times \sqrt{27}}$$

$$= \frac{30}{\sqrt{1188}}$$

$$\angle ABC = 29°$$

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Finds correct expressions for $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\left\|\overrightarrow{BA}\right\|$ and $\left\|\overrightarrow{BC}\right\|$, or equivalent merit |
| **1** | Finds correct expressions for $\overrightarrow{BA}$ and $\overrightarrow{BC}$ OR finds correct $\overrightarrow{BA}$ and $\left\|\overrightarrow{BA}\right\|$ OR finds correct $\overrightarrow{BC}$ and $\left\|\overrightarrow{BC}\right\|$ |

**Question 13** (15 marks)

a) Given that $|z-1|=2$ and $|w+2|=1$, find the maximum possible value of $|z-w|$. **2**
Justify your answer.

***Sample solution 1:***

$|z-w|=|(z-1)-(w+2)+3|$

Triangle inequality: $|x \pm y| \leq |x|+|y|$

Hence:

$|z-w| \leq |(z-1)+3|+|(w+2)|$

$|z-w| \leq |(z-1)|+|3|+|(w+2)|$

$|z-w| \leq 2+3+1$

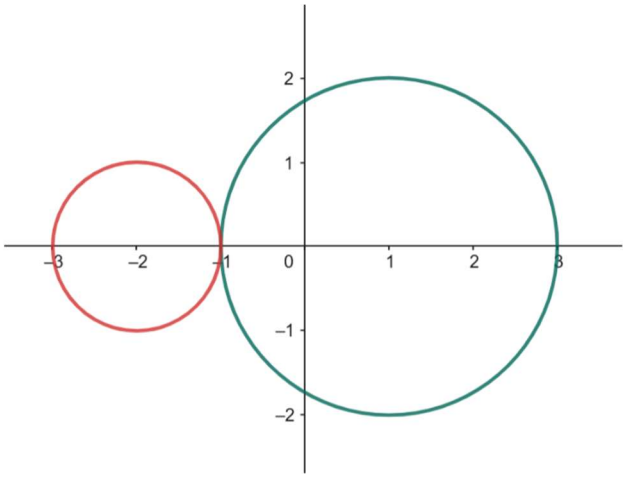
$|z-w| \leq 6$

***Sample solution 2:***

Maximum value of $|z-w|$ occurs when $z$ and $w$ lie in a straight line.

From the diagram, this happens when $z=3$ and $w=-3$.

Hence $|z-w| \leq |3-(-3)| = 6$.




| Marks | Criteria |
|---|---|
| **2** | Provides correct solution with suitable justification |
| **1** | Provides a correct triangle inequality statement or draws both circles correctly on the same plane or equivalent measure. |

b) Let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be two non-zero vectors. **2**

Prove that if the vectors $\underset{\sim}{u}+\underset{\sim}{v}$ and $\underset{\sim}{u}-\underset{\sim}{v}$ are perpendicular, then $\left|\underset{\sim}{u}\right| = \left|\underset{\sim}{v}\right|$.

***Sample solution:***

$$\left(\underset{\sim}{u}+\underset{\sim}{v}\right)\cdot\left(\underset{\sim}{u}-\underset{\sim}{v}\right) = 0$$

$$\underset{\sim}{u}\cdot\underset{\sim}{u}-\underset{\sim}{u}\cdot\underset{\sim}{v}+\underset{\sim}{v}\cdot\underset{\sim}{u}-\underset{\sim}{v}\cdot\underset{\sim}{v} = 0$$

$$\left|\underset{\sim}{u}\right|^2-\left|\underset{\sim}{v}\right|^2 = 0$$

$$\left|\underset{\sim}{u}\right|^2 = \left|\underset{\sim}{v}\right|^2$$

$$\left|\underset{\sim}{u}\right| = \left|\underset{\sim}{v}\right| \quad \text{since } \left|\underset{\sim}{u}\right|, \left|\underset{\sim}{v}\right| \geq 0$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct proof |
| **1** | Provides correct expansion of $\left(\underset{\sim}{u}+\underset{\sim}{v}\right)\cdot\left(\underset{\sim}{u}-\underset{\sim}{v}\right)$ and equates this to zero |

c) A line $L$ has the vector equation $\underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$. **3**

A sphere $S$ has the equation $(x-4)^2 + (y-5)^2 + (z-1)^2 = 9$.

Show that $L$ and $S$ do not have any points of intersection.

***Sample solution:***

$L$: $x = 1 + 2\lambda$

$y = 2 - \lambda$

$z = 3 + 2\lambda$

For points of intersection, sub into equations of $S$.

$$(1 + 2\lambda - 4)^2 + (2 - \lambda - 5)^2 + (3 + 2\lambda - 1)^2 = 9$$
$$(2\lambda - 3)^2 + (-3 - \lambda)^2 + (2 + 2\lambda)^2 = 9$$
$$4\lambda^2 - 12\lambda + 9 + 9 + 6\lambda + \lambda^2 + 4 + 8\lambda + 4\lambda^2 = 9$$
$$9\lambda^2 + 2\lambda + 13 = 0$$

$$\Delta = b^2 - 4ac$$
$$= 2^2 - 4(9)(13)$$
$$= -464$$
$$< 0 \quad \therefore \text{ no solutions for } \lambda$$

Hence $L$ and $S$ do not intersect.

