HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2026

STUDENT NUMBER:_____ **STUDENT NAME:**_____ **TEACHER:**_____

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided separately
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination room.

Total marks – 100

Section I Pages 2 – 5

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 6 – 11

90 marks

Attempt Questions 11 – 16

Start each question in a new writing booklet

Write your student number on every writing booklet

Question	*1-10*	*11*	*12*	*13*	*14*	*15*	*16*	*Total*
Total	*/10*	*/16*	*/15*	*/15*	*/15*	*/14*	*/15*	*/100*

This assessment task constitutes 30% of the Higher School Certificate Course School Assessment

Outcomes assessed: MEX12-1, MEX12-2, MEX12-3, MEX12-4, MEX12-5, MEX12-7, MEX12-8.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section.

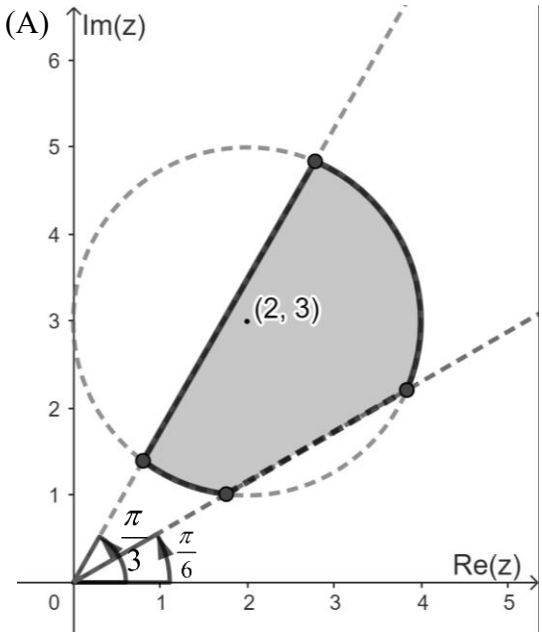
Use the Objective Response answer sheet for Questions 1 – 10

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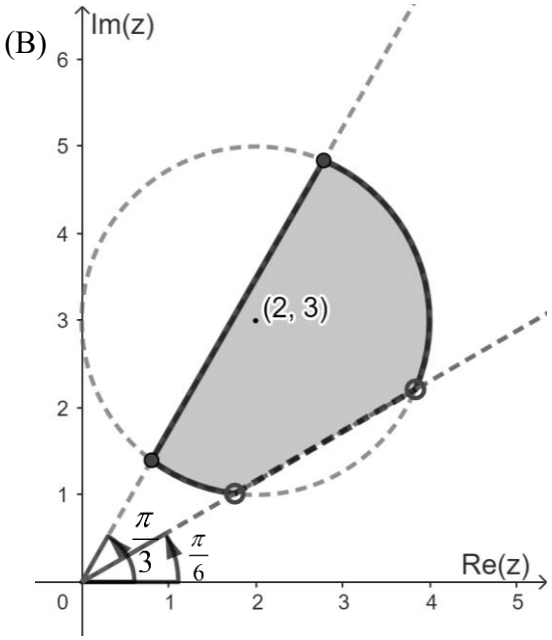
1. Let R be the region in the complex plane defined by $|z-2-3i| \le 2$ and $\frac{\pi}{6} < \text{Arg}(z) \le \frac{\pi}{3}$.

Which diagram best represents the shaded region?

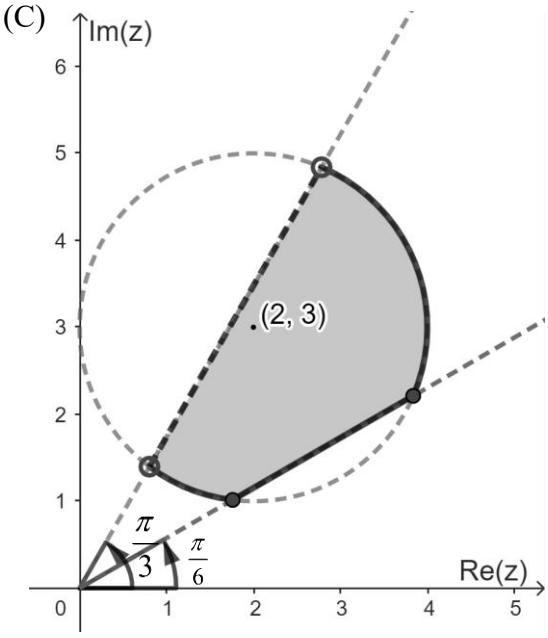
(A)



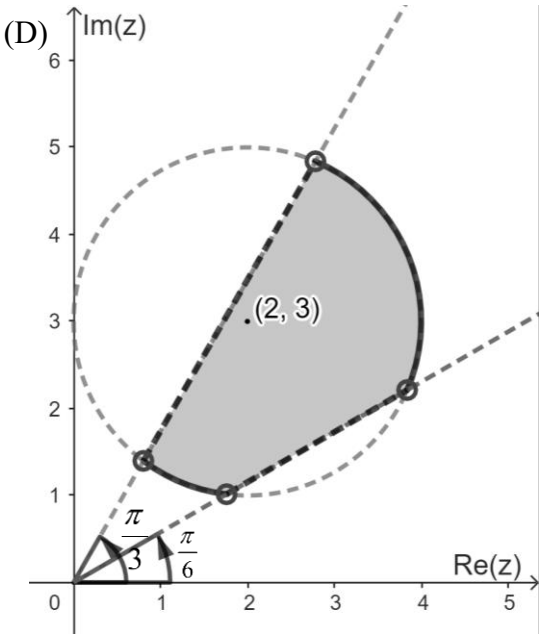
(B)



(C)



(D)



2.

Consider the proposition.

'If each term in a geometric series is smaller than the previous term, then $0 < r < 1$'.

Given the first term and common ratio in each of the following geometric series, which one disproves the proposition?

(A) $T_1 = 2, \quad r = 2$

(B) $T_1 = 8, \quad r = 0.5$

(C) $T_1 = -2, \quad r = 2$

(D) $T_1 = 8, \quad r = -0.5$

3. Consider the following statement.

'If I have a scientific calculator in my bag and I study Extension 2 Mathematics, then I am a problem solver.'

Which of the following is the contrapositive of this statement?

(A) If I am a problem solver, then I have a scientific calculator in my bag and I study Extension 2 Mathematics.

(B) If I am not a problem solver, then I do not have a scientific calculator in my bag or I do not study Extension 2 Mathematics.

(C) If I am a problem solver, then I do not have a scientific calculator in my bag or I do not study Extension 2 Mathematics.

(D) If I am not a problem solver, then I do not have a scientific calculator in my bag but I study Extension 2 Mathematics.

4. Which of the following expressions would have an anti-derivative which includes $\sin^{-1}\big(f(x)\big)$?

(A) $\dfrac{2x}{\sqrt{x^4 - 2x^2 + 1}}$

(B) $\dfrac{2x}{\sqrt{-x^4 - 4x^2 + 5}}$

(C) $\dfrac{2x}{x^4 - 2x^2 + 1}$

(D) $\dfrac{2x}{-x^4 - 4x^2 + 5}$

5. If $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{AB} = \underset{\sim}{b}$ and $\overrightarrow{BO} = \underset{\sim}{c}$, consider the statement:

"If $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 0$ then the points O, A and B form a triangle."

The converse of the statement is:

(A) If the points O, A and B do not form a triangle then $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} \neq 0$.

(B) If $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} \neq 0$ then O, A and B do not form a triangle.

(C) If the points O, A and B form a triangle then $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} \neq 0$.

(D) If the points O, A and B form a triangle then $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 0$.

6. A particle of mass 100 kilograms is moving in three-dimensional space, and is subject to a constant force.

Its displacement in metres from a fixed point after t seconds is

$$\underset{\sim}{r} = \begin{pmatrix} 4t^2 + 4t + 1 \\ 3t - 4 \\ 5 - 3t^2 \end{pmatrix}.$$

What is the magnitude of the force acting on the particle?

(A) 200 N

(B) 648 N

(C) 1000 N

(D) 1374 N

7 A particle moving in a straight line obeys $v^2 = 9\left(5 + 4x - x^2\right)$ where x is the displacement from the origin in metres and v is its velocity in ms^{-1}. Where is the centre of the motion?

(A) $x = -1$

(B) $x = 0$

(C) $x = 2$

(D) $x = 5$

8. A stone is thrown horizontally with speed $V = \sqrt{2gH}$ ms^{-1} from the top point O of a vertical tower of height H metres, standing on horizontal ground. It moves in a vertical plane under gravity where the acceleration due to gravity is g ms^{-2}. At time t seconds the stone has position vector $Vt\underset{\sim}{i} - \frac{1}{2}gt^2\underset{\sim}{j}$ relative to O. The stone hits the ground at time T seconds at a distance X metres from the foot of the tower. What is the value of X ?

(A) $\frac{1}{2}H$

(B) H

(C) $2H$

(D) $4H$

9.

Let $g(x)$ be a function with first derivative given by $g'(x) = \int_0^x e^{-t^2}\,dt$.

Which of the following must be true on the interval $0 < x < 2$?

(A) $g(x)$ is increasing and the graph of $g(x)$ is concave up.

(B) $g(x)$ is increasing and the graph of $g(x)$ is concave down.

