Student Number: | | | | | | | | | |

Class: _____

2026 **YEAR 12 TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Student Number and Maths Class at the top of this page

Total marks: 100

Section I – 10 marks (pages 3–7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 9–21)

- Attempt Questions 11–15
- Allow about 2 hours and 45 minutes for this section

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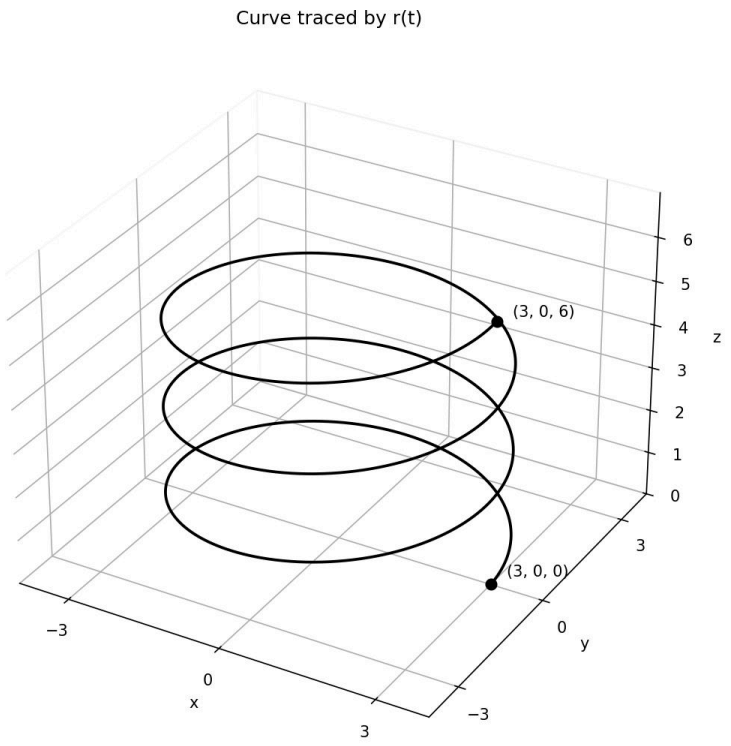
Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1

The diagram below shows a curve traced by the vector-valued function $\underset{\sim}{r}(t)$.



Which of the following best represents $\underset{\sim}{r}(t)$?

A. $\underset{\sim}{r}(t) = 3\cos t\,\underset{\sim}{i} + 3\sin t\,\underset{\sim}{j} + \frac{t}{\pi}\underset{\sim}{k}, \quad 0 \le t \le 6\pi$

B. $\underset{\sim}{r}(t) = t\cos t\,\underset{\sim}{i} + t\sin t\,\underset{\sim}{j} + \frac{t}{\pi}\underset{\sim}{k}, \quad 0 \le t \le 6\pi$

C. $\underset{\sim}{r}(t) = 3\cos t\,\underset{\sim}{i} + 3\sin t\,\underset{\sim}{j} + \frac{2t}{\pi}\underset{\sim}{k}, \quad 0 \le t \le 3\pi$

D. $\underset{\sim}{r}(t) = 3\cos t\,\underset{\sim}{i} + 3\sin t\,\underset{\sim}{j} + \frac{t}{2\pi}\underset{\sim}{k}, \quad 0 \le t \le 6\pi$

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2 Given $P = z$, $Q = z + \bar{z}$, $R = iz$. Which statement about the angle $\angle QPR$ is true?

A. Always 90°, for any z

B. Always 45°, for any z

C. Always 60°, for any z

D. None of the above

3 Let α and β be the roots of $z^2 - 2z + 4 = 0$. What is the smallest positive integer n for which $\alpha^n + \beta^n$ is a negative real number?

A. $n = 1$

B. $n = 2$

C. $n = 3$

D. $n = 4$

4 Let the complex numbers z_1, z_2, z_3 and z_4 represent the vertices of a quadrilateral A, B, C, D respectively.
Given that $z_1 - z_2 + z_3 - z_4 = 0$ and $|z_1 - z_2| = |z_2 - z_3|$, what specific type of quadrilateral is $ABCD$?

A. Parallelogram

B. Rectangle

C. Rhombus

D. Trapezium

5 The spheres $\left|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right| = p$ and $\left|\begin{pmatrix} x - l \\ y - m \\ z - n \end{pmatrix}\right| = q$, where p, q are distinct prime numbers, touch at exactly one point.

Let $k = \sqrt{l^2 + m^2 + n^2}$. Which of the following statements could be true?

A. $p + q + k = 0$

B. $|p - q| = k$

C. $p^2 + q^2 = k^2$

D. $p \times q = k$

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6 A particle undergoes simple harmonic motion. The magnitude of its maximum acceleration is three times the magnitude of its maximum velocity.

What is the period of the motion?

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. π

D. 2π

7 A Year 12 student makes the following statement:

If I achieve a Band 6 in both Mathematics Extension 2 and English Advanced, then I will enrol in at least one of a Bachelor of Medicine or a Bachelor of Music.

Consider the following situations.

1. The student does not achieve a Band 6 in Mathematics Extension 2 but enrols in a Bachelor of Medicine.

2. The student achieves a Band 6 in both Mathematics Extension 2 and English Advanced but enrols in neither a Bachelor of Medicine nor a Bachelor of Music.

3. The student does not achieve a Band 6 in Mathematics Extension 2, does not enrol in a Bachelor of Medicine, but enrols in a Bachelor of Music.

4. The student achieves a Band 6 in both Mathematics Extension 2 and English Advanced and enrols in both a Bachelor of Medicine and a Bachelor of Music.

In which of the situations is the student's statement true?

A. 1 and 3 only

B. 2 and 4 only

C. 1, 3 and 4 only

D. 1, 2, 3 and 4

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8 Let

$$S(x) : \text{“}x \text{ is a student”},$$

$$M(x) : \text{“}x \text{ studies Mathematics Extension 2”},$$

and

$$H(x, y) : \text{“}x \text{ has helped } y\text{”}.$$

The given statement is

$$\forall x \left(S(x) \rightarrow \exists y \left(M(y) \wedge H(y, x)\right)\right).$$

Which **ONE** of the following is the negation of the given statement?

A. Every student has been helped by at least one Mathematics Extension 2 student.

B. There is at least one student who has not been helped by any Mathematics Extension 2 student.

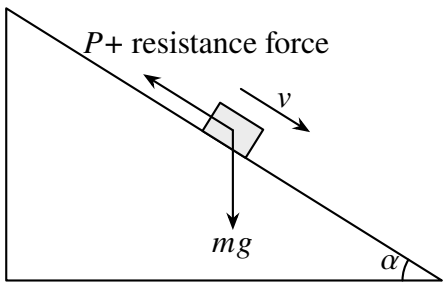
C. Every Mathematics Extension 2 student has helped at least one student.

D. There is at least one Mathematics Extension 2 student who has not helped any student.

9 A particle of mass m kg moves down a line of greatest slope of a plane inclined at an angle α to the horizontal. A constant force of magnitude P newtons acts up the plane.

The resistance to the motion is kv^2 newtons per kilogram of the particle's mass, where v is the speed of the particle and k is a positive constant.

The diagram is not to scale.



Taking the positive direction to be down the plane, which of the following differential equations describes the motion of the particle?

A. $m\dfrac{dv}{dt} = mg\sin\alpha - P - kv^2$

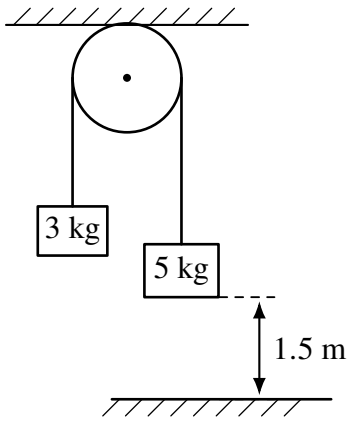
B. $m\dfrac{dv}{dt} = mg\sin\alpha - P - mkv^2$

C. $m\dfrac{dv}{dt} = mg\cos\alpha - P - kv^2$

D. $m\dfrac{dv}{dt} = mg\cos\alpha - P - mkv^2$

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10 Particles of mass 3 kg and 5 kg are attached to the ends of a light inextensible string that passes over a fixed smooth pulley, as shown. Initially, the 5 kg particle is 1.5 m above the ground, and the system is released from rest.



The 5 kg particle descends and hits the ground. The string then becomes slack, and the 3 kg particle continues to move vertically upwards.

Provided that the 3 kg particle does not hit the pulley, what is the maximum height, in metres, that it rises above its initial position?

A. $\frac{15}{8}$m

B. $\frac{15}{9}$m

C. $\frac{15}{16}$m

D. $\frac{15}{18}$m

End of Multiple Choice

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Section II

90 marks
Attempt Questions 11–15
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (19 marks) Use the Question 11 Writing Booklet

(a) Let $z = \sqrt{3} - i$ and $w = 1 + i\sqrt{3}$.

(i) Express $\dfrac{z^3}{w^2}$ in the form $a + ib$ where a and b are real. **1**

(ii) Find the smallest positive integer n such that $(\frac{z}{w})^n$ is a real number. **2**

(b) Let $\omega = e^{\dfrac{i\pi}{4}}$.

(i) Show that **1**

$$1 + \omega + \omega^2 + \omega^3 = \frac{\omega^4 - 1}{\omega - 1}.$$

(ii) Hence evaluate $1 + \omega + \omega^2 + \omega^3$, giving your answer in the form $a + ib$ where a and b are real. **2**

(c) Given $z = e^{\dfrac{i\pi}{7}}$.

(i) Show that $z^{14} = 1$. **1**

(ii) Show that **2**

$$z^2 + z^4 + z^6 + z^8 + z^{10} + z^{12} = -1.$$

(iii) Hence show that **2**

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}.$$

Question 11 continues....

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(d) Find the possible value(s) of λ if the angle between $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 1 \\ 2 \\ \lambda \end{bmatrix}$ is $\cos^{-1}\left(\frac{1}{3}\right)$. **3**

(e) Prove by contradiction that **2**

$$\sqrt{5} + \sqrt{7} > \sqrt{23}.$$

(f) Find **3**

$$\int_3^4 \frac{x+1}{\sqrt{x^2+2x+3}}\, dx.$$

Examination continues. . .