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Forms a simplified quadratic equation in $\lambda$ |
| **1** | Forms parametric equations for the components of $L$ and substitutes these into the equation for $S$, or equivalent merit |

d) By considering $e^{i(A+B)}$ and using Euler's formula, prove the trigonometric identity: **2**

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

***Sample solution:***

$e^{i(A+B)} = e^{iA} \times e^{iB}$

Using Euler's formula:

$$\begin{aligned}\cos(A+B) + i\sin(A+B) &= (\cos A + i\sin A)(\cos B + i\sin B) \\ &= \cos A\cos B + i\cos A\sin B + i\sin A\cos B - \sin A\sin B \\ &= \cos A\cos B - \sin A\sin B + i(\cos A\sin B + \sin A\cos B)\end{aligned}$$

Equating real parts:

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution |
| **1** | Provides two different expressions for $e^{i(A+B)}$ in modulus-argument form, or equivalent merit<br>Completes a valid proof which equates $e^{i(A+B)}$ to $cis(A+B)$ |

e) Evaluate $\int_2^5 \frac{x-1}{\sqrt{5+4x-x^2}}\,dx$. **3**

***Sample solution:***

$$\int_2^5 \frac{x-1}{\sqrt{5+4x-x^2}}\,dx = -\frac{1}{2}\int_2^5 \frac{4-2x-2}{\sqrt{5+4x-x^2}}\,dx$$

$$= -\frac{1}{2}\int_2^5 (4-2x)(5+4x-x^2)^{-\frac{1}{2}} - \frac{2}{\sqrt{9-(x^2-4x+4)}}\,dx$$

$$= -\frac{1}{2}\left[2(5+4x-x^2)^{\frac{1}{2}} - 2\sin^{-1}\left(\frac{x-2}{3}\right)\right]_2^5$$

$$= -\frac{1}{2}\left(2\sqrt{0} - 2\sin^{-1}1 - \left(2\left(\sqrt{9} - 2\sin^{-1}0\right)\right)\right)$$

$$= -\frac{1}{2}\left(-2\left(\frac{\pi}{2}\right) - 6\right)$$

$$= \frac{\pi}{2} + 3$$

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Splits the integral into the sum of two integrals and correctly integrates one, or equivalent merit |
| **1** | Completes the square for $5+4x-x^2$ OR writes $x-1=\frac{1}{2}(4-2x-2)$, or equivalent merit |

f) Let $z$ be a complex number such that $|z| = 1$. **3**

Prove that $\frac{z-1}{z+1}$ is a purely imaginary number, given $z \neq -1$.

***Sample solution:***

$$\frac{z-1}{z+1} = \frac{x-1+iy}{x+1+iy} \times \frac{x+1-iy}{x+1-iy}$$

$$= \frac{x-(1-iy)}{(x+1)+iy} \times \frac{x+(1-iy)}{(x+1)-iy}$$

$$= \frac{x^2-(1-iy)^2}{(x+1)^2-(iy)^2}$$

$$= \frac{x^2-(1-2iy-y^2)}{(x+1)^2+y^2}$$

$$= \frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$$

$$\text{Re}\left(\frac{z-1}{z+1}\right) = \frac{x^2+y^2-1}{(x+1)^2+y^2}$$

But $|z| = \sqrt{x^2+y^2} = 1$

$$x^2+y^2 = 1$$

$$\therefore \text{Re}\left(\frac{z-1}{z+1}\right) = \frac{1-1}{(x+1)^2+y^2} = 0$$

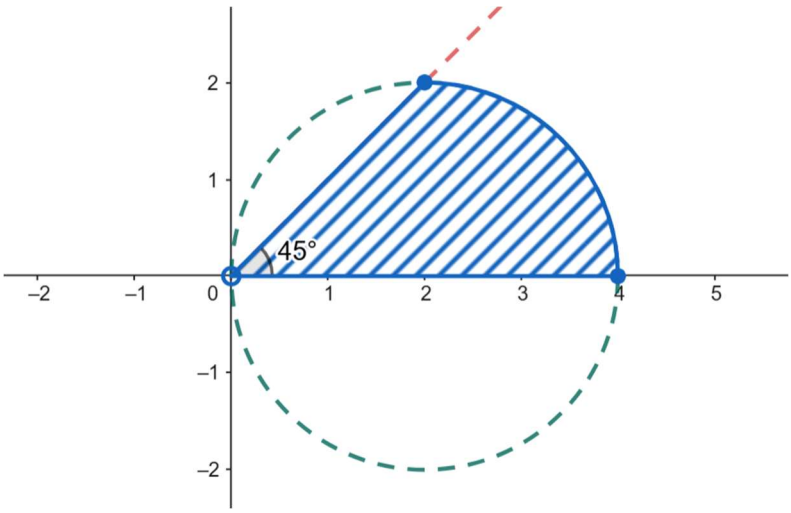
i.e. $\frac{z-1}{z+1}$ is purely imaginary