(C) $g(x)$ is decreasing and the graph of $g(x)$ is concave up.

(D) $g(x)$ is decreasing and the graph of $g(x)$ is concave down.

10.

The complex number $z = x + iy$ satisfies $|z| \leq y + 1$ and $|z| \leq x + 1$.

What is the maximum possible value of $|z|$?

(A) $1 - \sqrt{2}$

(B) $1 + \sqrt{2}$

(C) $2 - \sqrt{2}$

(D) $2 + \sqrt{2}$

END OF SECTION I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) **START A NEW WRITING BOOKLET**

(a) Express $\dfrac{4-3i}{3+2i}$ in the form $x+iy,$ where x and y are real numbers. **2**

(b) Evaluate $\int \cos^3 x \sin^2 x \ dx.$ **2**

(c) (i) Write the complex number $-1+\sqrt{3}i$ in exponential form. **2**

(ii) Hence, find the exact value of $\left(-1+\sqrt{3}i\right)^8$, giving your answer in the form $x+iy.$ **2**

(d) A triangle PQR is formed in three-dimensional space with vertices $P(3, -2, 4)$, $Q(-2, 1, 5)$ and $R(5, -1, 2)$. Find the size of $\angle QRP$, giving your answer to the nearest degree. **3**

(e) Let $\underset{\sim}{r}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ \lambda \in R.$ be the vector equation of the line L_1. A second line L_2 passes through the point $P(-2, 4)$ and is parallel to L_1. Find the equation of the line L_2 in the form $y = mx + c.$ **2**

(f) Using the substitution $t = \tan\dfrac{x}{2},$ find $\int \dfrac{dx}{1-\sin x}.$ **3**

End of Question 11

Question 12 (15 marks) **START A NEW WRITING BOOKLET**

(a) (i) For positive real numbers a and b, prove that $\frac{a+b}{2} \geq \sqrt{ab}$. **2**

(ii) Given $a \geq b > 0$, prove that $\frac{1}{a} \leq \frac{1}{b}$. **1**

(iii) Given positive real numbers p and q, consider $f(x) = \frac{x}{(x+p)(x+q)}$.

By using parts (i) and (ii), or otherwise, prove that for $x > 0$ **3**

$$f(x) \leq \frac{1}{(\sqrt{p}+\sqrt{q})^2}.$$

(b) (i) Find integers A, B and C such that:

$$\frac{x^2+3x+18}{(x^2+4)(x-5)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-5}$$ **3**

(ii) Hence find $\int \frac{x^2+3x+18}{(x^2+4)(x-5)}\,dx$ **2**

(c) A particle moving along the x axis. At time t seconds it has displacement x metres from the origin O, velocity v ms^{-1} and acceleration a ms^{-2} given by $a = -\frac{1}{50}(10-x)^3$. Initially the particle is at O and moving to the right of O with speed 10 ms^{-1}.

(i) Use integration to show that $v = \frac{1}{10}(10-x)^2$. **2**

(ii) Hence use integration to show that $x = \frac{10t}{1+t}$. **2**

End of Question 12

Question 13 (15 marks) **START A NEW WRITING BOOKLET**

(a) Prove by contradiction that $\log_4 7$ is irrational. **3**

(b) A sequence is defined recursively as $u_0 = 5,\ u_n = 2u_{n-1} + 1$ for $n \geq 1$. **4**

Use mathematical induction to prove that $u_n = 3\left(2^{n+1}\right) - 1$ for all positive integers n.

(c) (i) Solve the equation $z^5 = 1,$ giving the roots in modulus/argument form. **2**

(ii) Hence show that $z^4 + z^5 + z^2 + z + 1 = \left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$. **3**

(iii) Show that $4\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = 5$. **1**

(iv) Hence show that $x = \cos\frac{2\pi}{5}$ is a root of the equation $8x^3 - 8x^2 - 8x + 3 = 0$. **2**

End of Question 13

Question 14 (15 marks) **START A NEW WRITING BOOKLET**

(a) The point of intersection of the line through $A\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $B\begin{pmatrix}4\\5\\0\end{pmatrix}$ with the line through $C\begin{pmatrix}0\\9\\10\end{pmatrix}$ and $D\begin{pmatrix}1\\7\\7\end{pmatrix}$ is E.

(i) Prove that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$. **1**

(ii) Find the coordinates of E. **2**

(b) Given that $z = 1 - \sqrt{3}\,i$ find the two square roots of z in the from $a + ib$. **3**

(c) Consider the complex number ω, which is the non-real solution to $z^5 - 1 = 0$ with the smallest principal argument. **3**

Using the fact that $z = \omega^2 + \dfrac{1}{\omega^2}$ is a root of $z^2 + z - 1 = 0$, show that

$$\cos\frac{4\pi}{5} = -\frac{\sqrt{5}+1}{4}.$$

(d) Let points A and B on the complex plane represent the complex numbers $z_A = 1 + i$ and $z_B = -4$ respectively.

The isosceles triangle ABC is right-angled at C.

(i) Find the possible values of z_C, represented by the point C. **4**

(ii) The line between, but not including the points A and B, has equation **2**

$$\arg(z + x_1 + iy_1) - \arg(z + x_2 + iy_2) = \theta.$$

Find the values of x_1, y_1, x_2, y_2 and θ.

End of Question 14

Question 15 (14 marks) **START A NEW WRITING BOOKLET**

(a) A particle is undergoing simple harmonic motion on the x-axis about the origin. It is initially at its extreme positive position. The amplitude of the motion is 10 and the particle returns to its initial position every 7 seconds.

(i) Write an equation for the position of the particle at time t seconds. **1**

(ii) How long does the particle take to move from a rest position to the point halfway between the rest position and the equilibrium position? **1**

(b) A particle of mass m kg is set in motion with speed u ms^{-1} and moves in a straight line before coming to rest. After time t seconds the particle has displacement x metres from its starting point O, velocity v ms^{-1} and acceleration a ms^{-2}. The resultant force acting on the particle directly opposes its motion and has a magnitude $m(1 + v)$ Newtons so that $a = -(1+v)$.

(i) Find expressions for x in terms of v, v in terms of t and x in terms of t. **6**

(ii) Show that $x + v + t = u$. **1**

(iii) Find the distance travelled and the time taken by the particle in coming to rest. **2**

(c) Use integration by parts to evaluate $\int_0^1 \sin^{-1} x \, dx$. **3**

End of Question 15

Question 16 (15 marks) **START A NEW WRITING BOOKLET**

(a) If $a>1, b>1$ and $c>1$ such that $\frac{a}{b} \geq \frac{c}{a}$, show that $\frac{\ln a}{\ln b} \geq \frac{\ln c}{\ln a}$. **4**

(b) Let $I_n = \int_1^e (\ln x)^n \, dx$ for $n = 0, 1, 2, ...$

(i) Prove that $I_{n+1} = e - (n+1) I_n$ for $n = 0, 1, 2, ...$ and hence prove $I_2 = e - 2$. **3**

(ii) Use mathematical induction to prove that $I_n = (-1)^{n+1} n! + (-1)^n \, e \, n! \sum_{r=2}^{n} \frac{(-1)^r}{r!}$ **3**

for all positive integers $n \geq 2$.

(c) Given three vectors $\vec{a}, \vec{b},$ and $\vec{c}$, the vectors $\vec{u}, \vec{v}, \vec{w}$ are defined as

$$\vec{u} = (\vec{b} \cdot \vec{c})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}$$
$$\vec{v} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$
$$\vec{w} = (\vec{b} \cdot \vec{a})\vec{c} - (\vec{c} \cdot \vec{b})\vec{a}$$

The vectors $\vec{a}, \vec{b}, \vec{c}$ form a triangle.

(i) Prove $\vec{u}, \vec{v}, \vec{w}$ also form a triangle. **2**

(ii) Calculate $\vec{u} \cdot \vec{c}$. **1**

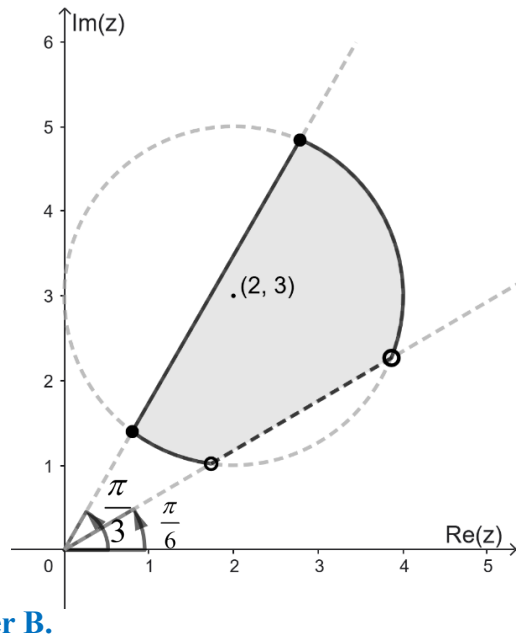
(iii) Prove that the two triangles formed by $\vec{a}, \vec{b}, \vec{c}$ and $\vec{u}, \vec{v}, \vec{w}$ are similar. **2**

END OF ASSESSMENT

SOLUTIONS TO QUESTIONS FOR EXT 2 TRIAL PAPER 2026

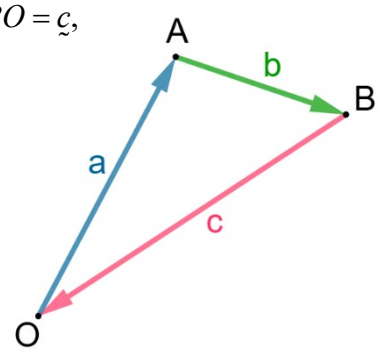
Section I Multiple Choice

Question	Answers
1	**B**
2	**C**
3	**B**
4	**B**
5	**D**
6	**C**
7	**C**
8	**C**
9	**A**
10	**D**

	Suggested Solution	Markers' Comments
1.	$\lvert z-2-3i\rvert \le 2$ and $\frac{\pi}{6} < \text{Arg}(z) \le \frac{\pi}{3}$ Circle centre (2, 3) Radius = 2 units including points on circumference. A region between the two rays including points on the ray $y=\sqrt{3}x$, but excluding points on $y=\frac{1}{\sqrt{3}}x$ Intersecting points of circle and rays are included along $y=\sqrt{3}x$ but excluded on the ray $y=\frac{1}{\sqrt{3}}x$.  **Answer B.**	92.5% students correct.
2.	The proposition is in the form $P \Rightarrow Q$, where: P: each term in a geometric series is smaller than the previous term Q: $0 < r < 1$.	57.5% students correct.