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Question 12 (19 marks) Use the Question 12 Writing Booklet

(a) Evaluate **2**

$$\int_0^{\pi/2} \sin^2\theta \cos^3\theta \, d\theta,$$

(b) Using partial fractions, find **4**

$$\int \frac{x}{(x+2)(x^2+4x+5)}\, dx.$$

(c) The vector equations of lines ℓ_1 and ℓ_2 are given below.

$$\ell_1 : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\ell_2 : \underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

(i) Show that ℓ_1 and ℓ_2 do not intersect. **3**

(ii) Find the exact value of λ for which the distance between ℓ_1 and the origin, O, is minimised. **2**

(d) Given that $z = \cos\theta + i\sin\theta$, and n is an integer show that **3**

$$\frac{z^{2n}-1}{z^{2n}+1} = i\tan(n\theta)$$

and hence find all complex numbers z on the unit circle for which this expression is real.

Question 12 continues....

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(e)

(i) Sketch on an Argand diagram the locus of points satisfying **2**

$$Arg\left(\frac{z+2}{z-2}\right) = \frac{\pi}{3}$$

(ii) Hence shade the region where **1**

$$\frac{\pi}{3} \leq Arg\left(\frac{z+2}{z-2}\right) \leq \frac{2\pi}{3}$$

(f) In the Argand diagram, the point A represents $z = 4 + 2i$. The point B lies on the line $\text{Im}(z) = 6$ and the point C lies on the line $\text{Im}(z) = -4$. **2**

It is given that $AC = 2\,AB$ and $\angle BAC = \frac{\pi}{3}$, and that C lies to the *left* of A in the Argand diagram.

Find the exact complex number representing C.

Examination continues. . .

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Question 13 (17 marks) Use the Question 13 Writing Booklet

(a) Let n be a positive odd integer. Prove, by contradiction, that $\sqrt{2n}$ is an irrational number. **2**

(b) In an Argand diagram, the points A, B and C represent the complex numbers z_1, z_2 and z_3, respectively. The triangle ABC is equilateral, and its vertices are labelled in anticlockwise order.

The triangle ABC is rotated anticlockwise through an angle α about A, where

$$0 < \alpha < \frac{\pi}{3}.$$

Under this rotation, B and C map to B' and C', which represent the complex numbers z_4 and z_5, respectively.

The line $B'C'$ intersects BC at P and AC at Q. The line AB' intersects BC at R, as shown in the diagram.

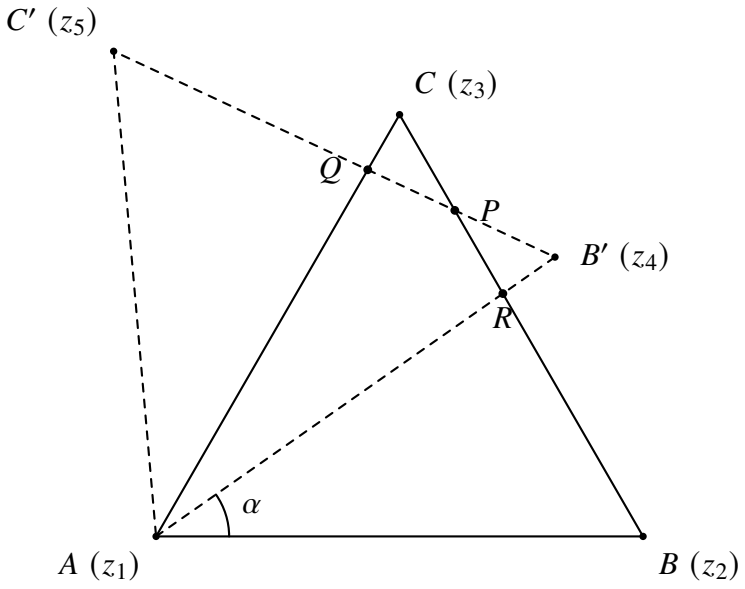


Diagram not to scale

(i) Find the complex number z_3, in terms of z_1 and z_2. **1**

(ii) Find the complex number z_5, in terms of z_1, z_2 and α. **1**

(iii) By considering the ratio $AR : AB$, show that the complex number z_R represented by the point R is **3**

$$z_R = z_1 + \frac{\sqrt{3}\, e^{i\alpha}}{\sqrt{3}\cos\alpha + \sin\alpha}\,(z_2 - z_1).$$

Question 13 continues....

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(c)

(i) Let $f(x)$ and $g(x)$ be continuous functions defined on $\mathbb{R}$. It is given that $f(x)$ is an even function and $g(x)$ is an odd function. Prove that **2**

$$\int_{-c}^{c} \frac{f(x)}{1+e^{g(x)}}\,dx = \int_{0}^{c} f(x)\,dx,$$

where c is a constant.

(ii) Hence or otherwise, evaluate **3**

$$\int_{-1}^{1} \frac{3x^2+1}{\left(1+e^{\sin^3 x}\right)\left(x^6+2x^4+x^2+1\right)}\,dx.$$

(d) A particle of unit mass moves in a straight line against a resisting force whose magnitude is numerically equal to

$$v + \lambda v^3,$$

newtons, where v is the speed of the particle and λ is a positive constant.

Initially, the particle is at the origin and has speed Q, where $Q > 0$. Let x denote the displacement of the particle from the origin and let t denote the time elapsed.

(i) Show that v is related to the displacement x by the formula **2**

$$x = \frac{1}{\sqrt{\lambda}}\left[\tan^{-1}\left(\sqrt{\lambda}\,Q\right) - \tan^{-1}\left(\sqrt{\lambda}\,v\right)\right].$$

(ii) Show that the time t which has elapsed when the particle is travelling with speed v is given by **3**

$$t = \frac{1}{2}\log_e\left(\frac{Q^2\left(1+\lambda v^2\right)}{v^2\left(1+\lambda Q^2\right)}\right).$$

Examination continues. . .

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Question 14 (17 marks) Use the Question 14 Writing Booklet

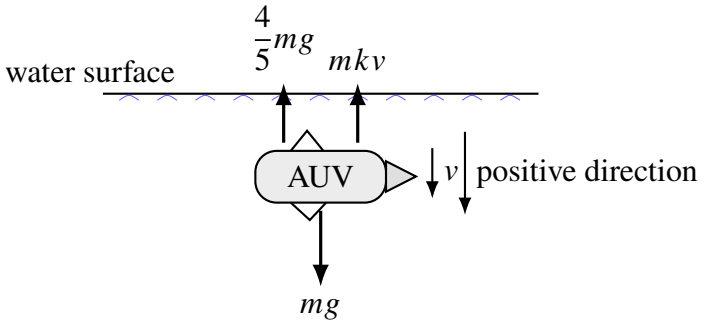
(a) An autonomous underwater vehicle (AUV) of mass m kg is descending vertically through seawater. Take the downward direction as positive.

The AUV is fully submerged in seawater and experiences:

- its weight, of magnitude mg, acting downwards;
- a constant upward buoyant force of magnitude $\frac{4}{5}mg$; and
- a resistive force of magnitude mkv, where v is its downward speed and $k > 0$.

At time $t = 0$, the AUV is descending with speed u m s^{-1}. Its thrusters are initially switched off. Assume that the AUV remains vertical and moves in a straight line.

The diagram is not drawn to scale.



While the thrusters are switched off, the speed v satisfies the following equation

$$\frac{dv}{dt} = \frac{g}{5} - kv.$$

(i) Find the terminal speed of the AUV, in terms of k, while its thrusters are switched off. **1**

(ii) Show that **2**

$$v = \frac{g}{5k} + \left(u - \frac{g}{5k}\right) e^{-kt}.$$

(iii) If x is the distance descended after t seconds, show that **2**

$$x = \frac{g}{5k}t + \frac{1}{k}\left(u - \frac{g}{5k}\right)\left(1 - e^{-kt}\right).$$

Question 14 continues....

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(iv) In normal operation, the AUV switches on its thrusters when its speed first reaches the value halfway between its initial speed and its terminal speed. Assume $u \neq \frac{g}{5k}$. The thrusters then exert a constant upward force of magnitude $\frac{1}{5}mg$.

(a) Find the time, in terms of k, at which the thrusters switch on. **2**

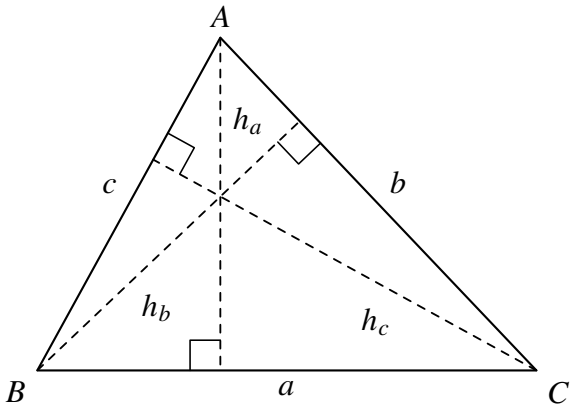
(b) After the thrusters switch on, by considering the new equation of the motion, show that the speed approaches zero as $t \to \infty$. **2**

(b) Let a, b and c be the side lengths of triangle ABC. **3**

Let h_a, h_b and h_c denote the altitudes from vertices A, B and C to their respective opposite sides. Let the area of triangle ABC be M.

It is given that $a^2 + b^2 \geq 2ab$ for positive real numbers a and b. **Do not prove this.**

Diagram not drawn to scale



Show that

$$\left(\frac{a^2 + b^2}{c}\right) h_c + \left(\frac{b^2 + c^2}{a}\right) h_a + \left(\frac{c^2 + a^2}{b}\right) h_b \geq 12M.$$

Question 14 continues....

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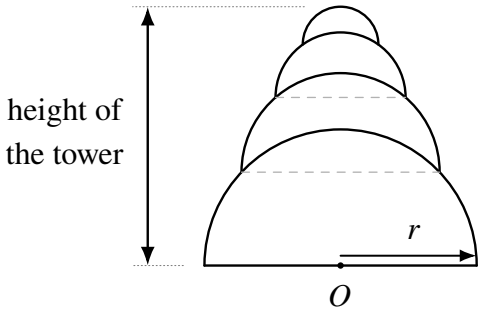
(c)

(i) Let x, y and k be positive real numbers. Show that **2**

$$x + y\sqrt{k} \leq \sqrt{x^2 + y^2}\sqrt{k+1}.$$

(ii) A hemispherical bubble of radius r forms the lowest chamber of a bubble tower. A smaller hemispherical bubble is placed on it so that the circular base of the new hemisphere is a horizontal cross-section of the hemisphere immediately below it. Further, smaller hemispherical bubbles are placed in the same way. The process continues until the tower consists of n hemispherical chambers. **3**

The diagram below shows a possible tower with $n = 4$ and the geometry at one stage of the construction.