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Makes significant progress towards a correct solution |
| **1** | Expresses $\frac{z-1}{z+1}$ in terms of $x$ and $y$ and multiplies top and bottom by the conjugate of the denominator |

**Question 14** (15 marks)

a) (i) Sketch the region in the complex plane where both inequalities $|z-2| \leq 2$ and $0 \leq \arg(z) \leq \frac{\pi}{4}$ are satisfied. **2**

***Sample solution:***




| Marks | Criteria |
|---|---|
| **2** | Provides correct solution (no penalty if entire circle and ray are solid lines) |
| **1** | Correctly sketches one of the required regions, or equivalent merit |

(ii) Find the exact area of this region. **2**

***Sample solution:***

$$\text{Area} = \frac{1}{4} \times \pi \times 2^2 + \frac{1}{2} \times 2 \times 2$$
$$= (\pi + 2)\ \text{u}^2$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution (note that a student cannot gain full marks if their diagram in (i) did not result in a composite area) |
| **1** | Finds a correct expression for the area of the quadrant or the area of the triangle |

b) Solve $z^2 - (2+i)z + (3+i) = 0$. **2**

***Sample solution:***

$$z = \frac{2 + i \pm \sqrt{(2+i)^2 - 4(3+i)}}{2}$$

$$= \frac{2 + i \pm \sqrt{4 + 4i - 1 - 12 - 4i}}{2}$$

$$= \frac{2 + i \pm \sqrt{-9}}{2}$$

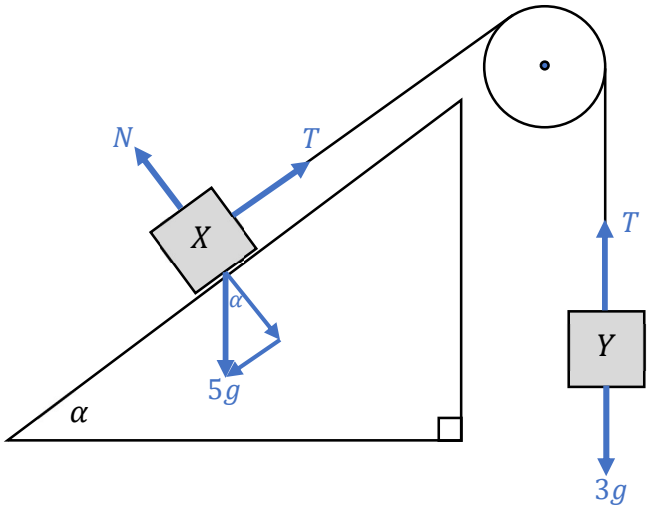
$$= \frac{2 + i \pm 3i}{2}$$

$$= \frac{2 + 4i}{2}, \frac{2 - 2i}{2}$$

$$= 1 + 2i, 1 - i$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution |
| **1** | Correctly substitutes into the quadratic formula and expands the discriminant, or equivalent merit |

c) An object, $X$, of mass 5 kg lies on a smooth plane inclined at an angle $\alpha$ to the horizontal, where $\sin\alpha = \frac{3}{5}$. Object $X$ is connected by a light inextensible string, which passes over a smooth pulley at the top of the plane, to a second object, $Y$, of mass 3 kg that hangs freely. The system is released from rest.




(i) Show that both objects are in equilibrium. **2**

***Sample solution:***

Consider the forces in the direction of motion for each object. Since object $X$ isn't falling through the plane, the forces on $X$ perpendicular to the plane will cancel each other out.

Object $X$:

$$F = 5g\sin\alpha - T$$
$$5a = 5g \times \frac{3}{5} - T$$
$$5a = 3g - T$$

Object $Y$:

$$F = 3g - T$$
$$3a = 3g - T$$

Since acceleration for each object is equal:

$$5a = 3a$$
$$a = 0$$

Since the acceleration on each object is zero, the system is in equilibrium.

| Marks | Criteria |
|---|---|
| **2** | Provides a correct and thorough solution |
| **1** | Decomposes the forces on object $X$ into components parallel and perpendicular to the plane OR finds an expression for the resultant force on object $Y$ OR completes solution but leaves out both tension forces, or equivalent merit |

(ii) Find the tension in the string. Use $g = 9.8$ ms$^{-2}$. **1**

***Sample solution:***

$$3g - T = 0$$

$$T = 3g$$
$$= 3 \times 9.8$$
$$= 29.4 \text{ N}$$

| Marks | Criteria |
|---|---|
| **1** | Provides correct answer |

d) The points $A$, $B$ and $C$ in the complex plane represent the complex numbers $z_1$, $z_2$ and $z_3$ respectively. **3**

If $\frac{z_2-z_1}{z_3-z_1} = i$, show that triangle $ABC$ is a right-angled isosceles triangle.