	Suggested Solution	Markers' Comments
2.	**Continues:** To disprove the proposition we need to find P and not Q occurring simultaneously, so each term in a geometric series is smaller than the previous term AND r is not between 0 and 1. Writing out the first three terms in each series: A: $2,\ 4,\ 8$, which is not in descending order. Eliminate A. B: $8,\ 4,\ 2$, which is in descending order but has $0 < r < 1$. Eliminate B. C: $-2,\ -4,\ -8$, which is in descending order and where r is not between 0 and 1. Correct. D: $8,\ -4,\ 2$, which is not in descending order. Eliminate D. **Answer C.** ***Alternative solution :*** The proposition is in the form $P \Rightarrow Q$ where: P: each term in a geometric series is smaller than the previous term $Q: 0 < r < 1$. $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$. Writing out the first three terms in each series: A: $2,\ 4,\ 8,\ \ (\neg P \wedge \neg Q)$. Eliminate A. B: $8,\ 4,\ 2, (P \wedge\ Q)$. Eliminate B. C: $-2,\ -4,\ -8, (P \wedge \neg Q)$. Correct. D: $8,\ -4,\ 2, (\neg P \wedge \neg Q)$. Eliminate D. **Answer C.**	
3.	Original statement: If I have a scientific calculator in my bag and I study Extension 2 Mathematics, then I am a problem solver. i.e. $(P \cap Q) \Rightarrow R$ P = { I have a scientific calculator in my bag } Q = { I study Extension 2 Mathematics} R = { I am a problem solver} Contraposition: $\neg R \Rightarrow \neg(P \cap Q)$ $\neg R \Rightarrow \neg P \cup \neg Q$ $\neg R \Rightarrow \neg P$ or $\neg Q$ If I am not a problem solve, then I do not have a scientific calculator in my bag **or** I do not study Extension 2 Mathematics. **(B)**	95% students correct.

	Suggested Solution	Markers' Comments
4.	The integrand has to be able to be written in the form $\frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}$. A: $\frac{2x}{\sqrt{x^4-2x^2+1}}=\frac{2x}{\sqrt{(x^2-1)^2}}$ $=\frac{\frac{d}{dx}(x^2-1)}{\lvert x^2-1\rvert}$ INCORRECT. FYI: The anti-derivative is $\frac{(x^2-1)\ln\lvert x^2-1\rvert}{\lvert x^2-1\rvert}+c$. B: $\frac{2x}{\sqrt{-x^4-4x^2+5}}$ $=\frac{\frac{d}{dx}(x^2+2)}{\sqrt{3^2-(x^2+2)^2}}$ CORRECT. FYI: The anti-derivative is $\sin^{-1}\left(\frac{x^2+2}{3}\right)+c$. C: $\frac{2x}{x^4-2x^2+1}$ $=\frac{\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$ INCORRECT. FYI: The anti-derivative is $-\frac{1}{x^2-1}+c$. D: $\frac{2x}{-x^4-4x^2+5}$ $=\frac{\frac{d}{dx}(x^2+2)}{3^2-(x^2+2)^2}$ INCORRECT . FYI: The anti-derivative is $\frac{\ln(x^2+5)-\ln\lvert x^2-1\rvert}{6}+c$. **Answer B.** ***Alternative solution I:*** The integrand has to be able to be written in the form $\frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}$. The denominator has to be the square root of the difference of a constant and the square of a function of x, and the numerator has to be the derivative of the function of x.	

	Suggested Solution	Markers' Comments
4.	**Continues** A: $\dfrac{2x}{\sqrt{x^4-2x^2+1}}=\dfrac{2x}{\sqrt{(x^2-1)^2}}$. Eliminate A since the denominator is not the square root of the difference of a constant and the square of a function of x. B: $\dfrac{2x}{\sqrt{-x^4-4x^2+5}}=\dfrac{\dfrac{d}{dx(x^2+2)}}{\sqrt{3^2-(x^2+2)^2}}$ CORRECT, since the denominator is the square root of the difference of a constant and the square of a function of x, and the numerator is the derivative of the function of x. C: $\dfrac{2x}{x^4-2x^2+1}=\dfrac{\dfrac{d}{dx}(x^2-1)}{(x^2-1)^2}$ Eliminate C since the denominator is not the square root of the difference of a constant and the square of a function of x. D: $\dfrac{2x}{-x^4-4x^2+5}=\dfrac{\dfrac{d}{dx(x^2+2)}}{3^2-(x^2+2)^2}$. Eliminate D since the denominator is not the square root of the difference of a constant and the square of a function of x. **Answer B.**	
5.	If $\overrightarrow{OA}=\underset{\sim}{a}$, $\overrightarrow{AB}=\underset{\sim}{b}$ and $\overrightarrow{BO}=\underset{\sim}{c}$,  Original statement: "If $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}=0$ then the points O, A and B form a triangle." i.e. P = $\{\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}=0\}$ Q = {the points O, A and B form a triangle} Converse statement: If Q, then P If the points O, A and B form a triangle, then $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}=0$ **Answer D.**	92.5% students correct.

	Suggested Solution	Markers' Comments
6.	$\underset{\sim}{r} = \begin{pmatrix} 4t^2+4t+1 \\ 3t-4 \\ 5-3t^2 \end{pmatrix}$ $\underset{\sim}{v} = \begin{pmatrix} 8t+4 \\ 3 \\ -6t \end{pmatrix}$ $\underset{\sim}{a} = \begin{pmatrix} 8 \\ 0 \\ -6 \end{pmatrix}$ $\left\lvert \underset{\sim}{a} \right\rvert = \sqrt{8^2+0^2+6^2}$ $= 10 \text{ ms}^{-2}$ $F = m \times a$ $= 100 \times 10$ $= 1000 \text{ N}$ **Answer C.**	92.5% students correct.
7.	$v^2 = 9\left(5+4x-x^2\right)$ $v^2 = -9\left(x^2-4x-5\right)$ $v^2 = -9\left(x^2-4x+4-5-4\right)$ $v^2 = -9\left((x-2)^2-9\right)$ $v^2 = 9\left(9-(x-2)^2\right)$ Hence centre is at $x=2$ **Answer C.**	95% students correct.
8.	$\frac{1}{2}gT^2 = H \text{ and } X = VT = \sqrt{2gH}\,T$ $\Rightarrow \frac{2H}{g} = \frac{X^2}{2gH}$ $\therefore X = 2H$ **Answer C.**	Poorly done (40% students correct.)
9.	$g'(x) = \int_0^x e^{-t^2}\,dt.$ As $e^{-t^2} > 0,\ 0 \le t < 2,$ then $g'(x) = \int_0^x e^{-t^2}\,dt > 0$ i.e. $g(x)$ is increasing. $g''(x) = \frac{d}{dx}\int_0^x e^{-t^2}\,dt = e^{-x^2} > 0,\ 0 \le t < 2,$ i.e. $g(x)$ is concave up. **Answer A**	Poorly done (27.5% students correct.) Many students did not consider that the exponential function $e^{-t^2} > 0$ for all real t values which makes both $g'(x)$ and $g''(x)$ positive.