A tower with $n = 4$

Geometry at one stage

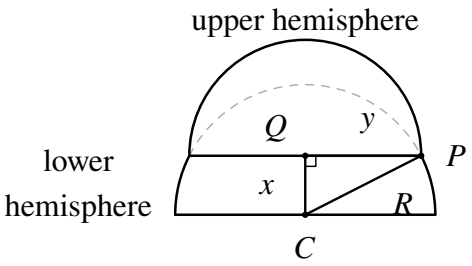


Diagram not drawn to scale

At the stage shown, C is the centre of the lower hemisphere, $CP = R$, $CQ = x$ and $QP = y$. Hence,

$$x^2 + y^2 = R^2.$$

Using mathematical induction and the result from part c (i), prove that the maximum possible height of a tower consisting of n hemispherical chambers, whose lowest hemisphere has radius r, is

$$r\sqrt{n}.$$

Examination continues. . .

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Question 15 (18 marks) Use the Question 15 Writing Booklet

(a) An aircraft is approaching an airport with constant velocity

$$\underset{\sim}{v} = 0.4\underset{\sim}{i} - 0.4\underset{\sim}{j} - 0.1\underset{\sim}{k} \quad \text{km s}^{-1}.$$

(You may assume that the velocity already contains the effects of gravity, air resistance, thrust and buoyancy)

The aircraft is first observed by the control tower at time $t = 0$ seconds. Sixty seconds later, the aircraft passes directly above a navigation beacon. The position vector of the navigation beacon relative to the base of the control tower is

$$\underset{\sim}{i} + 2\underset{\sim}{j} \quad \text{km},$$

and the aircraft is at an altitude of 2 km at this time.

Assume that the aircraft continues to travel with constant velocity until it lands.

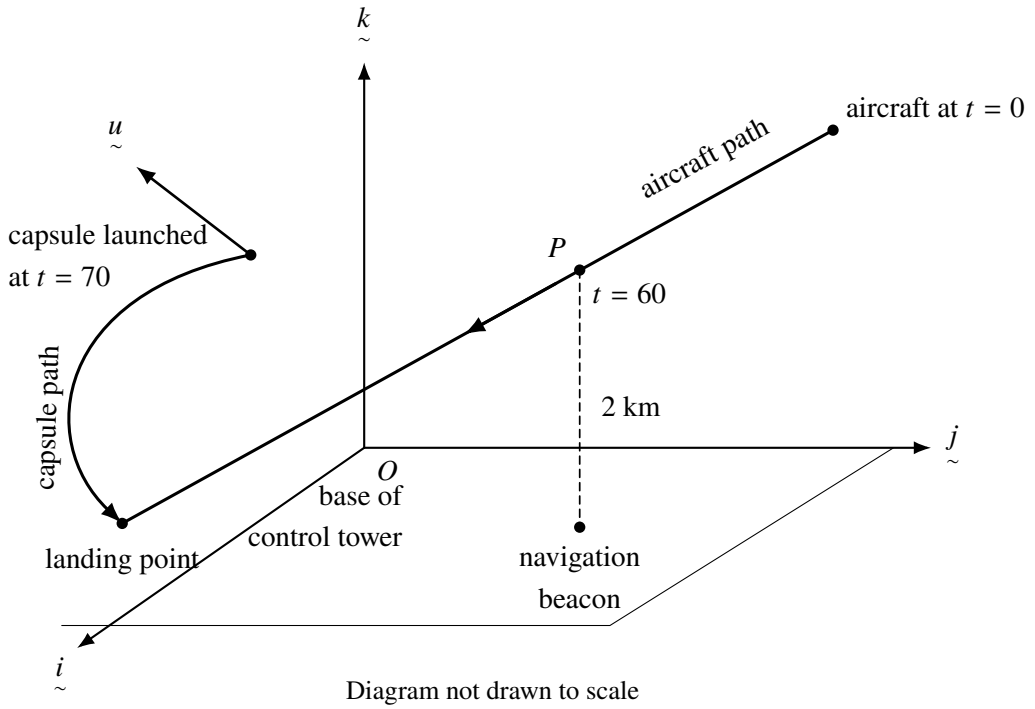


Diagram not drawn to scale

(i) Show that the position vector of the aircraft relative to the base of the control tower at time t seconds is **2**

$$\underset{\sim}{r}(t) = (0.4t - 23)\underset{\sim}{i} + (26 - 0.4t)\underset{\sim}{j} + (8 - 0.1t)\underset{\sim}{k}.$$

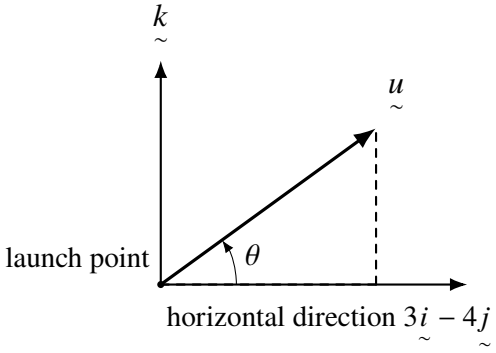
(ii) Hence, find the time taken for the aircraft to land and the position vector of the landing point relative to the base of the control tower at this time. **2**

Question 15 continues....

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At time $t = 70$ seconds, an emergency supply capsule is launched from an emergency response helicopter whose position vector relative to the base of the control tower is $3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ in km at this time.

The capsule has initial speed $\frac{1}{\sqrt{2}}$ km s^{-1}. The horizontal component of its initial velocity is in the direction $3\underset{\sim}{i} - 4\underset{\sim}{j}$, and the capsule is projected at an angle θ above the horizontal.



The initial velocity of the capsule can be written as

$$\underset{\sim}{u} = \frac{1}{\sqrt{2}}\cos\theta\left(\frac{3}{5}\underset{\sim}{i} - \frac{4}{5}\underset{\sim}{j}\right) + \frac{1}{\sqrt{2}}\sin\theta\,\underset{\sim}{k}. \qquad \textbf{Do not prove this}$$

During its flight, the capsule has constant acceleration

$$\underset{\sim}{a} = \frac{1}{225}\left(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k}\right) \quad \text{km s}^{-2},$$

due to the combined effects of gravity, air resistance,thrust and buoyancy.

(iii) Determine the angle of projection θ so that the emergency supply capsule arrives at the aircraft's landing position 5 seconds after the aircraft has landed. **4**

Question 15 continues....

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(b) For each positive integer n, let

$$I_n = \int_0^{\pi} e^{-x}(\pi - x)^n \, dx.$$

It is given that $I_1 = \pi + \dfrac{1}{e^{\pi}} - 1$. **Do not prove this.**

(i) Show that **2**

$$I_{n+1} = \pi^{n+1} - (n+1)I_n.$$

(ii) By using repeated application of (b) (i), show that **3**

$$(-1)^n \frac{I_n}{n!} + \frac{1}{e^{\pi}} = \sum_{k=0}^{n} (-1)^k \frac{\pi^k}{k!}.$$

i.e.

$$(-1)^n \frac{I_n}{n!} + \frac{1}{e^{\pi}} = 1 - \pi + \frac{\pi^2}{2!} - \frac{\pi^3}{3!} + \frac{\pi^4}{4!} - \cdots + (-1)^n \frac{\pi^n}{n!}.$$

[You may not use mathematical induction to show this.]

It is given that

$$\lim_{n \to \infty} \frac{\pi^n}{n!} = 0. \qquad \textbf{Do not prove this}$$

(iii) Hence, or otherwise, show that **2**

$$\frac{1}{e^{\pi}} = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^k}{k!}.$$

i.e.

$$\frac{1}{e^{\pi}} = 1 - \pi + \frac{\pi^2}{2!} - \frac{\pi^3}{3!} + \frac{\pi^4}{4!} - \cdots$$

Question 15 continues....

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(c) The diagram shows a pyramid with **parallelogram** base $OABC$ and vertex D. **3**
Let M be the midpoint of AB, and let L be the point on CD such that

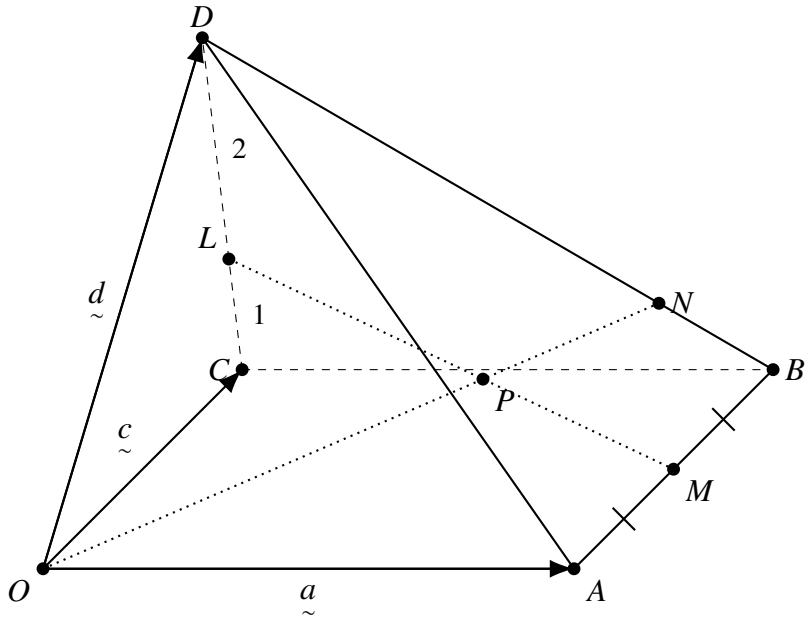
$$CL : LD = 1 : 2.$$

Taking O as the origin, let

$$\overrightarrow{OA} = \underset{\sim}{a}, \quad \overrightarrow{OC} = \underset{\sim}{c}, \quad \overrightarrow{OD} = \underset{\sim}{d},$$

Let P be the point on the line segment LM such that the ray from O through P intersects the edge BD at N.