***Sample solution:***

$z_2 - z_1 = \overrightarrow{AB}$

$z_3 - z_1 = \overrightarrow{AC}$

$$\left|\frac{z_2 - z_1}{z_3 - z_1}\right| = |i|$$

$$\frac{|z_2 - z_1|}{|z_3 - z_1|} = 1$$

$$|z_2 - z_1| = |z_3 - z_1|$$

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right|$$

Hence $\Delta ABC$ is isosceles

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg(i)$$

$$\arg(z_2 - z_1) - \arg(z_3 - z_1) = \frac{\pi}{2}$$

$$\arg(\overrightarrow{AB}) - \arg(\overrightarrow{AC}) = \frac{\pi}{2}$$

$$\angle BAC = \frac{\pi}{2}$$

Hence $\Delta ABC$ is right-angled at $A$

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Correctly shows that $\Delta ABC$ is isosceles OR that $\Delta ABC$ is right-angled at $A$ |
| **1** | Correctly expresses $z_2 - z_1$ and $z_3 - z_1$ in terms of $A$, $B$ and $C$ |

e) Prove by mathematical induction that, for all non-negative integers $n$, **3**

$$(x+a)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} a^r$$

***Sample solution:***

* Prove true for $n = 0$

$LHS = (x+a)^0$

$= 1$

$RHS = \binom{0}{0} x^0 a^0$

$= 1$

$\therefore$ true for $n = 0$

* Assume true for $n = k$

i.e. assume:

$$(x+a)^k = \binom{k}{0} x^k + \binom{k}{1} x^{k-1} a + \binom{k}{2} x^{k-2} a^2 + \cdots + \binom{k}{k} a^k$$

* Prove true for $n = k+1$, if true for $n = k$

i.e. prove:

$$(x+a)^{k+1} = \binom{k+1}{0} x^{k+1} + \binom{k+1}{1} x^k a + \binom{k+1}{2} x^{k-1} a^2 + \cdots + \binom{k+1}{k+1} a^{k+1}$$

$LHS = (x+a)(x+a)^k$

$= (x+a)\left(\binom{k}{0} x^k + \binom{k}{1} x^{k-1} a + \binom{k}{2} x^{k-2} a^2 + \cdots + \binom{k}{k} a^k\right)$ by assumption

$= \left(\binom{k}{0} x^{k+1} + \binom{k}{1} x^k a + \cdots + \binom{k}{k} x a^k\right) + \left(\binom{k}{0} x^k a + \binom{k}{1} x^{k-1} a^2 + \cdots + \binom{k}{k} a^{k+1}\right)$ **

$= \binom{k}{0} x^{k+1} + \left(\binom{k}{0} + \binom{k}{1}\right) x^k a + \left(\binom{k}{1} + \binom{k}{2}\right) x^{k-1} a^2 + \cdots + \left(\binom{k}{k-1} + \binom{k}{k}\right) x a^k$

$+ \binom{k}{k} a^{k+1}$

But:

$$\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}, \qquad \binom{n}{0}=\binom{n+1}{0}=1, \qquad \binom{n}{n}=\binom{n+1}{n+1}=1$$

Hence:

$$LHS=\binom{k+1}{0}x^{k+1}+\binom{k+1}{1}x^{k}a+\binom{k+2}{2}x^{k-1}a^{2}+\cdots+\binom{k+1}{k}xa^{k}+\binom{k+1}{k+1}a^{k+1}$$

Hence true for $n=k+1$ if true for $n=k$.

Since true for $n=0$, $\therefore$ true for all $n\geq 0$ by mathematical induction.

| Marks | Criteria |
|---|---|
| **3** | Provides correct proof |
| **2** | Establishes the inductive step, or equivalent merit |
| **1** | Establishes the base case OR progresses in inductive step up to **, or equivalent merit |

**Question 15** (15 marks)

a) Let $z$ be a root of the equation $z^5 = -1$.

(i) Show that the roots of this equation are given by $z_k = e^{i\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)}$, for $k = 0, 1, 2, 3, 4$. **2**

***Sample solution:***

$$z^5 = -1$$

$$r^5 \operatorname{cis} 5\theta = \operatorname{cis} \pi$$

$$r^5 = 1 \qquad 5\theta = \pi + 2k\pi, \qquad k = 0, 1, 2, 3, 4$$

$$r = 1 \qquad \theta = \frac{\pi + 2k\pi}{5}$$

$$z_k = \operatorname{cis}\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)$$

$$z_k = e^{i\left(\frac{\pi}{5}+\frac{2k}{5}\right)}$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution |
| **1** | Correctly shows $r = 1$ OR correctly shows $\theta = \frac{\pi+2k\pi}{5}$ |

(ii) Hence, prove that $\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{3\pi}{5}\right) = -\frac{1}{4}$. **3**

***Sample solution:***

$$z_0 = e^{\frac{i\pi}{5}}$$

$$z_1 = e^{\frac{3i\pi}{5}}$$

$$z_2 = e^{i\pi} = -1$$

$$z_3 = e^{\frac{7i\pi}{5}} = e^{-\frac{3i\pi}{5}}$$

$$z_4 = e^{\frac{9i\pi}{5}} = e^{-\frac{i\pi}{5}}$$

Consider one factorisation:

$$z^5 + 1 = 0$$

$$(z+1)\left(z - e^{\frac{i\pi}{5}}\right)\left(z - e^{-\frac{i\pi}{5}}\right)\left(z - e^{\frac{3i\pi}{5}}\right)\left(z - e^{-\frac{3i\pi}{5}}\right) = 0$$

$$(z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right) = 0$$

Consider another factorisation:

$$z^5 + 1 = 0$$

$$(z+1)(z^4 - z^3 + z^2 - z + 1) = 0$$

Hence:

$$z^4 - z^3 + z^2 - z + 1 = \left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$$