	Suggested Solution	Markers' Comments
10.	The inequalities are symmetrical in x and y, so the maximum occurs when $y = x$. $\|z\| = \sqrt{x^2 + x^2}$ $= \sqrt{2x^2}$ $\therefore \sqrt{2x^2} = x + 1$ $2x^2 = x^2 + 2x + 1$ $x^2 - 2x - 1 = 0$ $x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$ $= \frac{2 \pm 2\sqrt{2}}{2}$ $= 1 \pm \sqrt{2}$ $y = 1 \pm \sqrt{2}$ $\|z\|_{\max}$ when $x = y = 1 + \sqrt{2}$ since $\|1 + \sqrt{2}\| > \|1 - \sqrt{2}\|$ $\|z\|_{\max} = \sqrt{2}(1 + \sqrt{2})$ using the 45° exact triangle $= \sqrt{2} + 2$ **Answer D.** ***Alternative solution I:*** $\|z\| \le y + 1 \quad (1)$ $\|z\| \le x + 1 \quad (2)$ $(2) - (1)$: $0 \le x - y \to y \le x \quad (3)$ $(1) - (2)$: $0 \le y - x \to y \ge x \quad (4)$ From (3) and (4): $y = x \quad (5)$ $\|z\| = \sqrt{x^2 + y^2}$ From (5): $\|z\| = \sqrt{x^2 + x^2}$ $= \sqrt{2x^2}$ sub in (2): $\therefore \sqrt{2x^2} = x + 1$ $2x^2 = x^2 + 2x + 1$ $x^2 - 2x - 1 = 0$ $x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$ $= \frac{2 \pm 2\sqrt{2}}{2}$ $= 1 \pm \sqrt{2}$ $y = 1 \pm \sqrt{2}$ $\|z\|_{\max}$ when $x = y = 1 + \sqrt{2}$ since $\|1 + \sqrt{2}\| > \|1 - \sqrt{2}\|$ $\|z\|_{\max} = \sqrt{2}(1 + \sqrt{2})$ using the 45° exact triangle $= \sqrt{2} + 2$	Poorly done (32.5% students correct.) Many students (52,5%) chose (B) not realising that it is the coordinates of z and not the modulus of z.

	Suggested Solution	Markers' Comments
10.	**Answer D.**	

Alternative solution II:

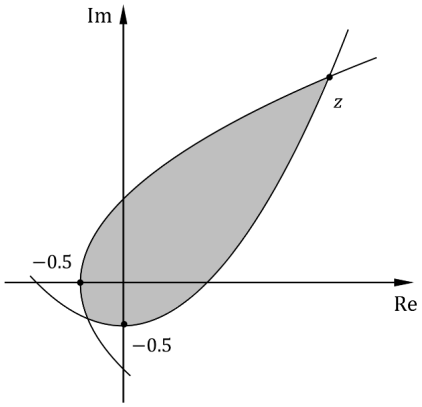
$$|z| \le x + 1$$
$$\sqrt{x^2 + y^2} \le x + 1$$
$$x^2 + y^2 \le x^2 + 2x + 1$$
$$y^2 \le 2x + 1$$
$$2x \ge y^2 - 1$$
$$x \ge \frac{y^2 - 1}{2}$$

Which is a concave right parabola with vertex $\left(-\frac{1}{2},\ 0\right)$.

Similarly

$$|z| \le y + 1 \ \rightarrow\ y \ge \frac{x^2 - 1}{2}$$

which is a concave up parabola with vertex $\left(0,\ -\frac{1}{2}\right)$.



The region is graphed at right, with $|z|$ a maximum at the top right.

Using symmetry, the maximum value of $|z|$ occurs at the point of intersection in the first quadrant where $y = x$.

$$\therefore x = \frac{x^2 - 1}{2}$$
$$2x = x^2 - 1$$
$$x^2 - 2x - 1 = 0$$
$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{2 \pm 2\sqrt{2}}{2}$$
$$= 1 \pm \sqrt{2}$$
$$y = 1 \pm \sqrt{2}$$
$$|z|_{\max} = \sqrt{2}\left(1 + \sqrt{2}\right) \quad \text{using the } 45° \text{ exact triangle}$$
$$= \sqrt{2} + 2$$

Answer D.

SECTION 11

11. (a)

$$\frac{4-3i}{3+2i}=\frac{4-3i}{3+2i}\times\frac{3-2i}{3-2i}$$

$$=\frac{(4-3i)(3-2i)}{9+4} \quad ①$$

$$=\frac{12-17i-6}{13}$$

$$=\frac{6}{13}-\frac{17}{13}i \quad ①$$

Mostly well done. (87.5% correct)

11. (b)

$$\int \cos^3 x\sin^2 x\ dx=\int \cos x\cos^2 x\sin^2 x\ dx$$

$$=\int \cos x\left(1-\sin^2 x\right)\sin^2 x\ dx \quad ①$$

$$=\int \cos x\left(\sin^2 x-\sin^4 x\right)\ dx$$

$$=\int \cos x\sin^2 x-\cos x\sin^4 x\ dx$$

$$=\frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+C \quad ①$$

Mostly well done (95% students correct.)

11. (c)(i)

$$\left|-1+\sqrt{3}i\right|=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}$$

$$=2$$

$$\text{Arg}\left(-1+\sqrt{3}i\right)=\tan^{-1}\frac{\sqrt{3}}{-1} \quad \textbf{(QII)}$$

$$=\pi-\frac{\pi}{3}$$

$$=\frac{2\pi}{3} \quad ①$$

$$\therefore -1+\sqrt{3}i=2e^{\frac{2\pi}{3}i} \quad ①$$

Mostly well done. (87.5% correct)

(c)(ii)

$$\left(-1+\sqrt{3}i\right)^8=\left(2e^{\frac{2\pi}{3}i}\right)^8 \quad \text{or} \quad \left(2\text{cis}\frac{2\pi}{3}\right)^8$$

$$=2^8e^{\frac{16\pi}{3}i} \qquad =2^8\text{cis}\left(8\left(\frac{2\pi}{3}\right)\right)$$

$$=256e^{\left(5\pi+\frac{\pi}{3}\right)i} \qquad =256\ \text{cis}\frac{16\pi}{3}$$

$$\textbf{(QIII)} \quad ①$$

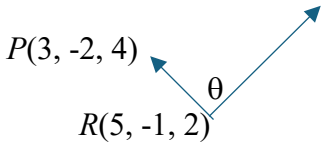
$$=256\left(-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$$

$$=256\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$

$$=-128-128\sqrt{3}i \quad ①$$

Mostly well done. (87.5% correct)
Area of concern:

(1) students not realising $\frac{16\pi}{3}$ is in QIII.

11.	(d) $Q(-2, 1, 5)$  $\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$ and $\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$ $= \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ $= \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ $= \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ ① $= \begin{bmatrix} -7 \\ 2 \\ 3 \end{bmatrix}$ ① $\cos\theta = \dfrac{\overrightarrow{RP} \bullet \overrightarrow{RQ}}{\left\|\overrightarrow{RP}\right\|\left\|\overrightarrow{RQ}\right\|}$ $= \dfrac{14 - 2 + 6}{\sqrt{(-2)^2 + (-1)^2 + (2)^2} \times \sqrt{(-7)^2 + (2)^2 + (3)^2}}$ $= \dfrac{18}{\sqrt{9} \times \sqrt{62}}$ $= \dfrac{6}{\sqrt{62}}$ $\therefore \theta = 40.36^\circ$ $\therefore \theta = 40^\circ$ (to nearest degree) ①	Mostly well done. (75% correct)
11.	(e) L_1: $\underset{\sim}{r}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\lambda \in R$. Given: $L_2 \parallel L_1$, $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ① i.e. $x = -2 + 2\mu$ — ① $y = 4 + 3\mu$ — ② From ① $2\mu = x + 2$ $\mu = \dfrac{x}{2} + 1$ — ①' Sub ①' into ② $y = 4 + 3\left(\dfrac{x}{2} + 1\right)$ $y = 4 + \dfrac{3x}{2} + 3$ $\therefore y = \dfrac{3}{2}x + 7$ ①	Mostly well done. (85% correct)

11. (f) $\int \frac{dx}{1-\sin x}$ where $t=\tan\frac{x}{2}$ and $\sin x=\frac{2t}{1+t^2}$

$$\frac{x}{2}=\tan^{-1}t$$

$$x=2\tan^{-1}t$$

$$\frac{dx}{dt}=\frac{2}{1+t^2}$$

$$dx=\frac{2}{1+t^2}\,dt$$

$$=\int\frac{\frac{2}{1+t^2}}{1-\frac{2t}{1+t^2}}\,dt \qquad ①$$

$$=\int\frac{\frac{2}{1+t^2}}{\frac{(1+t^2)-2t}{1+t^2}}\,dt$$

$$=\int\frac{2}{t^2-2t+1}\,dt$$

$$=2\int\frac{1}{(t-1)^2}\,dt \qquad ①$$

$$=2\int(t-1)^{-2}\,dt$$

$$=2\left[\frac{(t-1)^{-1}}{-1}\right]+C$$

$$=-\frac{2}{t-1}+C$$

$$=-\frac{2}{\tan\frac{x}{2}-1}+C \qquad ①$$

Mostly well done. (67.5% correct)
Area of concern:
(1) students did not answer in terms of x.