Diagram not drawn to scale



Using vector method, determine the ratio

$$DN : NB.$$

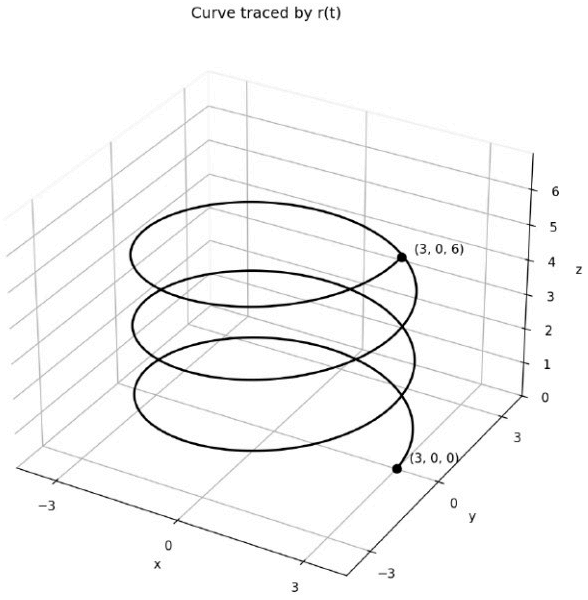
End of Exam

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Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
A	D	B	C	B	B	C	B	B	A

1

The diagram below shows a curve traced by the vector-valued function $\underset{\sim}{r}(t)$.



Which of the following best represents $\underset{\sim}{r}(t)$?

A. $\underset{\sim}{r}(t) = 3\cos t\,\underset{\sim}{i} + 3\sin t\,\underset{\sim}{j} + \frac{t}{\pi}\underset{\sim}{k}, \quad 0 \le t \le 6\pi$ $t = 6\pi$ $p = 3$

B. $\underset{\sim}{r}(t) = t\cos t\,\underset{\sim}{i} + t\sin t\,\underset{\sim}{j} + \frac{t}{\pi}\underset{\sim}{k}, \quad 0 \le t \le 6\pi$ ✗

C. $\underset{\sim}{r}(t) = 3\cos t\,\underset{\sim}{i} + 3\sin t\,\underset{\sim}{j} + \frac{2t}{\pi}\underset{\sim}{k}, \quad 0 \le t \le 3\pi$ $t = 3\pi$ ✗ $p = 3/2$

D. $\underset{\sim}{r}(t) = 3\cos t\,\underset{\sim}{i} + 3\sin t\,\underset{\sim}{j} + \frac{t}{2\pi}\underset{\sim}{k}, \quad 0 \le t \le 6\pi$ $t = 12\pi$ ✗

2 Given $P = z$, $Q = z + \bar{z}$, $R = iz$. Which statement about the angle $\angle QPR$ is true?

$z = 1 \quad \bar{z} = 1$
$iz = i$

A. Always 90°, for any z

B. Always 45°, for any z

C. Always 60°, for any z

D. None of the above

---

3 Let α and β be the roots of $z^2 - 2z + 4 = 0$. What is the smallest positive integer n for which $\alpha^n + \beta^n$ is a negative real number?

$(z-1)^2 = 3i^2$
$z = 1 \pm \sqrt{3}i$
$\alpha = 2\operatorname{cis}\frac{\pi}{3} \quad \beta = 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$
$\alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3}$

A. $n = 1$

B. $n = 2$

C. $n = 3$

D. $n = 4$

4 Let the complex numbers z_1, z_2, z_3 and z_4 represent the vertices of a quadrilateral A, B, C, D respectively.
Given that $z_1 - z_2 + z_3 - z_4 = 0$ and $|z_1 - z_2| = |z_2 - z_3|$, what specific type of quadrilateral is $ABCD$?

A. Parallelogram

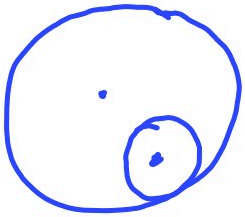
B. Rectangle

C. Rhombus

D. Trapezium

5 The spheres $\left|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right| = p$ and $\left|\begin{pmatrix} x-l \\ y-m \\ z-n \end{pmatrix}\right| = q$, where p, q are distinct prime numbers, touch at exactly one point.

Let $k = \sqrt{l^2 + m^2 + n^2}$. Which of the following statements could be true?



A. $p + q + k = 0$

B. $|p - q| = k$

C. $p^2 + q^2 = k^2$

D. $p \times q = k$

6 A particle undergoes simple harmonic motion. The magnitude of its maximum acceleration is three times the magnitude of its maximum velocity.

What is the period of the motion?

$x = a\cos(nt+\alpha)$
$\dot{x} = -na\sin(nt+\alpha)$
$\ddot{x} = -n^2 a\cos nt$
$n^2 a = 3(na)$
$\therefore n = 3$
$P = \frac{2\pi}{3}$

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. π

D. 2π

.....

7 A Year 12 student makes the following statement:

If I achieve a Band 6 in both Mathematics Extension 2 and English Advanced, then I will enrol in at least one of a Bachelor of Medicine or a Bachelor of Music.

P
Q
$P \Rightarrow Q$ all T except P is T and Q is F

Consider the following situations.

✓ **1.** The student does not achieve a Band 6 in Mathematics Extension 2 but enrols in a Bachelor of Medicine.

✗ **2.** The student achieves a Band 6 in both Mathematics Extension 2 and English Advanced but enrols in neither a Bachelor of Medicine nor a Bachelor of Music.

✓ **3.** The student does not achieve a Band 6 in Mathematics Extension 2, does not enrol in a Bachelor of Medicine, but enrols in a Bachelor of Music.

✓ **4.** The student achieves a Band 6 in both Mathematics Extension 2 and English Advanced and enrols in both a Bachelor of Medicine and a Bachelor of Music.

In which of the situations is the student's statement true?

A. 1 and 3 only

B. 2 and 4 only

C. 1, 3 and 4 only

D. 1, 2, 3 and 4

.....

.....

.....

.....

8 Let

$$S(x) : \text{“}x \text{ is a student”},$$

$$M(x) : \text{“}x \text{ studies Mathematics Extension 2”},$$

and

$$H(x, y) : \text{“}x \text{ has helped } y\text{”}.$$

The given statement is

$$\neg\left(\forall x\,(S(x) \rightarrow \exists y\,(M(y) \wedge H(y,x)))\right).$$

$\equiv \exists x, S(x) \wedge \forall y\,(\neg M(y) \vee \neg H(y,x))$

Which **ONE** of the following is the negation of the given statement?

A. Every student has been helped by at least one Mathematics Extension 2 student.

B. There is at least one student who has not been helped by any Mathematics Extension 2 student.

C. Every Mathematics Extension 2 student has helped at least one student.

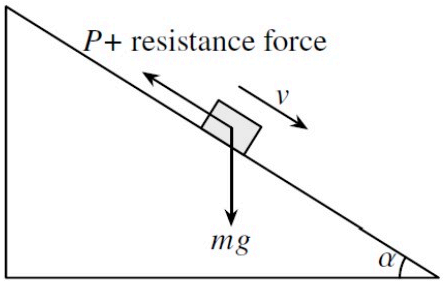
D. There is at least one Mathematics Extension 2 student who has not helped any student.

---

9 A particle of mass m kg moves down a line of greatest slope of a plane inclined at an angle α to the horizontal. A constant force of magnitude P newtons acts up the plane.

The resistance to the motion is kv^2 newtons per kilogram of the particle's mass, where v is the speed of the particle and k is a positive constant.

The diagram is not to scale.



Taking the positive direction to be down the plane, which of the following differential equations describes the motion of the particle?

A. $m\dfrac{dv}{dt} = mg\sin\alpha - P - kv^2$

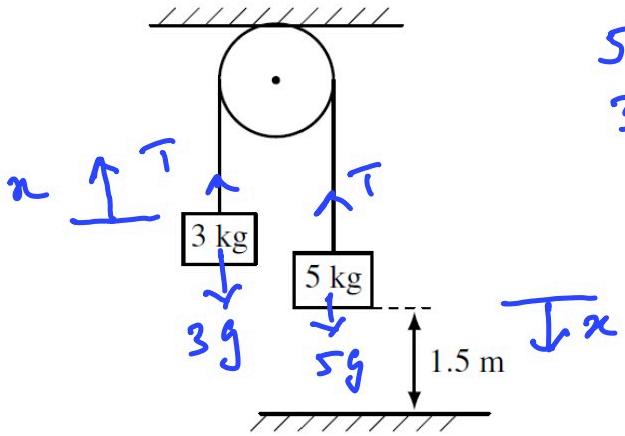
B. $m\dfrac{dv}{dt} = mg\sin\alpha - P - mkv^2$

C. $m\dfrac{dv}{dt} = mg\cos\alpha - P - kv^2$

D. $m\dfrac{dv}{dt} = mg\cos\alpha - P - mkv^2$

---

10 Particles of mass 3 kg and 5 kg are attached to the ends of a light inextensible string that passes over a fixed smooth pulley, as shown. Initially, the 5 kg particle is 1.5 m above the ground, and the system is released from rest.



$5\ddot{x} = 5g - T$

$3\ddot{x} = -3g + T$

$8\ddot{x} = 2g$

$\ddot{x} = \frac{1}{4}g$

The 5 kg particle descends and hits the ground. The string then becomes slack, and the 3 kg particle continues to move vertically upwards.

Provided that the 3 kg particle does not hit the pulley, what is the maximum height, in metres, that it rises above its initial position?

A. $\frac{15}{8}$m

B. $\frac{15}{9}$m

C. $\frac{15}{16}$m

D. $\frac{15}{18}$m

$\int_0^v v\,dv = \int_0^x \frac{1}{4}g\,dx$

$v^2 = \frac{1}{2}gx$

At $x = 1.5$

$v^2 = \frac{3g}{4}$

For 3 kg particle: $\ddot{x} = -g$

$\int_{\frac{\sqrt{3g}}{2}}^{0} v\,dv = \int_0^d -g\,dx$

$-\frac{3g}{8} = -g(d)$

$d = \frac{3}{8}$ ∴ Height above $= \frac{3}{8} + \frac{3}{2}$

$= \frac{15}{8}$ m

Question 11 (19 marks) Use the Question 11 Writing Booklet

(a) Let $z = \sqrt{3} - i$ and $w = 1 + i\sqrt{3}$.