Consider the term in $z^2$ on each side:

$$z^2 = z^2 + 4z^2\cos\frac{\pi}{5}\cos\frac{3\pi}{5} + z^2$$

$$1 = 2 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$$

$$-1 = 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$$

$$-\frac{1}{4} = \cos\frac{\pi}{5}\cos\frac{3\pi}{5}$$

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Provides two different factorisations of $z^5 + 1 = 0$ and equates them, or equivalent merit |
| **1** | Provides one correct factorisation of $z^5 + 1 = 0$, or equivalent merit |

b) The polynomial $P(z) = z^4 - 4z^3 + 14z^2 - 36z + 45$ has a purely imaginary root. Find all the roots of $P(z) = 0$. **3**

***Sample solution:***

Complex roots appear in conjugate pairs, so let the roots be $a + bi$, $a - bi$, $ki$, $-ki$

Sum of roots:

$4 = a + bi + a - bi + ki - ki$

$4 = 2a$

$a = 2$

Roots 3 at a time:

$36 = (2 + bi)(2 - bi)(ki) + (2 + bi)(2 - bi)(-ki) + (2 + bi)(ki)(-ki) + (2 - bi)(ki)(-ki)$

$36 = 2k^2 + k^2bi + 2k^2 - k^2bi$

$36 = 4k^2$

$k^2 = 9$

$k = \pm 3$

Product of roots:

$45 = (2 + bi)(2 - bi)(3i)(-3i)$

$45 = 9(4 + b^2)$

$5 = 4 + b^2$

$b^2 = 1$

$b = \pm 1$

Hence the roots are $3i$, $-3i$, $2 + i$, $2 - i$

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Finds two of the three required coefficients, or finds the imaginary roots and develops the equation $z^2 - 4z + 5$ or equivalent merit |
| **1** | Defines the four roots and forms one correct equation involving sums/products of roots, or finds the imaginary root and its conjugate or equivalent merit |

c) Let $a$, $b$, $c$ be the side lengths of a triangle. **3**

Prove that the following inequality must hold:

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\geq 3$$

***Sample solution:***

By triangle inequality:

$a+b>c \Rightarrow a+b-c>0$

$b+c>a \Rightarrow b+c-a>0$

$a+c>b \Rightarrow a+c-b>0$

Let:

$x=b+c-a>0$

$y=c+a-b>0$

$z=a+b-c>0$

Then:

$x+y=2c \Rightarrow c=\frac{1}{2}(x+y)$

$x+z=2b \Rightarrow b=\frac{1}{2}(x+z)$

$y+z=2a \Rightarrow a=\frac{1}{2}(y+z)$

RTP:

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\geq 3$$

$$LHS=\frac{\frac{1}{2}(y+z)}{x}+\frac{\frac{1}{2}(x+z)}{y}+\frac{\frac{1}{2}(x+y)}{z}$$

$$=\frac{1}{2}\left(\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{x}{z}+\frac{z}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)\right)$$

By AM/GM inequality:

$$\frac{x}{y}+\frac{y}{x} \geq 2\sqrt{\frac{x}{y}\times\frac{y}{x}}$$

$$\frac{x}{y}+\frac{y}{x} \geq 2$$

Hence:

$$LHS = \frac{1}{2}\left(\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{x}{z}+\frac{z}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)\right)$$

$$\geq \frac{1}{2}(2+2+2)$$

$$\geq 3 \quad \text{as required}$$

| Marks | Criteria |
|---|---|
| **3** | Provides correct proof |
| **2** | Makes significant progress towards correct solution |
| **1** | Correctly applies triangle inequality or AM/GM inequality at an appropriate point in proof, or equivalent merit |

d) (i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

***Sample solution:***

Consider $\int_0^a f(u)\,du$

Let $u = a - x$

$\frac{du}{dx} = -1$

$du = -dx$

$u = 0 \Rightarrow x = a$

$u = a \Rightarrow x = 0$

$$\int_0^a f(x)\,dx = \int_0^a f(u)\,du$$
$$= \int_a^0 f(a-x)(-dx)$$
$$= \int_0^a f(a-x)\,dx$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution |
| **1** | Correctly finds new bounds of integration and finds $\frac{du}{dx}$ in order to implement integration by substitution, or equivalent merit |

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan(x)}\,dx$. **2**

***Sample solution:***

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan(x)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot(x)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan(x)}{\tan(x)+1}\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan(x)}\,dx + \int_0^{\frac{\pi}{2}} \frac{\tan(x)}{1+\tan(x)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\tan(x)}{1+\tan(x)}\,dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,dx$$

$$= \frac{1}{2}[x]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-0\right)$$