12. (a)(i) $(\sqrt{a}-\sqrt{b})^2\geq 0$

$$\Rightarrow a-2\sqrt{ab}+b\geq 0$$

$$\Rightarrow a+b\geq 2\sqrt{ab}$$

$$\Rightarrow \frac{a+b}{2}\geq\sqrt{ab}$$

(a)(ii) $\frac{1}{a}-\frac{1}{b}=\frac{b-a}{ab}$

≤ 0, since $a\geq b\Rightarrow b-a\leq 0$

Hence, $\frac{1}{a}\leq\frac{1}{b}$

12. (a)(iii)

$$f(x) = \frac{x}{(x+p)(x+q)}$$

$$\frac{1}{f(x)} = \frac{(x+p)(x+q)}{x}$$

$$= \frac{x^2 + px + qx + pq}{x}$$

$$\frac{1}{f(x)} = x + (p+q) + \frac{pq}{x}$$

$$\frac{1}{f(x)} = (p+q) + x + \frac{pq}{x}$$

But $\frac{a+b}{2} \geq \sqrt{ab} \quad from(i)$

$$a + b \geq 2\sqrt{ab}$$

So if $a = x$ and $b = \frac{pq}{x}$

$$x + \frac{pq}{x} \geq 2\sqrt{x \times \frac{pq}{x}}$$

$$x + \frac{pq}{x} \geq 2\sqrt{pq}$$

$$\frac{1}{f(x)} \geq p + q + 2\sqrt{pq}$$

$$\frac{1}{f(x)} \geq p + 2\sqrt{pq} + q$$

$$\frac{1}{f(x)} \geq \left(\sqrt{p}\right)^2 + 2\sqrt{pq} + \left(\sqrt{q}\right)^2$$

$$\frac{1}{f(x)} \geq \left(\sqrt{p} + \sqrt{q}\right)^2 \geq 0$$

Hence: $f(x) \leq \frac{1}{\left(\sqrt{p} + \sqrt{q}\right)^2}$, using part (ii)

12. (b) (i) $\frac{x^2+3x+18}{(x^2+4)(x-5)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-5}$

$(x^2 + 3x + 18) = (Ax + B)(x - 5) + C(x^2 + 4)$

Let $x = 5 \Rightarrow 58 = 29C$

$C = \frac{58}{29} = 2$

Let $x = 0 \Rightarrow 18 = -5B + 4(2)$

$-5B = 10$

$B = -2$

Let $x = 1$

	$22 = (A-2)(-4) + 2(5)$ $22 = -4A + 8 + 10$ $4 = -4A$ $A = -1$ $A = -1, \quad B = -2$ and $C = 2$ (ii) $\int \frac{x^2+3x+18}{(x^2+4)(x-5)}\, dx$ $= \int \frac{-x-2}{x^2+4} + \frac{2}{x-5}\, dx$ (From part (i)) $= \int -\frac{x}{x^2+4}\, dx + \int -\frac{2}{x^2+4} dx + \int \frac{2}{x-5}\, dx$ $= -\frac{1}{2}\int \frac{2x}{x^2+4}\, dx - 2\int \frac{1}{x^2+4} dx + \int \frac{2}{x-5}\, dx$ $= -\frac{1}{2}\ln\lvert x^2+4\rvert - 2 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + 2\ln\lvert x-5\rvert + c$ $= -\frac{1}{2}\ln\lvert x^2+4\rvert - \tan^{-1}\left(\frac{x}{2}\right) + 2\ln\lvert x-5\rvert + c$	**Some students left out the middle term**
12.	(c)(i) $\frac{1}{2}\frac{d}{dx}v^2 = -\frac{1}{50}(10-x)^3$ $v^2 = \frac{1}{100}(10-x)^4 + c$ $t=0, x=0, v=+10 \Rightarrow c=0$ and $v^2 = \frac{1}{100}(10-x)^4,\ v>0$ $\therefore\ v = \frac{1}{10}(10-x)^2$ Or $v\frac{dv}{dx} = \frac{-1}{50}(10-x)^3$ $\int v.dv = \frac{-1}{50}\int (10-x)^3.dx$ $\left[\frac{v^2}{2}\right] = \frac{-1}{50}\left[\frac{(10-x)^4}{4\times(-1)}\right] + c$ $\frac{v^2}{2} = \frac{1}{200}(10-x)^4 + c$ When $t=0,\ x=0,\ v^2 = 10$	Need to give justification for discarding the negative case

$$\frac{10^2}{2} = \frac{1}{200}(10-0)^4 + c$$

$$\frac{100}{2} = \frac{1}{200}10000 + c$$

$$50 = 50 + c$$

$$c = 0$$

$$\frac{v^2}{2} = \frac{1}{200}(10-x)^4 + 0$$

$$v^2 = \frac{1}{100}(10-x)^4$$

$$v = \pm\frac{1}{10}(10-x)^2$$

When $t=0$ moving right, $v>0$

$$v = \frac{1}{10}(10-x)^2$$

12. (c)(ii)

$$\frac{dx}{dt} = \frac{1}{10}(10-x)^2$$

$$\int \frac{1}{(10-x)^2}dx = \frac{1}{10}\int dt$$

$$\frac{1}{10-x} = \frac{1}{10}t + c$$

$$t=0, x=0 \Rightarrow c = \frac{1}{10}$$

$$\therefore \frac{10}{10-x} = 1+t$$

$$1 - \frac{x}{10} = \frac{1}{1+t}$$

$$\frac{x}{10} = 1 - \frac{1}{1+t}$$

$$x = \frac{10t}{1+t}$$

13. (a) To prove: $\log_4 7$ is irrational

Assume $\log_4 7$ is rational ①

i.e. $\log_4 7 = \frac{p}{q}$ where $p, q \in Z^+$ and p, q are relative primes

$q \log_4 7 = p$

$\log_4 7^q = p$

$\therefore 7^q = 4^p$ ①

LHS $= 7^q$

$= 7 \times 7 \times 7 \times \ldots$ i.e. product of q number of 7 with all factors are 7

RHS $= 4^p$

$= 4 \times 4 \times 4 \times \ldots$ i.e. product of p number of 4 with all factors are 4

Hence LHS $\neq$ RHS, which is a contradiction to the statement that $\log_4 7$ is rational ①

$\therefore \log_4 7$ is irrational.

Mostly well done. (85% correct)
Area of concern:
(1) some students are not realising the contradiction is on the original statement. Hence their statement and conclusion are confusing.

13. (b) To prove: $u_n = 3\left(2^{n+1}\right) - 1$ for all positive integers n

given $u_0 = 5$, $u_n = 2u_{n-1} + 1$ for $n \geq 1$.

Proof:

Step 1: to prove true for $n = 1$.

From closed definition: $u_1 = 3\left(2^{1+1}\right) - 1$

$= 3\left(2^2\right) - 1$

$= 12 - 1$

$\therefore u_1 = 11$

From recursive definition: $u_1 = 2u_{1-1} + 1$

$= 2u_0 + 1$

$= 2(5) + 1$

$\therefore u_1 = 11$

which agrees with the closed definition.

Hence true for $n = 1$.

Step 2: Assume true for $n = k$, $k \in Z^+$,

$u_k = 3\left(2^{k+1}\right) - 1$

RTP: when $n = k + 1$, $u_{k+1} = 3\left(2^{k+2}\right) - 1$

From recursive definition::

$u_{k+1} = 2u_{(k+1)-1} + 1$

$= 2u_k + 1$

$= 2\left[3\left(2^{k+1}\right) - 1\right] + 1$

(from assumption)

$= 2 \times 3 \times 2^{k+1} - 2 + 1$

Poorly done. (50% correct)
Area of concern:
(1) some students finding the smallest n value as 0 whilst the proof is for $n \geq 0$.
(2) some students do not realise how significant the assumption of $n = k$ is true for the proof of $n = k + 1$.
(3) some students do not realise how significant the two different types of equations given and the fact that one of them is recursive.

	$= 3\times\left(2\times 2^{k+1}\right)-1$ $\therefore u_{k+1} = 3\left(2^{k+2}\right)-1$ which agrees with the closed definition. If true for $n = k$, hence proven true for $n = k + 1$. Step 3: Since proven true for $n = 1$, hence proven true for $n = 1 + 1 = 2$, $n = 2 + 1 = 3$, and so on. Therefore true for all positive integers n.	
13.	(c)(i) $z^5 = 1$ where $z = \cos\theta + i\sin\theta,\ -\pi \le \theta \le \pi$ $z^n = \cos n\theta + i\sin n\theta$ (De Moirve's theorem) $\cos 5\theta + i\sin 5\theta = 1$ Equating real terms: $\cos 5\theta = 1$ $5\theta = 2k\pi,\ k \in Z$ $\therefore \theta = \frac{2k\pi}{5}$ where $k = 0,\ \pm 1,\ \pm 2$ $\therefore z_1 = \text{cis } 0 = \cos 0 + i\sin 0 = 1$ $z_2 = \text{cis}\frac{2\pi}{5}$ $z_3 = \text{cis}\frac{4\pi}{5}$ $z_4 = \text{cis}\left(-\frac{2\pi}{5}\right) = \overline{z}_2$	Mostly well done. (92.5% correct) Area of concern: (1) some students are not stating De Moivre's theorem when they used it.

13.	(c)(i)cont. $z_5 = \text{cis}\left(-\frac{4\pi}{5}\right) = \overline{z}_3$ Hence the roots of $z^5 = 1$ are $\text{cis } 0,\ \text{cis}\frac{2\pi}{5},\ \text{cis}\left(-\frac{2\pi}{5}\right),\ \text{cis}\frac{4\pi}{5},\ \text{cis}\left(-\frac{4\pi}{5}\right)$ (ii) $z^5 = 1$ $z^5 - 1 = 0$ $(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5) = 0$ Since $z_4 = \overline{z}_2$ and $z_5 = \overline{z}_3$ $(z-1)(z-z_2)(z-\overline{z}_2)(z-z_3)(z-\overline{z}_3) = 0$ $(z-1)\left[z^2-(z_2+\overline{z}_2)+z_2\overline{z}_2\right]\left[z^2-(z_3+\overline{z}_3)+z_3\overline{z}_3\right] = 0$ where $z_2 + \overline{z}_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$ $z_2 + \overline{z}_2 = 2\cos\frac{2\pi}{5}$	Poorly done. (45% correct) Area of concern: (1) students are not showing how they obtain coefficient of z or the constant 1 of the quadratic factors.