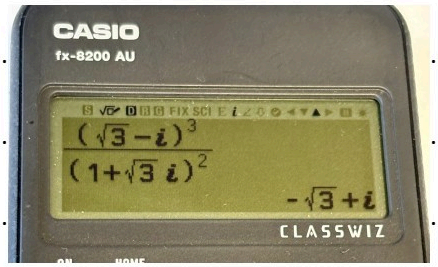
(i) Express $\frac{z^3}{w^2}$ in the form $a + ib$ where a and b are real. **1**

(ii) Find the smallest positive integer n such that $(\frac{z}{w})^n$ is a real number. **2**

(i) method 1 Notice $w = iz$

$\therefore \frac{z^3}{w^2} = \frac{z^3}{-z^2} = -z = -\sqrt{3} + i$ ✓

method 2



method 3 $z = 2\,\mathrm{cis}\left(-\frac{\pi}{6}\right)$

$w = 2\,\mathrm{cis}\left(\frac{\pi}{3}\right)$

$\frac{z^3}{w^2} = 2\,\mathrm{cis}\left(-\frac{\pi}{2} - \frac{2\pi}{3}\right)$

$= 2\,\mathrm{cis}\left(\frac{5\pi}{6}\right)$

$= -\sqrt{3} + i$

(ii)

method 1

$\frac{z}{w} = -i$ ✓

$\therefore \left(\frac{z}{w}\right)^2 = 1 \quad \therefore n = 2$ ✓

(a)(i) and (b)(ii)

Some students spent too much time carrying out unnecessary arithmetic calculations. A calculator should be used where possible.

Method 2

$\frac{z}{w} = \text{cis}\left(-\frac{\pi}{6} - \frac{\pi}{3}\right) = \text{cis}\left(-\frac{\pi}{2}\right)$

$\therefore \left(\frac{z}{w}\right)^n = \text{cis}\left(-\frac{n\pi}{2}\right)$ ✓

since $\left(\frac{z}{w}\right)^n$ is real

$\sin\left(\frac{n\pi}{2}\right) = 0$

$\frac{n\pi}{2} = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$

$n = 2k$: $k = 0, \pm 1, \pm 2, \ldots$

$\therefore n = 2$ ✓

(b) Let $\omega = e^{\frac{i\pi}{4}}$.

(i) Show that **1**

$$1 + \omega + \omega^2 + \omega^3 = \frac{\omega^4 - 1}{\omega - 1}.$$

(ii) Hence evaluate $1 + \omega + \omega^2 + \omega^3$, giving your answer in the form $a + ib$ where a and b are real. **2**

(i) $(1+\omega+\omega^2+\omega^3)(1-\omega)$

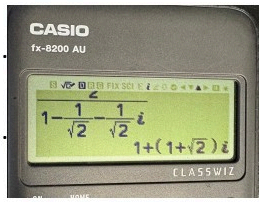
$= 1+\omega+\omega^2+\omega^3-\omega-\omega^2-\omega^3-\omega^4$ ✓

$= 1-\omega^4$

since $\omega \neq 1$

$1+\omega+\omega^2+\omega^3 = \frac{1-\omega^4}{1-\omega}$

(ii) $1+\omega+\omega^2+\omega^3 = \frac{1-e^{i\pi}}{1-e^{\frac{i\pi}{4}}}$ ✓



$= \frac{2}{1-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i}$

$= 1 + (1+\sqrt{2})i$ ✓

(b)(i)

Full working must be shown. Simply stating $1 + \omega + \omega^2 + \omega^3 = \dfrac{\omega^4 - 1}{\omega - 1}$ by GP, without justification, is not sufficient.

(c)

This part was generally done very well.

(c) Given $z = e^{\frac{i\pi}{7}}$.

(i) Show that $z^{14} = 1$. **1**

(ii) Show that **2**

$$z^2 + z^4 + z^6 + z^8 + z^{10} + z^{12} = -1.$$

(iii) Hence show that **2**

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}.$$

(i) $z^{14} = e^{i(2\pi)} = \cos(2\pi) + i\sin(2\pi)$

$= 1$ ✓

(ii)

$z^2 + z^4 + z^6 + z^8 + z^{10} + z^{12}$ is a GP

with $a = r = z^2$ and $n = 6$

$\therefore z^2 + z^4 + z^6 + \cdots + z^{12} = \dfrac{z^2((z^2)^6 - 1)}{z^2 - 1}$ ✓

$= \dfrac{z^{14} - z^2}{z^2 - 1}$

$= \dfrac{1 - z^2}{z^2 - 1}$ ✓

$= -1 \quad (z^2 \neq 1)$

(iii) $z^2 + z^4 + z^6 + z^8 + z^{10} + z^{12} = -1$

$\div z^7 \ (\neq 0)$ both sides

$$z^{-5} + z^{-3} + z^{-1} + z + z^3 + z^5 = \frac{-1}{z^7}$$ ✓

$z^7 = e^{i\pi} = -1$

and $z^n + z^{-n} = \text{Cis}\frac{n\pi}{7} + \text{Cis}\left(-\frac{n\pi}{7}\right)$

$= 2\cos\frac{n\pi}{7}$ ✓

$\therefore \ 2\cos\frac{5\pi}{7} + 2\cos\frac{\pi}{7} + 2\cos\frac{3\pi}{7} = \frac{-1}{7} = 1$

$\therefore \ \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$

Method 2 $\quad z = \text{Cis}\frac{\pi}{7}$

$z^2 + z^4 + \cdots + z^{12} = -1$

$\text{Cis}\frac{2\pi}{7} + \text{Cis}\frac{4\pi}{7} + \cdots + \text{Cis}\frac{12\pi}{7} = -1$

Equating the real parts

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cdots + \cos\frac{12\pi}{7} = -1$ ✓

$\cos\frac{2\pi}{7} = -\cos\frac{5\pi}{7}, \quad \cos\frac{6\pi}{7} = -\cos\frac{\pi}{7}$

$\cos\frac{4\pi}{7} = -\cos\frac{3\pi}{7} \quad \cos\frac{8\pi}{7} = -\cos\frac{\pi}{7}$

$\cos\frac{10\pi}{7} = -\cos\frac{2\pi}{7}$, $\cos\frac{12\pi}{7} = -\cos\frac{5\pi}{7}$ ✓

$\therefore -2\left(\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}\right) = -1$

$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$

Using $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}$ without proving may not be advisable. The result could instead have been obtained directly using the GP formula.

(d) Find the possible value(s) of λ if the angle between $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 2 \\ \lambda \end{pmatrix}$ is $\cos^{-1}\left(\frac{1}{3}\right)$.

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \cos\theta = \frac{1}{3}$

$\frac{2-2+2\lambda}{\sqrt{9}\sqrt{5+\lambda^2}} = \frac{1}{3}$ ✓

$2\lambda = \sqrt{5+\lambda^2}$ and $\lambda > 0$

$4\lambda^2 = 5+\lambda^2$ ✓

$\lambda = \pm\sqrt{\frac{5}{3}}$

$\lambda > 0 \quad \therefore \lambda = \sqrt{5/3}$ ✓

(e) Prove by contradiction that **2**

$$\sqrt{5} + \sqrt{7} > \sqrt{23}.$$

Assume $\sqrt{5} + \sqrt{7} \leq \sqrt{23}$ ✓

$12 + 2\sqrt{35} \leq 23$

$2\sqrt{35} \leq 11$

$\sqrt{140} \leq \sqrt{121}$ which is a contradiction ✓

$\therefore \sqrt{5} + \sqrt{7} > \sqrt{23}$

(f) Find

$$\int_3^4 \frac{x+1}{\sqrt{x^2+2x+3}}\,dx.$$

$$= \frac{1}{2}\int_3^4 (2x+2)(x^2+2x+3)^{-1/2}\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2+2x+3)^{1/2}}{1/2}\right]_3^4$$

$$= \sqrt{27} - \sqrt{18}$$

$$= 3\sqrt{3} - 3\sqrt{2}$$

(f)

Many students used substitution. However, this question can be solved more simply and efficiently by applying the reverse chain rule.

Question 12

a) $\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^3\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta (1 - \sin^2\theta) \, d\theta$

$= \int_0^{\frac{\pi}{2}} (\sin^2\theta \cos\theta - \sin^4\theta \cos\theta) \, d\theta$ (1)

$= \left[\frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5} \right]_0^{\frac{\pi}{2}}$

$= \frac{\sin^3\frac{\pi}{2}}{3} - \frac{\sin^5\frac{\pi}{2}}{5} - 0 + 0$

$= \frac{1}{3} - \frac{1}{5}$

$= \frac{2}{15}$ (1)

OR let $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$du = \cos\theta \, d\theta$

when $\theta = 0$, $u = 0$

$\theta = \frac{\pi}{2}$, $u = 1$

$\therefore \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \cos\theta \, d\theta$

$= \int_0^1 u^2 (1 - u^2) \, du$ (1)

$= \int_0^1 (u^2 - u^4) \, du$

$= \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^1$

$= \frac{1}{3} - \frac{1}{5} - 0 + 0$

$= \frac{2}{15}$ (1)

Question was done very well.
Please show the substitution step with the $-0+0$.

b) $\dfrac{x}{(x+2)(x^2+4x+5)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4x+5}$ (1)

$x = A(x^2+4x+5) + (Bx+C)(x+2)$

when $x=-2$

$-2 = A((-2)^2+4(-2)+5) + (B(-2)+C)(-2+2)$

$-2 = A$

let $x=0$

$0 = -2(0^2+4(0)+5) + (B(0)+C)(0+2)$

$0 = -10+2C$

$C = 5$

let $x=1$

$1 = -2(1^2+4(1)+5) + (B(1)+5)(1+2)$

$1 = -2(10) + 3B + 15$

$1 = -5 + 3B$

$3B = 6$

$B = 2$

$\therefore \int \dfrac{x}{(x+2)(x^2+4x+5)}\,dx$

$= \int \dfrac{-2}{x+2} + \dfrac{2x+5}{x^2+4x+5}\,dx$ (1)

$= -2\ln|x+2| + \int \dfrac{2x+4}{x^2+4x+5}\,dx + \int \dfrac{1}{x^2+4x+5}\,dx$ (1)

$= -2\ln|x+2| + \ln|x^2+4x+5| + \int \dfrac{dx}{1+(x+2)^2}$

$= \ln\left|\dfrac{x^2+4x+5}{(x+2)^2}\right| + \tan^{-1}(x+2) + C$ (1)

Done well, all students had the absolute value.

c) i) equating l_1 and l_2

$1+\lambda = 3+2\mu$ ①

$2-\lambda = -1+\mu$ ②

$-1+2\lambda = 4-\mu$ ③

} ①

② + ③:

$1+\lambda = 3$

$\lambda = 2$

sub. into ②.