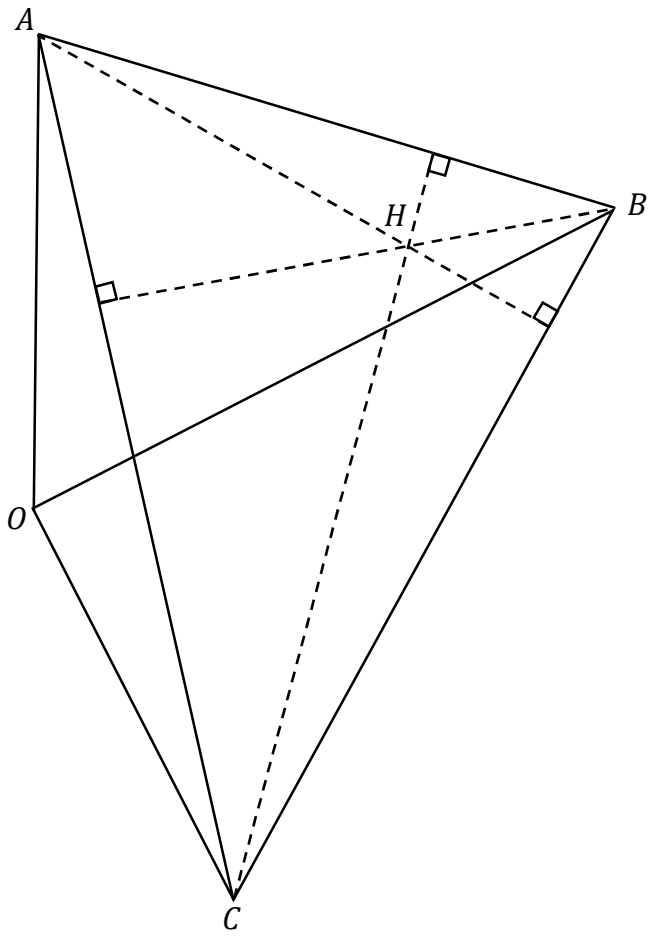
$$= \frac{\pi}{4}$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct solution |
| **1** | Correctly uses result from part (i) and uses complementary angle results and reciprocal ratios to form a different expression for the integral in terms of $\tan x$, or equivalent merit |

**Question 16** (15 marks)

a) In the tetrahedron $OABC$, the edges $OA$, $OB$ and $OC$ are mutually perpendicular. **4**
Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$.

The altitude of a triangle is the perpendicular line segment drawn from any vertex to the opposite side. Every triangle has three altitudes, which intersect at a single point called the orthocentre. Let $H$ be the orthocentre of triangle $ABC$.




Similarly, the altitude of a pyramid is the perpendicular line segment drawn from any vertex to the opposite face. Consider the altitude of pyramid $OABC$, drawn from $O$ and perpendicular to triangle $ABC$. Use vector methods to show that this altitude passes through $H$.

***Sample solution 1:***

Let $\overrightarrow{OH} = \underset{\sim}{h}$

RTP: $\overrightarrow{OH}$ is perpendicular to $\Delta ABC$

i.e. need to show that $OH \perp AB$, $OH \perp AC$, $OH \perp BC$

Since $OA$, $OB$ and $OC$ are mutually perpendicular, $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

Since $H$ lies on the altitudes of $\Delta ABC$, $\overrightarrow{AH} \cdot \overrightarrow{BC} = \overrightarrow{BH} \cdot \overrightarrow{AC} = \overrightarrow{CH} \cdot \overrightarrow{AB} = 0$

$$\overrightarrow{AH} \cdot \overrightarrow{BC} = \left(\underset{\sim}{h} - \underset{\sim}{a}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{b}\right)$$
$$0 = \underset{\sim}{h} \cdot \underset{\sim}{c} - \underset{\sim}{h} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{a} \cdot \underset{\sim}{b}$$
$$0 = \underset{\sim}{h} \cdot \underset{\sim}{c} - \underset{\sim}{h} \cdot \underset{\sim}{b}$$
$$\underset{\sim}{h} \cdot \underset{\sim}{c} = \underset{\sim}{h} \cdot \underset{\sim}{b}$$

$$\overrightarrow{BH} \cdot \overrightarrow{AC} = \left(\underset{\sim}{h} - \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right)$$
$$0 = \underset{\sim}{h} \cdot \underset{\sim}{c} - \underset{\sim}{h} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{a}$$
$$0 = \underset{\sim}{h} \cdot \underset{\sim}{c} - \underset{\sim}{h} \cdot \underset{\sim}{a}$$
$$\underset{\sim}{h} \cdot \underset{\sim}{c} = \underset{\sim}{h} \cdot \underset{\sim}{a}$$

$$\overrightarrow{CH} \cdot \overrightarrow{AB} = \left(\underset{\sim}{h} - \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{a} - \underset{\sim}{b}\right)$$
$$0 = \underset{\sim}{h} \cdot \underset{\sim}{a} - \underset{\sim}{h} \cdot \underset{\sim}{b} - \underset{\sim}{c} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{b}$$
$$0 = \underset{\sim}{h} \cdot \underset{\sim}{a} - \underset{\sim}{h} \cdot \underset{\sim}{b}$$
$$\underset{\sim}{h} \cdot \underset{\sim}{a} = \underset{\sim}{h} \cdot \underset{\sim}{b}$$