Similarly, $z_3 + \bar{z}_3 = 2\cos\frac{4\pi}{5}$

$$z_2\bar{z}_2 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)\left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$$

$$= \cos^2\frac{2\pi}{5} - i^2\sin^2\frac{2\pi}{5}$$

$$= \cos^2\frac{2\pi}{5} - (-1)\sin^2\frac{2\pi}{5}$$

$$= \cos^2\frac{2\pi}{5} + \sin^2\frac{2\pi}{5}$$

$\therefore z_2\bar{z}_2 = 1$ and similarly, $z_3\bar{z}_3 = 1$

$$\therefore (z-1)\left(z^2 - 2\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5} + 1\right) = 0$$

$$z^5 - 1 = (z-1)\left(z^2 - 2\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5} + 1\right)$$

$$(z-1)\left(z^4 + z^5 + z^2 + z + 1\right)$$

$$= (z-1)\left(z^2 - 2\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5} + 1\right)$$

$$\therefore z^4 + z^5 + z^2 + z + 1 = \left(z^2 - 2\cos\frac{2\pi}{5} + 1\right) \times \left(z^2 - 2\cos\frac{4\pi}{5} + 1\right)$$

(as required)

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13. (c)(iii) $z^4 + z^5 + z^2 + z + 1 = \left(z^2 - 2\cos\frac{2\pi}{5} + 1\right) \times \left(z^2 - 2\cos\frac{4\pi}{5} + 1\right)$

Substituting $z = 1$,

$$(1)^4 + (1)^5 + (1)^2 + (1) + 1 = \left((1)^2 - 2\cos\frac{2\pi}{5} + 1\right) \times \left((1)^2 - 2\cos\frac{4\pi}{5} + 1\right)$$

$$5 = \left(2 - 2\cos\frac{2\pi}{5}\right)\left(2 - 2\cos\frac{4\pi}{5}\right)$$

$$5 = 2\left(1 - \cos\frac{2\pi}{5}\right) \times 2\left(1 - \cos\frac{4\pi}{5}\right)$$

$$\therefore 4\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = 5$$ (as required)

Mostly well done. (75% correct)

(iv) $4\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = 5$

Mostly well done. (62.5% correct)
Area of concern:

[Hint: $\cos\frac{4\pi}{5}=\cos 2\left(\frac{2\pi}{5}\right)$ i.e. double angle]

$$4\left(1-\cos\frac{2\pi}{5}\right)\left(1-\cos 2\left(\frac{2\pi}{5}\right)\right)=5$$

$$4\left(1-\cos\frac{2\pi}{5}\right)\left(1-\left(1-2\sin^2\left(\frac{2\pi}{5}\right)\right)\right)=5$$

$$4\left(1-\cos\frac{2\pi}{5}\right)\times 2\sin^2\left(\frac{2\pi}{5}\right)=5$$

$$8\left(1-\cos\frac{2\pi}{5}\right)\sin^2\left(\frac{2\pi}{5}\right)=5$$

$$8\left(1-\cos\frac{2\pi}{5}\right)\left(1-\cos^2\frac{2\pi}{5}\right)=5$$

Let $x=\cos\frac{2\pi}{5}$, $\quad 8(1-x)\left(1-x^2\right)=5$

$$8\left(1-x-x^2+x^3\right)=5$$

$$8x^3-8x^2-8x+8=5$$

$$\therefore 8x^3-8x^2-8x+3=0$$

Hence $\cos\frac{2\pi}{5}$ is a root of the equation

(1) some students did not identify that $\cos\frac{4\pi}{5}=\cos 2\left(\frac{2\pi}{5}\right)$

14. (a)(i)

The direction vectors of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are $\begin{pmatrix}3\\6\\-3\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\-3\end{pmatrix}$ respectively.

$$\begin{pmatrix}3\\6\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-3\end{pmatrix}=3-12+9$$

$$=0$$

$\therefore AB$ is perpendicular to CD. ①

Alternative solution I (if they find E first and come back to this question):

$$\overrightarrow{AB}=\begin{pmatrix}3\\6\\-3\end{pmatrix}$$

$$\overrightarrow{CE}=\begin{pmatrix}3-0\\3-9\\1-10\end{pmatrix}$$

$$=\begin{pmatrix}3\\-6\\-9\end{pmatrix}$$

$$\overrightarrow{AB}\cdot\overrightarrow{CE}=\begin{pmatrix}3\\6\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\-6\\-9\end{pmatrix}$$

$$=9-36+27$$

$$=0$$ ①

$\therefore AB$ is perpendicular to CE, so AB is perpendicular to CD

Done well by most students

) since C, D and E are collinear.

(a)(ii)

$$\underset{\sim}{r}_{AB} = \begin{pmatrix}4\\5\\0\end{pmatrix} + \lambda\begin{pmatrix}-3\\-6\\3\end{pmatrix}$$

$$= \begin{pmatrix}4-3\lambda\\5-6\lambda\\3\lambda\end{pmatrix}$$

$$\underset{\sim}{r}_{CD} = \begin{pmatrix}0\\9\\10\end{pmatrix} + \mu\begin{pmatrix}-1\\2\\3\end{pmatrix}$$

$$= \begin{pmatrix}-\mu\\9+2\mu\\10+3\mu\end{pmatrix}$$

At the point of intersection

$$\begin{pmatrix}4-3\lambda\\5-6\lambda\\3\lambda\end{pmatrix} = \begin{pmatrix}-\mu\\9+2\mu\\10+3\mu\end{pmatrix} \quad ①$$

$\therefore 4-3\lambda = -\mu \quad (1)$

$5-6\lambda = 9+2\mu \quad (2)$

$3\lambda = 10+3\mu \quad (3)$

$(1)+(3)$: $\quad 4 = 10+2\mu$

14. (a)(ii) $(1)+(3)$: $4 = 10 + 2\mu$

$2\mu = -6$

$\mu = -3$

The point E is $\begin{pmatrix} -(-3) \\ 9+2(-3) \\ 10+3(-3) \end{pmatrix} = (3,3,1).$ ①

Note that there are many ways in which this solution can be set out - far too many to list. They all end with the same answer. One alternative solution is shown next page.

Alternative solution I:

$\underset{\sim}{r}_{AB} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 4+\lambda \\ 5+2\lambda \\ -\lambda \end{pmatrix}$

$\underset{\sim}{r}_{CD} = \begin{pmatrix} 0 \\ 9 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} \mu \\ 9-2\mu \\ 10-3\mu \end{pmatrix}$

At the point of intersection

$\begin{pmatrix} 4+\lambda \\ 5+2\lambda \\ -\lambda \end{pmatrix} = \begin{pmatrix} \mu \\ 9-2\mu \\ 10-3\mu \end{pmatrix}$

$\therefore 4+\lambda = \mu \quad (1)$

$5+2\lambda = 9-2\mu \quad (2)$

$-\lambda = 10-3\mu \quad (3)$

sub (1) in (3): $-\lambda = 10-3(4+\lambda)$

$= 10-12-3\lambda$

$2\lambda = -2$

$\lambda = -1$

The point E is $\begin{pmatrix} 4+(-1) \\ 5+2(-1) \\ -(-1) \end{pmatrix} = (3,3,1).$

①

①

14. (b)

Write $z = 1 - \sqrt{3}\,i$ in polar form $r(\cos\theta + i\sin\theta)$

$r = \sqrt{1+3} = \sqrt{4} = 2$

$\tan\theta = \dfrac{-\sqrt{3}}{1} = -\sqrt{3}$, $\theta = \dfrac{-\pi}{3}$

$z = 2\left(\cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)\right)$

$$\gamma = (2)^{\frac{1}{2}}\left(\cos\frac{\left(-\frac{\pi}{3}+2k\pi\right)}{2} + i\sin\frac{\left(-\frac{\pi}{3}+2k\pi\right)}{2}\right),$$

$k = 0, 1$

14. (b)continues

For $k = 0$, $\gamma = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{6}\right) + i\sin\left(-\dfrac{\pi}{6}\right)\right)$

$= \sqrt{2}\left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)$

$= \dfrac{\sqrt{6} - \sqrt{2}\,i}{2}$

For $k = 1$, $\gamma = \sqrt{2}\left(\cos\left(\dfrac{5\pi}{6}\right) + i\sin\left(\dfrac{5\pi}{6}\right)\right)$

$= \sqrt{2}\left(-\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right) = \dfrac{-\sqrt{6} + \sqrt{2}\,i}{2}$

$= \dfrac{-\sqrt{6} + \sqrt{2}\,i}{2}$

The square roots are $\dfrac{\sqrt{6} - \sqrt{2}\,i}{2}$ and $\dfrac{-\sqrt{6} + \sqrt{2}\,i}{2}$

$\dfrac{\sqrt{6}}{2} - \dfrac{\sqrt{2}i}{2}$, $\dfrac{-\sqrt{6}}{2} + \dfrac{\sqrt{2}i}{2}$

14. (c)

$\omega = \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$

$\omega^2 + \dfrac{1}{\omega^2} = \omega^2 + \overline{\omega^2}$ since $|\omega^2| = |\omega|^2 = |1|^2 = 1$