$2-2 = -1+\mu$

$\mu = 1$

} ①

when $\lambda = 2$, $\mu = 1$ for ①

LHS $= 1+\lambda$

$= 1+2$

$= 3$

RHS $= 3+2(1)$

$= 5$

$\neq$ LHS

} ① the setting out of this step was done very poorly.

$\therefore$ the lines do not intersect.

OR using ① & ②:

$\mu = \frac{1}{3}$, $\lambda = \frac{8}{3}$

for ③:

LHS $= \frac{13}{3}$

RHS $= \frac{11}{3}$

$\neq$ LHS

ii) $\begin{pmatrix} 1+\lambda \\ 2-\lambda \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$ ①

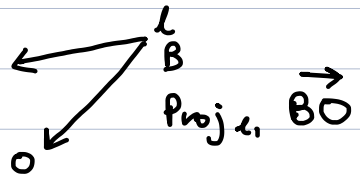
$(1+\lambda) + (-1)(2-\lambda) + 2(-1+2\lambda) = 0$

$1+\lambda - 2+\lambda - 2+4\lambda = 0$

$6\lambda = 3$

$\lambda = \frac{1}{2}$ ①

OR



$$\text{Proj}_{\begin{pmatrix}1\\-1\\2\end{pmatrix}}\begin{pmatrix}-1\\-2\\1\end{pmatrix} = \frac{\begin{pmatrix}-1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}}{1^2+(-1)^2+2^2}\begin{pmatrix}1\\-1\\2\end{pmatrix} \quad (1)$$

$$= \frac{3}{6}\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

perpendicular: $\begin{pmatrix}1\\2\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$\therefore \lambda = \frac{1}{2}$ (1)

OR

$$\text{distance} = \sqrt{(1+\lambda)^2 + (2-\lambda)^2 + (-1+2\lambda)^2} = \sqrt{6(\lambda^2-\lambda+1)} \quad (1)$$

minimise $\lambda^2-\lambda+1$

$$\lambda = \frac{-b}{2a} = \frac{1}{2}$$

should show why it is a minimum

as leading coefficient > 0 concave up parabola,

$\therefore \lambda = \frac{1}{2}$ is minimum (1)

d) $\dfrac{z^{2n}-1}{z^{2n}+1} = \dfrac{z^n - z^{-n}}{z^n + z^{-n}}$

$$= \frac{\text{cis}(n\theta) - \text{cis}(-n\theta)}{\text{cis}(n\theta) + \text{cis}(-n\theta)} \quad \text{(De Moivre's Theorem)}$$

$$= \frac{\cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + i\sin(n\theta)}{\cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)} \quad (1) \text{ proof}$$

$$= \frac{2i\sin(n\theta)}{2\cos(n\theta)} = i\tan(n\theta)$$

OR $\dfrac{z^{2n}-1}{z^{2n}+1} = i\tan(n\theta)$

$$\text{LHS} = \frac{(\cos\theta + i\sin\theta)^{2n} - 1}{(\cos\theta + i\sin\theta)^{2n} + 1}$$

$$= \frac{\cos 2n\theta + i\sin 2n\theta - 1}{\cos 2n\theta + i\sin 2n\theta + 1}$$

$$= \frac{1 - 2\sin^2 n\theta + i\,2\sin n\theta\cos n\theta - 1}{2\cos^2 n\theta - 1 + i\,2\sin n\theta\cos n\theta + 1}$$

$$= \frac{2\sin n\theta\,(-\sin n\theta + i\cos n\theta)}{2\cos n\theta\,(\cos n\theta + i\sin n\theta)}$$

$$= \frac{\tan n\theta\,(i)(\cos n\theta + i\sin n\theta)}{\cos n\theta + i\sin n\theta}$$

$$= i\tan(n\theta)$$

① proof

for real $\tan n\theta = 0$ ①

$$n\theta = 0 \pm k\pi, \quad k \in \mathbb{Z}$$

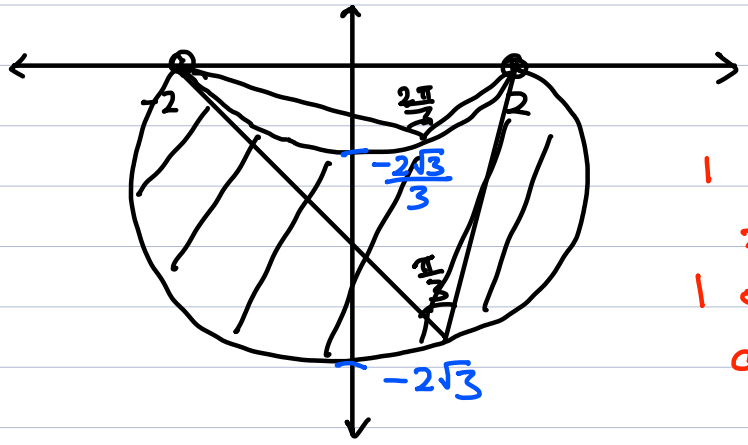
$$\theta = \frac{k\pi}{n}$$

$$\therefore z = \cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}$$

① need to state what k is.

$k = 0, 1, 2, 3, \ldots, 2n-1$

e)

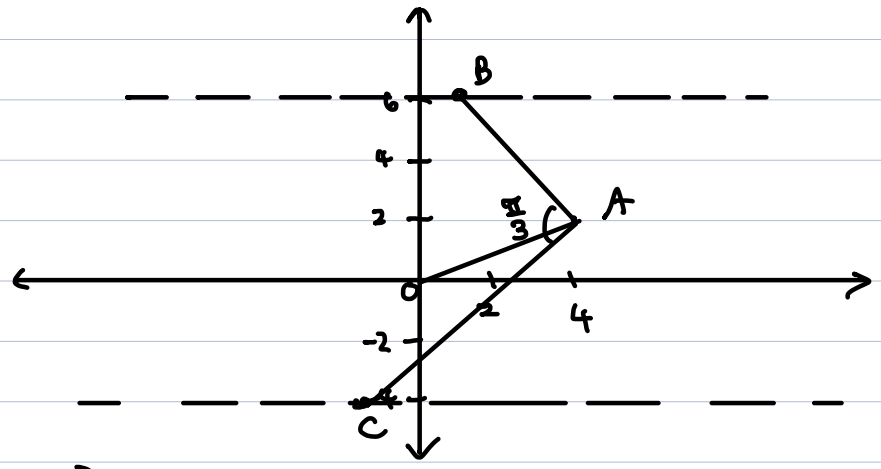


1 arc below x-axis

1 open circle and angle

1 region

f)



let $\vec{OB} = x + 6i$ $\qquad \vec{OA} = 4 + 2i$

$\vec{AB} = x - 4 + 6i - 2i$

$= x - 4 + 4i$

$\vec{AC} = y - 4 + i(-4 - 2)$

$= y - 4 - 6i$

$\vec{AC} = \vec{AB} \times 2 \text{ cis } \frac{\pi}{3}$

$y - 4 - 6i = 2(x - 4 + 4i)\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

$= 2(x - 4 + 4i)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= (x - 4 + 4i)(1 + i\sqrt{3})$ (1)

$= x - 4 + (x - 4)(i\sqrt{3}) + 4i - 4\sqrt{3}$

$y - 4 - 6i = x - 4 - 4\sqrt{3} + i(4 + \sqrt{3}x - 4\sqrt{3})$

equating real and imaginary parts

Re: $y - 4 = x - 4 - 4\sqrt{3}$

Im: $-6 = 4 + \sqrt{3}x - 4\sqrt{3}$

$\sqrt{3}x = -10 + 4\sqrt{3}$

$x = 4 - \frac{10}{\sqrt{3}}$

sub. into Re

$\therefore y - 4 = 4 - \frac{10}{3}\sqrt{3} - 4 - 4\sqrt{3}$

$y = 4 - \frac{22}{3}\sqrt{3}$

$\therefore \vec{OC} = -\frac{22\sqrt{3}}{3} + 4 - 4i$ (1)

Mathematics Extension 2 2026: Task 4

Solutions: Question 13

Question 13(a) *[2 marks]* Let n be a positive odd integer. Prove, by contradiction, that $\sqrt{2n}$ is an irrational number.

Solution

Suppose, for contradiction, that $\sqrt{2n}$ is rational. Then

$$\sqrt{2n} = \frac{p}{q},$$

where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$. Squaring gives

$$p^2 = 2nq^2.$$

Thus p^2 is even, so p is even. Write $p = 2k$ for some $k \in \mathbb{Z}$. Then

$$p^2 = 4k^2 = 2nq^2 \quad \Longrightarrow \quad 2k^2 = nq^2.$$

The left-hand side is even. Since n is odd, q^2 must be even. Hence q must be even. Therefore p and q have a common factor of 2; that is, $\gcd(p, q) \neq 1$.

So $\gcd(p, q) = 1$ and $\gcd(p, q) \neq 1$: a contradiction. Hence

$$\boxed{\sqrt{2n} \text{ is irrational.}}$$

Marker's Comments

First mark
Awarded for correctly beginning the proof by contradiction: assume $\sqrt{2n} = p/q$, where p and q are coprime integers, and obtain

$$p^2 = 2nq^2.$$

Second mark
Awarded for completing a valid parity argument. From $p^2 = 2nq^2$, deduce that p is even and write $p = 2m$. Then

$$nq^2 = 2m^2.$$

Since n is odd, q^2, and hence q, must be even. Thus p and q are both even, contradicting $\gcd(p, q) = 1$.