Hence $\underset{\sim}{h} \cdot \underset{\sim}{a} = \underset{\sim}{h} \cdot \underset{\sim}{b} = \underset{\sim}{h} \cdot \underset{\sim}{c} = 0$

$$\overrightarrow{OH} \cdot \overrightarrow{AB} = \underset{\sim}{h} \cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right)$$
$$= \underset{\sim}{h} \cdot \underset{\sim}{b} - \underset{\sim}{h} \cdot \underset{\sim}{a}$$
$$= 0$$

$$\overrightarrow{OH} \cdot \overrightarrow{BC} = \underset{\sim}{h} \cdot \left(\underset{\sim}{c} - \underset{\sim}{b}\right)$$
$$= \underset{\sim}{h} \cdot \underset{\sim}{c} - \underset{\sim}{h} \cdot \underset{\sim}{b}$$
$$= 0$$

$$\overrightarrow{OH} \cdot \overrightarrow{AC} = \underset{\sim}{h} \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right)$$
$$= \underset{\sim}{h} \cdot \underset{\sim}{c} - \underset{\sim}{h} \cdot \underset{\sim}{a}$$
$$= 0$$

Hence $\overrightarrow{OH} \perp \Delta ABC$, so the altitude from $O$ passes through $H$.

***Sample solution 2:***

Let $P$ be the foot of the perpendicular from $O$ to $\Delta ABC$. Let $\overrightarrow{OP} = \underset{\sim}{p}$.

RTP: Point $P$ is point $H$

i.e. need to show that $AP \perp BC$, $BP \perp AC$, $CP \perp AB$

Since $OA$, $OB$ and $OC$ are mutually perpendicular, $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

Since $\overrightarrow{OP}$ is perpendicular to $\Delta ABC$, $\underset{\sim}{p} \cdot \overrightarrow{AB} = \underset{\sim}{p} \cdot \overrightarrow{AC} = \underset{\sim}{p} \cdot \overrightarrow{BC} = 0$

| | | |
|---|---|---|
| $\underset{\sim}{p} \cdot \overrightarrow{AB} = 0$ | $\underset{\sim}{p} \cdot \overrightarrow{AC} = 0$ | $\underset{\sim}{p} \cdot \overrightarrow{BC} = 0$ |
| $\underset{\sim}{p} \cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right) = 0$ | $\underset{\sim}{p} \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right) = 0$ | $\underset{\sim}{p} \cdot \left(\underset{\sim}{c} - \underset{\sim}{b}\right) = 0$ |
| $\underset{\sim}{p} \cdot \underset{\sim}{b} - \underset{\sim}{p} \cdot \underset{\sim}{a} = 0$ | $\underset{\sim}{p} \cdot \underset{\sim}{c} - \underset{\sim}{p} \cdot \underset{\sim}{a} = 0$ | $\underset{\sim}{p} \cdot \underset{\sim}{c} - \underset{\sim}{p} \cdot \underset{\sim}{b} = 0$ |

Hence $\underset{\sim}{p} \cdot \underset{\sim}{a} = \underset{\sim}{p} \cdot \underset{\sim}{b} = \underset{\sim}{p} \cdot \underset{\sim}{c}$

$$\begin{aligned}\overrightarrow{AP} \cdot \overrightarrow{BC} &= \left(\underset{\sim}{p} - \underset{\sim}{a}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{b}\right) \\ &= \underset{\sim}{p} \cdot \underset{\sim}{c} - \underset{\sim}{p} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{a} \cdot \underset{\sim}{b} \\ &= 0\end{aligned}$$

$$\begin{aligned}\overrightarrow{BP} \cdot \overrightarrow{AC} &= \left(\underset{\sim}{p} - \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right) \\ &= \underset{\sim}{p} \cdot \underset{\sim}{c} - \underset{\sim}{p} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{a} \\ &= 0\end{aligned}$$

$$\begin{aligned}\overrightarrow{CP} \cdot \overrightarrow{AB} &= \left(\underset{\sim}{p} - \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{b} - \underset{\sim}{a}\right) \\ &= \underset{\sim}{p} \cdot \underset{\sim}{b} - \underset{\sim}{p} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{a} \\ &= 0\end{aligned}$$

Since $P$ lies on all three altitudes of $\Delta ABC$, point $P$ must be point $H$.

| Marks | Criteria |
|---|---|
| **4** | Provides correct solution |
| **3** | Shows that $P$ lies on at least one of the three altitudes of $\Delta ABC$, or equivalent merit |
| **2** | Shows that $\underset{\sim}{h} \cdot \underset{\sim}{a} = \underset{\sim}{h} \cdot \underset{\sim}{b} = \underset{\sim}{h} \cdot \underset{\sim}{c}$ OR shows that $\underset{\sim}{p} \cdot \underset{\sim}{a} = \underset{\sim}{p} \cdot \underset{\sim}{b} = \underset{\sim}{p} \cdot \underset{\sim}{c}$ (method 2), or equivalent merit |
| **1** | Makes a clear statement about what property is required to be proved (i.e. relevant dot product equal to zero) OR states that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$ OR states that $\overrightarrow{AH} \cdot \overrightarrow{BC} = \overrightarrow{BH} \cdot \overrightarrow{AC} = \overrightarrow{CH} \cdot \overrightarrow{AB} = 0$, or equivalent merit |