$= 2\mathrm{Re}(\omega^2)$

$= 2\mathrm{Re}\left(\cos\dfrac{4\pi}{5} + i\sin\dfrac{4\pi}{5}\right)$ using DMT

$= 2\cos\dfrac{4\pi}{5}$ ①

$\therefore \left(2\cos\dfrac{4\pi}{5}\right)^2 + 2\cos\dfrac{4\pi}{5} - 1 = 0$

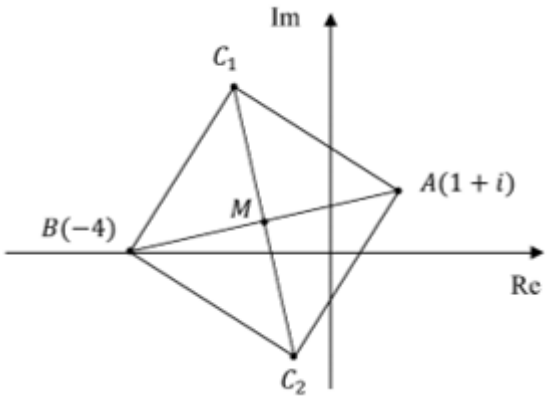
$4\cos^2\dfrac{4\pi}{5} + 2\cos\dfrac{4\pi}{5} - 1 = 0$

$\cos\dfrac{4\pi}{5} = \dfrac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$

$= \dfrac{-2 \pm \sqrt{20}}{8}$

$= \dfrac{-2 \pm 2\sqrt{5}}{8}$

	①	Must justify why the positive case is discarded
	①	

14.	(d)(i)  From the diagram we can see that C can lie in the second or third quadrants. The two possible right-angled triangles form a square centred at the midpoint of AB, which is marked as M. Since the diagonals of a square bisect at right-angles, we rotate MA 90° anticlockwise or clockwise about M to find the possible positions for C. To find the two possible values for z_C, we start at z_M and rotate $z_A - z_M$ 90° anticlockwise (multiply by i) or clockwise (multiply by $-i$). $z_M = \frac{1+i-4}{2}$ $= -\frac{3}{2}+\frac{i}{2}$ ① $z_A - z_M = 1+i-\left(-\frac{3}{2}+\frac{i}{2}\right)$	Common alternative methods were to - Use dot product since right angled at C $\left(\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}x\\y\end{pmatrix}\right)\cdot\left(\begin{pmatrix}-4\\0\end{pmatrix}-\begin{pmatrix}x\\y\end{pmatrix}\right)=0$ or - Pythagoar's Theorem $\lvert\overrightarrow{AC}\rvert^2+\lvert\overrightarrow{BC}\rvert^2=\lvert\overrightarrow{AB}\rvert^2$ $\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x+4)^2+(y-0)^2}=\sqrt{(1--4)^2+(1-0)^2}$ to find $\left(x+\frac{3}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{26}{4}$ and then use the fact that ABC is an isosceles triangle so AC=BC and

$|z - z_A| = |z - z_B|$

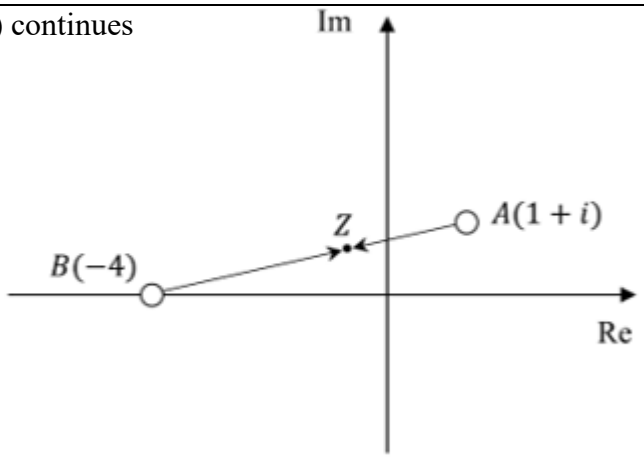
$|x + iy - (1+i)| = |x + iy - (-4 + 0i)|$

$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x+4)^2 + y^2}$ ①

To obtain $y = -5x - 7$ and substitute into equation of circle above to find $x=-2, y=3$ or $x=-1, y=-2$ ①

(d)(ii) The argument of the vector from B to any point Z on on the diagonal, representing the complex number z. is always π radians greater than the argument from A to Z. ①

14. (d)(ii) continues



So, $\arg(z - z_B) - \arg(z - z_A) = \pi$

$\therefore \arg(z - (-4 + 0i)) - \arg(z - (1 + i)) = \pi$

$\therefore \arg(z + 4 + 0i) - \arg(z - 1 - i) = \pi$

$\therefore x_1 = 4,\ y_1 = 0,\ x_2 = -1,\ y_2 = -1,\ \theta = \pi.$ ①

Common error $\theta \neq 0$

Alternatively

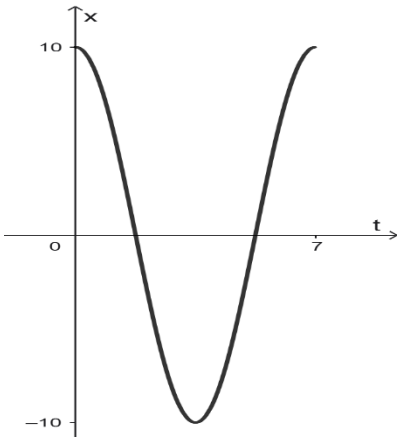
The argument of the vector from A to any point Z on the diagonal, representing the complex number z. is always π radians less than the argument from B to Z. ①

$\arg(z - z_A) - \arg(z - z_B) = -\pi$

$\therefore \arg(z - (1 + i)) - \arg(z - (-4 + 0i)) = -\pi$

$\therefore \arg(z - 1 - i) - \arg(z + 4 + 0i) = -\pi$

$\therefore x_1 = -1,\ y_1 = -1,\ x_2 = 4,\ y_2 = 0,\ \theta = -\pi.$

	①	
15.	(a)(i) Given: When $t = 0$, $x = x_{max}$, i.e. $v = 0$. ie. $x = a\cos nt + c$ Amplitude = 10, period = 7 s $\frac{2\pi}{n} = 7$ $n = \frac{2\pi}{7}$ Let the equilibrium position be at O,  Hence the position of particle at time t s is $x = 10\cos\frac{2\pi}{7}t$ ①	Mostly well done. (7.5% correct)

15.	(a)(ii) When $x = 5$, $10\cos\frac{2\pi}{7}t = 5$ $\cos\frac{2\pi}{7}t = \frac{1}{2}$ $\frac{2\pi}{7}t = \frac{\pi}{3}$ $\frac{2}{7}t = \frac{1}{3}$ $t = \frac{7}{6}$ s $\therefore$ the particle takes $1\frac{1}{6}$ seconds ①	Mostly well done. (55% correct) Area of concern: (1) careless errors caused many wrong answers.
15.	(i) $a = -(1+v)$ $v\frac{dv}{dx} = -(1+v)$ $\frac{dv}{dx} = -\frac{1+v}{v}$ $\int_u^v \frac{v}{1+v}\,dv = -\int_0^x 1\,dx$ $\int_u^v \frac{v+1-1}{1+v}\,dv = -\int_0^x 1\,dx$	Mostly well done. (55% correct) Area of concern: (1) careless errors caused many wrong answers. (2) few students forgot that the initioal velovity is u ms^{-1}, and wrote 0 which makes the subsequent working so much easier.

$$\int_u^v \frac{(v+1)}{1+v} - \frac{1}{1+v}\,dv = -\int_0^x 1\,dx$$

$$\int_u^v 1 - \frac{1}{1+v}\,dv = -\int_0^x 1\,dx$$

$$\left[v - \ln|1+v|\right]_u^v = -[x]_0^x$$

$$\left[v - \ln|1+v|\right] - \left[u - \ln|1+u|\right] = -[x-0]$$

$$v - \ln|1+v| - u + \ln|1+u| = -x$$

$$v - u - \left(\ln|1+v| - \ln|1+u|\right) = -x$$

$$v - u - \left(\ln\frac{|1+v|}{1+u}\right) = -x$$

15. (b)(i)cont. $\therefore x = u - v + \ln\dfrac{|1+v|}{1+u}$

$$a = -(1+v)$$

$$\frac{dv}{dt} = -(1+v)$$

$$\int_u^v \frac{1}{1+v}\,dv = -\int_0^t dt$$

$$\left[\ln|1+v|\right]_u^v = -[t]_0^t$$

$$\ln|1+v| - \ln|1+u| = -t$$

$$\ln\frac{|1+v|}{1+u} = -t$$

$$\frac{|1+v|}{1+u} = e^{-t}$$

$$|1+v| = (1+u)e^{-t} \text{ since initial } v > 0$$

$$1+v = (1+u)e^{-t}$$

$$\therefore v = (1+u)e^{-t} - 1$$

$$v = (1+u)e^{-t} - 1$$

$$\frac{dx}{dt} = (1+u)e^{-t} - 1$$

$$\int_0^x dx = \int_0^t (1+u)e^{-t} - 1\,dt$$

$$[x]_0^x = \left[-(1+u)e^{-t} - t\right]_0^t$$

$$x = \left[-(1+u)e^{-t} - t\right] - \left[-(1+u)e^{-0} - 0\right]$$

$$x = -(1+u)e^{-t} - t + (1+u)$$

	$\therefore x = u - (1+u)e^{-t} + 1 - t$ (b)(ii) From part (ii) $v = (1+u)e^{-t} - 1$ $(1+u)e^{-t} = v + 1$ — ① $x = u - (1+u)e^{-t} + 1 - t$ — ② Sub ① into ② $x = u - (v+1) + 1 - t$ $x = u - v - 1 + 1 - t$ $x = u - v - t$ $\therefore x + v + t = u$ (as required) (b)(iii) At rest, $v = 0$, $x = u - (0) + \ln\dfrac{\lvert 1+(0)\rvert}{1+u}$ [from (ii)]	Poorly done. (50% correct) Area of concern: (1) students did not know how to work this out. Poorly done. (40% correct) Area of concern: (1) students did not know how to work this out.