Issues observed

- The most common error was

  $$q^2 = \frac{2m^2}{n} = 2\left(\frac{m^2}{n}\right)$$

  followed by the unsupported conclusion that q^2 is even. It was not established that m^2/n is an integer. Why is this necessary? The definition of 'even' requires that $q^2 = 2k$ for some **integer** k. The argument was recoverable from here, but no one did this.
- Some responses proved that p was even but did not validly prove that q was even. These received the first mark only.
- Some responses did not use the fact that n is odd. The required inference is

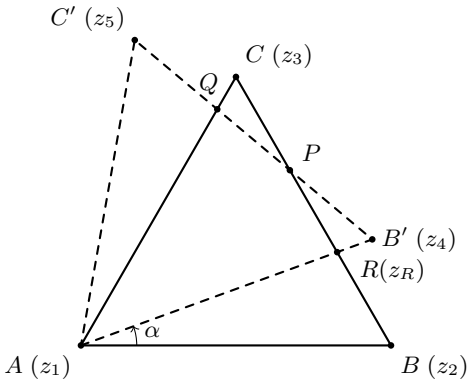
  $$nq^2 \text{ even}, \quad n \text{ odd} \quad \Longrightarrow \quad q^2 \text{ even}.$$
- The implication

  $$q^2 \mid 2m^2 \quad \Longrightarrow \quad q^2 \mid 2 \text{ or } q^2 \mid m^2$$

  is not valid in general. However, $q^2 \mid 2$ is justified if the response establishes $\gcd(q^2, m^2) = 1$ and applies the coprime form of Euclid's lemma (which no one did because it's not part of the course).
- A factor of 2 in $p^2/q^2 = 2n$ does not imply $\gcd(p^2, q^2) = 2$. Since p and q are coprime, $\gcd(p^2, q^2) = 1$.
- Some responses introduced $\sqrt{2}\,q$ as a denominator and treated its non-integrality as a contradiction. An algebraically introduced denominator is not required to be an integer.
- Isolated algebraic errors also occurred, such as replacing $4k + 2$ with $2(k + 1)$.

Question 13(b) In an Argand diagram, the points A, B, C represent the complex numbers z_1, z_2, z_3, respectively. The triangle ABC is equilateral, and its vertices are labelled in anticlockwise order.

The triangle is rotated anticlockwise through an angle α about A, where $0 < \alpha < \pi/3$. Under this rotation, B and C map to B' and C', representing z_4 and z_5, respectively. The line $B'C'$ intersects BC at P and AC at Q. The line AB' intersects BC at R.



(i) *[1 mark]* Find z_3 in terms of z_1 and z_2.

Solution

The vector AC is obtained by rotating AB anticlockwise through $\pi/3$. Hence

$$z_3 - z_1 = e^{i\pi/3}(z_2 - z_1),$$

so

$$\boxed{z_3 = z_1 + e^{i\pi/3}(z_2 - z_1).}$$

Single mark
Awarded for any expression equivalent to

$$z_3 = z_1 + e^{i\pi/3}(z_2 - z_1).$$

(ii) *[1 mark]* Find z_5 in terms of z_1, z_2 and α.

Solution

The rotation about A maps C to C', so

$$z_5 - z_1 = e^{i\alpha}(z_3 - z_1) = e^{i(\alpha+\pi/3)}(z_2 - z_1).$$

Therefore

$$\boxed{z_5 = z_1 + e^{i(\alpha+\pi/3)}(z_2 - z_1).}$$

Single mark
Awarded for any expression equivalent to

$$z_5 = z_1 + e^{i(\alpha+\pi/3)}(z_2 - z_1).$$

Issues observed

- The most frequent error was $z_5 = e^{i\alpha}z_3$, which rotates about the origin rather than about A.
- Several expanded answers contained incorrect signs or coefficients and were not equivalent to the correct expression.

Question 13(b)(iii) *[3 marks]* By considering the ratio $AR : AB$, show that

$$z_R = z_1 + \frac{\sqrt{3}e^{i\alpha}}{\sqrt{3}\cos\alpha + \sin\alpha}(z_2 - z_1).$$

Solution

In $\triangle ABR$,

$$\angle BAR = \alpha, \qquad \angle ABR = \frac{\pi}{3}, \qquad \angle ARB = \frac{2\pi}{3} - \alpha.$$

By the sine rule,

$$\frac{AR}{AB} = \frac{\sin(\pi/3)}{\sin(2\pi/3 - \alpha)}.$$

Now

$$\sin\left(\frac{2\pi}{3} - \alpha\right) = \frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha,$$

and hence

$$\frac{AR}{AB} = \frac{\sqrt{3}}{\sqrt{3}\cos\alpha + \sin\alpha}.$$

The vector AR is in the direction obtained by rotating AB through α. Therefore

$$z_R - z_1 = \frac{AR}{AB}e^{i\alpha}(z_2 - z_1).$$

Substituting the ratio gives

$$\boxed{z_R = z_1 + \frac{\sqrt{3}e^{i\alpha}}{\sqrt{3}\cos\alpha + \sin\alpha}(z_2 - z_1)}.$$

First mark
Awarded for correct, **meaningful** work towards $AR : AB$.

Second mark
Awarded for obtaining

$$\frac{AR}{AB} = \frac{\sqrt{3}}{\sqrt{3}\cos\alpha + \sin\alpha}.$$

Third mark
Awarded for applying the ratio with other elements of the problem to obtain

$$z_R - z_1 = \frac{AR}{AB}e^{i\alpha}(z_2 - z_1)$$

and hence the required expression for z_R.

Issues observed

- A frequent error was

  $$z_R - z_1 = e^{i\alpha}(z_2 - z_1),$$

  which incorrectly assumes $AR = AB$. The factor AR/AB is essential.
- Finding z_4 correctly was only preliminary work. By itself, it did not determine $AR : AB$.
- Some responses equated complex quotients having different arguments.
- Some responses rotated z_R rather than $z_R - z_1$, thereby rotating about the origin rather than about A.
- The required z_R formula was supplied in the question. Writing it without deriving it did not earn the third mark.

Question 13(c)(i) *[2 marks]*

Let $f(x)$ and $g(x)$ be continuous functions on $\mathbb{R}$, where f is even and g is odd. Prove that

$$\int_{-c}^{c} \frac{f(x)}{1+e^{g(x)}}\,dx = \int_{0}^{c} f(x)\,dx.$$

Solution 1: Reflecting the whole integral

Let

$$I = \int_{-c}^{c} \frac{f(x)}{1+e^{g(x)}}\,dx.$$

Make the substitution $u = -x$. Then $dx = -du$, and the limits $x = -c$ and $x = c$ become $u = c$ and $u = -c$, respectively. Therefore

$$\begin{aligned} I &= \int_{c}^{-c} \frac{f(-u)}{1+e^{g(-u)}}(-\,du) \\ &= \int_{-c}^{c} \frac{f(u)}{1+e^{-g(u)}}\,du \\ &= \int_{-c}^{c} \frac{f(x)}{1+e^{-g(x)}}\,dx \qquad \text{renaming the dummy variable} \end{aligned}$$

where $f(-u) = f(u)$ (even function), $g(-u) = -g(u)$ (odd function).

Since

$$\frac{1}{1+e^{-g(x)}} = \frac{e^{g(x)}}{1+e^{g(x)}},$$

we have

$$I = \int_{-c}^{c} \frac{e^{g(x)}f(x)}{1+e^{g(x)}}\,dx.$$

Adding this expression to the original expression for I gives

$$\begin{aligned} 2I &= \int_{-c}^{c} \frac{f(x)}{1+e^{g(x)}}\,dx + \int_{-c}^{c} \frac{e^{g(x)}f(x)}{1+e^{g(x)}}\,dx \\ &= \int_{-c}^{c} f(x)\left(\frac{1}{1+e^{g(x)}} + \frac{e^{g(x)}}{1+e^{g(x)}}\right)dx \\ &= \int_{-c}^{c} f(x)\,dx \\ &= 2\int_{0}^{c} f(x)\,dx \qquad \text{since } f \text{ is even.} \end{aligned}$$

Hence

$$\boxed{\int_{-c}^{c} \frac{f(x)}{1+e^{g(x)}}\,dx = \int_{0}^{c} f(x)\,dx}.$$

First mark
Awarded for a correct reflection of the integral, normally by using $u = -x$, together with $f(-x) = f(x)$ and $g(-x) = -g(x)$, to obtain

$$I = \int_{-c}^{c} \frac{f(x)}{1+e^{-g(x)}}\,dx = \int_{-c}^{c} \frac{f(x)e^{g(x)}}{1+e^{g(x)}}\,dx.$$

Splitting the integral at 0 and correctly reflecting the interval $[-c, 0]$ onto $[0, c]$ was accepted as an equivalent method.

Second mark
Awarded for validly adding the reflected and original forms to obtain

$$2I = \int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx,$$

and hence the required result. The final equality alone did not earn this mark if it followed from invalid algebra.

Basically: one mark for preparing the integral (splitting and using a sub., or using a sub. for reflection), one mark for extracting the final result.

OR

Solution 2: Splitting the integral at 0

Let

$$I = \int_{-c}^{c} \frac{f(x)}{1+e^{g(x)}}\,dx.$$

Splitting the integral at 0, we obtain

$$I = \int_{-c}^{0} \frac{f(x)}{1+e^{g(x)}}\,dx + \int_{0}^{c} \frac{f(x)}{1+e^{g(x)}}\,dx.$$

In the first integral, make the substitution $u = -x$. Then $dx = -du$, while $x = -c$ gives $u = c$, and $x = 0$ gives $u = 0$. Thus

$$\begin{aligned}
\int_{-c}^{0} \frac{f(x)}{1+e^{g(x)}}\,dx &= \int_{c}^{0} \frac{f(-u)}{1+e^{g(-u)}}(-\,du) \\
&= \int_{0}^{c} \frac{f(u)}{1+e^{-g(u)}}\,du \\
&= \int_{0}^{c} \frac{e^{g(u)}f(u)}{1+e^{g(u)}}\,du \\
&= \int_{0}^{c} \frac{e^{g(x)}f(x)}{1+e^{g(x)}}\,dx.
\end{aligned}$$

Substituting this into the expression for I gives

$$\begin{aligned}
I &= \int_{0}^{c} \frac{e^{g(x)}f(x)}{1+e^{g(x)}}\,dx + \int_{0}^{c} \frac{f(x)}{1+e^{g(x)}}\,dx \\
&= \int_{0}^{c} f(x)\left(\frac{e^{g(x)}+1}{1+e^{g(x)}}\right)dx \\
&= \int_{0}^{c} f(x)\,dx.
\end{aligned}$$

Therefore

$$\boxed{\int_{-c}^{c} \frac{f(x)}{1+e^{g(x)}}\,dx = \int_{0}^{c} f(x)\,dx}.$$

Issues observed

Most responses were solid.