b) Find $\int \frac{1}{(x^2+1)^2}\,dx$. **3**

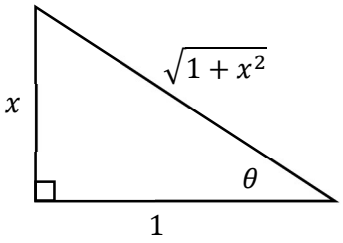
***Sample solution:***

Let:

$$x = \tan\theta$$

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$dx = \sec^2\theta\, d\theta$$




$$\begin{aligned}
\int \frac{1}{(x^2+1)^2}\,dx &= \int \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta\, d\theta \\
&= \int \frac{1}{\sec^4\theta}\sec^2\theta\, d\theta \\
&= \int \cos^2\theta\, d\theta \\
&= \frac{1}{2}\int 1+\cos 2\theta\, d\theta \\
&= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c \\
&= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + c \\
&= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\times\frac{x}{\sqrt{1+x^2}}\times\frac{1}{\sqrt{1+x^2}} + c \\
&= \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)} + c
\end{aligned}$$

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Finds a primitive in terms of $\theta$ |
| **1** | Uses substitution to express integral in terms of $\theta$ |

c) Prove that, for any real numbers $a$ and $b$, $a^2 + b^2 - ab - a - b + 1 \geq 0$. **2**

***Sample solution:***

$$\begin{aligned} a^2 + b^2 - ab - a - b + 1 &= \frac{1}{2}(2a^2 + 2b^2 - 2ab - 2a - 2b + 2) \\ &= \frac{1}{2}\big((a^2 - 2ab + b^2) + (a^2 - 2a + 1) + (b^2 - 2b + 1)\big) \\ &= \frac{1}{2}((a-b)^2 + (a-1)^2 + (b-1)^2) \\ &\geq 0 \quad \text{since } (a-b)^2 \geq 0, (a-1)^2 \geq 0, (b-1)^2 \geq 0 \end{aligned}$$

| Marks | Criteria |
|---|---|
| **2** | Provides correct proof |
| **1** | Expresses LHS as the sum of perfect squares |

d) Consider the integral $I_n = \int_0^1 (1-x^2)^n dx$ for integers $n \geq 0$.

(i) Show that for $n \geq 1$, the recurrence relation $I_n = \frac{2n}{2n+1} I_{n-1}$ holds. **3**

***Sample solution:***

Let: $u = (1-x^2)^n$ $\quad v' = 1$

$u' = n(1-x^2)^{n-1}(-2x)$ $\quad v = x$

$$I_n = [x(1-x^2)^n]_0^1 - \int_0^1 -2nx^2(1-x^2)^{n-1} dx$$

$$= 0 - 2n \int_0^1 -x^2(1-x^2)^{n-1} dx$$

$$= -2n \int_0^1 (1-x^2-1)(1-x^2)^{n-1} dx$$

$$= -2n \int_0^1 (1-x^2)^n - (1-x^2)^{n-1} dx$$

$$I_n = -2nI_n + 2nI_{n-1}$$

$$(1+2n)I_n = 2nI_{n-1}$$

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Correctly uses integration by parts to obtain a single integral term, or equivalent merit |
| **1** | Attempts to use integration by parts, or equivalent merit |

(ii) Hence, show that $I_n = \frac{4^n(n!)^2}{(2n+1)!}$. **3**

***Sample solution:***

$$I_0 = \int_0^1 1\,dx \qquad I_1 = \frac{2}{3}I_0 \qquad I_2 = \frac{4}{5}I_1 \qquad I_3 = \frac{6}{7}I_2$$
$$= [x]_0^1 \qquad = \frac{2}{3}\times 1 \qquad = \frac{4}{5}\times\frac{2}{3} \qquad = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}$$
$$= 1 \qquad = \frac{2}{3}$$

$$I_n = \underbrace{\frac{2n}{2n+1}\times\frac{2(n-1)}{2(n-1)+1}\times\frac{2(n-2)}{2(n-2)+1}\times\cdots\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}}_{n \text{ terms}}$$
$$= \frac{2^n(n(n-1)(n-2)\cdots\times 3\times 2\times 1}{(2n+1)(2n-1)(2n-3)\times 7\times 5\times 3}$$
$$= \frac{2^n(n!)}{(2n+1)!}\times\underbrace{(2n(2n-2)(2n-4)\cdots\times 6\times 4\times 2)}_{n \text{ terms}}$$
$$= \frac{2^n(n!)}{(2n+1)!}\times\left(2^n(n!)\right)$$
$$= \frac{2^{2n}(n!)^2}{(2n+1)!}$$
$$= \frac{4^n(n!)^2}{(2n+1)!}$$

| Marks | Criteria |
|---|---|
| **3** | Provides correct solution |
| **2** | Develops an expanded expression for $I_n$ |
| **1** | Finds values of $I_0$, $I_1$ and $I_2$ |

**End of paper**