15.	(b)(iv)cont. $x = u + \ln\dfrac{1}{1+u}$ $x = u + \ln(1+u)^{-1}$ $\therefore x = u - \ln(1+u)$ m i.e. the distance the particle travelled from O before coming to rest. At rest, $v = 0$, $x = u - \ln(1+u)$, $x + v + t = u$ [from (iii)] $[u - \ln(1+u)] + (0) + t = u$ ~~$[u - \ln(1+u)] + (0) + t = u$~~ $-\ln(1+u) + t = 0$ $\therefore t = \ln(1+u)$ s i.e. the time taken to come to rest.	
15.	(c) $\int_0^1 \sin^{-1} x\, dx = [uv]_0^1 - \int_0^1 u'v\, dx$ where $u = \sin^{-1} x$ $u' = \dfrac{1}{\sqrt{1-x^2}}$ $v' = 1$ $v = x$ $= \left[x \sin^{-1} x\right]_0^1 - \int_0^1 \dfrac{x}{\sqrt{1-x^2}}\, dx$	Mostly done. (62.5% correct) Area of concern: (1) students through careless integration obtain answer $\dfrac{\pi}{2} + 1$ rather than $\dfrac{\pi}{2} - 1$.(27.5%)

$$= \left[(1)\sin^{-1}(1) - 0\right] - \left(-\frac{1}{2}\right)\int_0^1 \frac{-2\times x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{2} + \frac{1}{2}\int_0^1 -2x\left(1-x^2\right)^{-\frac{1}{2}}\,dx$$

$$= \frac{\pi}{2} + \frac{1}{2}\left[\frac{2\left(1-x^2\right)^{\frac{1}{2}}}{1}\right]_0^1$$

$$= \frac{\pi}{2} + \left[\sqrt{1-x^2}\right]_0^1$$

$$= \frac{\pi}{2} + \left[\sqrt{1-(1)^2} - \sqrt{1-(0)^2}\right]$$

$$= \frac{\pi}{2} + [0-1]$$

$$\therefore \int_0^1 \sin^{-1} x\,dx = \frac{\pi}{2} - 1$$

16. (a)

$$\frac{a}{b} \geq \frac{c}{a}$$

$$\Rightarrow a^2 \geq b \quad :$$

$$\Rightarrow 1 \quad a^2 \geq 1 \quad (b) \quad :$$

$$\Leftrightarrow 2\,1 \quad \ln a \geq 1 \quad \ln b + 1 \quad \ln c$$

$$\Rightarrow 1 \quad \ln a \geq \frac{1 \quad \ln b + 1 \quad \ln c}{2}.$$

Now $(x+y)^2 = (x-y)^2 + 4xy \geq 4xy \Rightarrow (x+y)^2 \geq 4xy.$

Now for $x \geq 0$ and $y \geq 0$

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Substitute $x = \ln b$ and $y = \ln c$ gives

$$\frac{\ln b + \ln c}{2} \geq \sqrt{\ln b \ln c}.$$

Thus $\ln a \geq \sqrt{\ln b \ln c}.$

i.e.

$$(\ln a)^2 \geq \ln b \ln c$$

$$\frac{\ln a}{\ln b} \geq \frac{\ln c}{\ln a}.$$

Many students over simplified the process and/or made errors with $\ln bc \neq \ln b \ln c$

Or

taking the log of both sides of $\frac{a}{b} \geq \frac{c}{a}$

gives $\ln\frac{a}{b} \geq \ln\frac{c}{a}$

NOT $\frac{\ln a}{\ln b} \geq \frac{\ln c}{\ln a}$

16. (b)(i)

$$I_n = \int_1^e (\ln x)^n dx \quad , n = 0, 1, 2, \ldots$$

$$I_{n+1} = \int_1^e 1 \cdot (\ln x)^{n+1} dx$$

	$I_0 = \int_1^e 1\,dx = e-1$ $I_1 = e-(0+1)\,I_0$ $= e-(e-1)$ $= 1$ $I_2 = e-(1+1)I_1$ $= e-2$	Need proof of result, not just substitution $I_2 = e-(1+1)I_1$ not $I_2 = e-(2+1)I_1$

16.	(b)(ii) $I_n = (-1)^{n+1}\,n! + (-1)^n\, e\, n! \sum_{r=2}^{n} \frac{(-1)^r}{r!}$ Step 1: When $n = 2$, LHS = I_2 [from part (i)] $= e - 2$ RHS = $(-1)^{2+1}2! + (-1)^2 e \times 2!\left(\frac{1}{2!}\right)$ $= -2! + e$ $= e - 2$ $\therefore$ true for $n = 2$. Step 2: Assume true for $n = k$, $I_k = (-1)^{k+1}k! + +(-1)^k e\, k! \sum_{r=2}^{k} \frac{(-1)^r}{r!}$ RTP: When $n = k + 1$, $I_{k+1} = (-1)^{k+2}(k+1)! + +(-1)^{k+1} e\,(k+1)! \sum_{r=2}^{k+1} \frac{(-1)^{r+1}}{(r+1)!}$ From part (i), $I_{k+1} = e-(k+1)I_k$ $= e-(k+1)\left\{(-1)^{k+1}k! + (-1)^k e\, k! \sum_{r=2}^{k} \frac{(-1)^r}{r!}\right\}$ (from assumption) $= e + (-1)^{k+2}(k+1)! + (-1)^{k+1} e\,(k+1)! \sum_{r=2}^{k} \frac{(-1)^r}{r!}$ $= (-1)^{k+2}(k+1)! + (-1)^{k+1} e (k+1)! \left\{\sum_{r=2}^{k} \frac{(-1)^r}{r!} + \frac{1}{(-1)^{k+1}(k+1)!}\right\}$ $= (-1)^{k+2}(k+1)! + (-1)^{k+1} e (k+1)! \left\{\sum_{r=2}^{k} \frac{(-1)^r}{r!} + \frac{(-1)^{k+1}}{(k+1)!}\right\}$ $= \sum_{r=2}^{k+1} \frac{(-1)^r}{r!}$	**Many students did not write 'from assumtion'**

	If true for $n = k$, hence proven true for $n = k + 1$. Step 3: Since proven true for $n = 2$, $n = 2 + 1 = 3$, and so on, hence true for all positive integers of $n \geq 2$.	**Many students struggled to prove true for *n=k+1***
16.	(c)(i) $\vec{a},\vec{b},\vec{c}$ form a triangle $\Rightarrow \vec{a}+\vec{b}+\vec{c}=\vec{0}$ $\vec{u}+\vec{v}+\vec{w}=(\vec{b}\cdot\vec{c})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}+(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}+(\vec{b}\cdot\vec{a})\vec{c}-(\vec{c}\cdot\vec{b})\vec{a}$ $=(\vec{b}\cdot\vec{c})\vec{a}-(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}+(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}+(\vec{b}\cdot\vec{a})\vec{c}$ $=\vec{0}$ since $\vec{p}\cdot\vec{q}=\vec{q}\cdot\vec{p}$ Hence $\vec{u},\vec{v},\vec{w}$ also form a triangle	Many tried to expand terms such as $(\vec{b}\bullet\vec{c})\vec{a}$ incorrectly NOT the same as $\vec{a}\bullet(\vec{b}+\vec{c})=\vec{a}\bullet\vec{b}+\vec{a}\bullet\vec{c}$ Some did not recognise $\vec{a}+\vec{b}+\vec{c}=0$ as the condition for forming a triangle

16.	(c)(ii) $\vec{u}\cdot\vec{c}=\left((\vec{b}\cdot\vec{c})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}\right)\cdot\vec{c}$ $=(\vec{b}\cdot\vec{c})\vec{a}\cdot\vec{c}-(\vec{c}\cdot\vec{a})\vec{b}\cdot\vec{c}$ $=0$ (c)(iii) From (ii), $\vec{u}$ and $\vec{c}$ are perpendicular. Similarly, $\vec{v}$ and $\vec{a}$, and $\vec{w}$ and $\vec{b}$ are respectively perpendicular. The sides of the triangle formed with $\vec{u},\vec{v},\vec{w}$ are perpendicular to the sides of the triangle formed with $\vec{a},\vec{b},\vec{c}$, hence the two triangles are equiangular, that is are similar.	Done well by most Some tried to expand incorrectly $\vec{u}\bullet\vec{c}=0$, **so** $\vec{u}$ and $\vec{c}$ are perpendicular Similarly $\vec{v}\bullet\vec{a}=0$ and $\vec{w}\bullet\vec{b}=0$