- Some students made the claim that the function $\dfrac{f(x)}{1+e^{g(x)}}$ was itself either odd or even, and consequently applied an invalid symmetry argument.
- Some responses added fractions by adding their denominators, or used $\int e^{g(x)}\,dx = e^{g(x)}/g'(x)$. Neither operation is valid in general.
- When using the evenness of f, the correct conclusion is

  $$\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx.$$

  Several responses lost the factor 2 or used incorrect limits.

Question 13(c)(ii) *[3 marks]* Evaluate

$$\int_{-1}^{1} \frac{3x^2+1}{(1+e^{\sin^3 x})(x^6+2x^4+x^2+1)}\,dx.$$

Solution

Take

$$f(x) = \frac{3x^2+1}{x^6+2x^4+x^2+1}, \qquad g(x) = \sin^3 x.$$

Then f is even (rational function where numerator and denominator have only even-powered terms in x), and g is odd (composition of two odd functions), so part (i) gives

$$\int_{-1}^{1} \frac{3x^2+1}{(1+e^{\sin^3 x})(x^6+2x^4+x^2+1)}\,dx = \int_{0}^{1} \frac{3x^2+1}{x^6+2x^4+x^2+1}\,dx.$$

Since

$$x^6+2x^4+x^2+1 = 1+(x^3+x)^2,$$

and $\dfrac{d}{dx}(x^3+x) = 3x^2+1$, arctangent is the required primitive, so we have

$$\begin{aligned}\int_0^1 \frac{3x^2+1}{1+(x^3+x)^2}\,dx &= \left[\tan^{-1}(x^3+x)\right]_0^1 \\ &= \boxed{\tan^{-1}(2)}.\end{aligned}$$

Note: There is a logic applied to find the integrand here. The ratio of polynomials should trigger the following checks: logarithm $\dfrac{f'}{f}$, partial fraction decomposition, arctangent. Clearly the logarithmic avenue is a dead end. Also, a quick check of ± 1 as a potential zero of the denominator (rational zero theorem) also yields a dead end (i.e. the polynomial is not going to be easily factorisable. This does not rule out partial fractions, but it suggests that this is unlikely to be the most efficient approach. This leaves only arctangent as a candidate.

You should know that $\displaystyle\int \frac{g'(x)}{a^2+[g(x)]^2} = \frac{1}{a}\tan^{-1}\frac{g(x)}{a} + C$ for $a \neq 0$, **or at least be able to source it from the reference sheet.**

If we're right, then $g'(x) = 3x^2+1$, so $g(x) = x^3+x+c$, and we would require, for the denominator,

$$a^2+(x^3+x+c)^2$$

At this point, we could think of the expansion and deduce $c=0$ and $a^2=1$, but we could also take $a^2+(x^3+x+c)^2$ as a signal to try to make a perfect square out of the terms in x:

$$x^6+2x^4+x^2 = x^2(x^4+2x^2+1) = x^2(x^2+1)^2 = (x^3+x)^2$$

so the denominator is

$$1+x^6+2x^4+x^2 = 1+(x^3+x)^2$$

and we easily see that $\dfrac{d}{dx}(x^3+x) = 3x^2+1$ as required.
So,

$$\begin{aligned}\int_0^1 \frac{3x^2+1}{x^6+2x^4+x^2+1}\,dx &= \int_0^1 \frac{3x^2+1}{1+(x^3+x)^2}\,dx \\ &= \left[\tan^{-1}(x^3+x)\right]_0^1 \\ &= \tan^{-1}(2).\end{aligned}$$

First mark
Awarded for identifying

$$f(x) = \frac{3x^2+1}{x^6+2x^4+x^2+1} \quad \text{as even,}$$

$$g(x) = \sin^3 x \quad \text{as odd,}$$

and applying part (i) to reduce the given integral to

$$\int_0^1 \frac{3x^2+1}{x^6+2x^4+x^2+1}\,dx.$$

This mark could be awarded independently of an unsuccessful proof in part (i), because the result of part (i) was supplied.

Second mark
Awarded for the conceptual substitution step: rewrite

$$x^6+2x^4+x^2+1 = 1+(x^3+x)^2$$

and use

$$u = x^3+x, \qquad du = (3x^2+1)\,dx.$$

The denominator rewrite itself was accepted for this mark when it clearly put the integrand into the required substitution form.

No mark was awarded for unsuccessful substitution.

Third mark
Awarded for completing the evaluation and explicitly obtaining

$$\tan^{-1}(2).$$

Issues observed

- Simply deleting the exponential factor was not sufficient. The response had to identify the relevant even and odd functions, or otherwise make clear that part (i) was being applied.
- Substitutions such as $u = x^2$ or $u = x^3$ do not match the numerator and do not transform the integral into the required arctangent form.
- Letting u equal the reciprocal of the denominator, or performing unmotivated polynomial division.
- Problems with transformed limits for $u = x^3+x$.

Question 13(d) A particle of unit mass moves in a straight line against a resisting force of magnitude $v + \lambda v^3$, where v is its speed and $\lambda > 0$. Initially, the particle is at the origin with speed $Q > 0$. Let x be its displacement and t the elapsed time.

Since the resisting force acts opposite to the motion, Newton's second law gives

$$\frac{dv}{dt} = -(v + \lambda v^3) = -v(1 + \lambda v^2).$$

The force expression must carry a negative because it is a force acting against the direction of motion; the force is in line with velocity, but it is acting to reduce it over time.

(i) *[2 marks]* Show that

$$x = \frac{1}{\sqrt{\lambda}} \left[\tan^{-1}(\sqrt{\lambda} Q) - \tan^{-1}(\sqrt{\lambda} v) \right].$$

Solution

Using

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

we have

$$v\frac{dv}{dx} = -v(1 + \lambda v^2).$$

While the particle is moving, $v > 0$, so

$$dx = -\frac{dv}{1 + \lambda v^2}.$$

Using $x = 0$ when $v = Q$,

$$\begin{aligned} x &= \int_v^Q \frac{du}{1 + \lambda u^2} \\ &= \boxed{\frac{1}{\sqrt{\lambda}} \left[\tan^{-1}(\sqrt{\lambda} Q) - \tan^{-1}(\sqrt{\lambda} v) \right]}. \end{aligned}$$

First mark
Awarded for the mechanics step

$$a = \frac{dv}{dt} = v\frac{dv}{dx}$$

and hence obtaining

$$dx = -\frac{dv}{1 + \lambda v^2},$$

or an equivalent correctly bounded integral.

Second mark
Awarded for correctly integrating

$$\int \frac{dv}{1 + \lambda v^2} = \frac{1}{\sqrt{\lambda}} \tan^{-1}(\sqrt{\lambda} v) + C,$$

using $x = 0$ when $v = Q$, and obtaining the stated expression. Equivalent definite-integral or constant-of-integration methods were accepted.

Issues observed

- Some responses used $a = dv/dt$ throughout part (i) and never converted to displacement. The relation $a = v\,dv/dx$ was needed for the first mark.
- The resisting force gives negative acceleration. Omitting this sign reversed the limits or produced a negative displacement.
- After cancelling v, the required integrand is $1/(1 + \lambda v^2)$. Several responses retained an extra v, used $1/(v + \lambda v^3)$, or confused this with the time integral from part (ii).
- A common integration error was to use a factor $1/\lambda$ instead of $1/\sqrt{\lambda}$, or to write $\tan^{-1}(v/\sqrt{\lambda})$ rather than $\tan^{-1}(\sqrt{\lambda} v)$.
- The bounds must correspond to $(x, v) = (0, Q)$ initially and (x, v) later. Reversing only one pair of bounds caused a sign error.
- **Merely writing the supplied target after invalid working did not earn the second mark.**

(ii) *[3 marks]* Show that

$$t = \frac{1}{2}\log_e\left(\frac{Q^2(1+\lambda v^2)}{v^2(1+\lambda Q^2)}\right).$$

Solution

For $v > 0$,

$$\frac{dv}{dt} = -v(1+\lambda v^2) < 0.$$

Thus $dv/dt \neq 0$, so $v(t)$ is locally invertible while the particle is moving. Hence

$$\frac{dt}{dv} = -\frac{1}{v(1+\lambda v^2)}.$$

Since $t = 0$ when $v = Q$,

$$\int_0^t d\tau = \int_v^Q \frac{du}{u(1+\lambda u^2)}.$$

i.e.

$$t = \int_v^Q \frac{du}{u(1+\lambda u^2)}.$$

Now

$$\frac{1}{u(1+\lambda u^2)} = \frac{1}{u} - \frac{\lambda u}{1+\lambda u^2},$$

so

$$\begin{aligned} t &= \left[\log_e u - \frac{1}{2}\log_e(1+\lambda u^2)\right]_v^Q \\ &= \log_e\left(\frac{Q}{v}\right) - \frac{1}{2}\log_e\left(\frac{1+\lambda Q^2}{1+\lambda v^2}\right) \\ &= \boxed{\frac{1}{2}\log_e\left(\frac{Q^2(1+\lambda v^2)}{v^2(1+\lambda Q^2)}\right)}. \end{aligned}$$

Note: Although partial fraction decomposition was used by the majority of students to decompose $\dfrac{1}{u(1+\lambda u^2)}$, it was not necessarily the best approach.

If you are not being examined specifically on partial fraction decomposition, you may use inspection/trial-and-error to determine the coefficients. Here, we try initially,

$$\frac{1}{u} - \frac{1}{1+\lambda u^2}$$

and now we can see that if we replace the numerator of 1 in the second fraction with λu, we will get what we need.

$$\frac{1}{u} - \frac{\lambda u}{1+\lambda u^2} = \frac{1+\lambda u^2 - (\lambda u)u}{u(1+\lambda u^2)} = \frac{1}{u(1+\lambda u^2)}$$

First mark
Awarded for separating the equation of motion correctly to obtain

$$dt = -\frac{dv}{v(1+\lambda v^2)}$$

and setting up an equivalent definite integral using $t = 0$ when $v = Q$.

Second mark
Awarded for the correct decomposition and integration

$$\frac{1}{v(1+\lambda v^2)} = \frac{1}{v} - \frac{\lambda v}{1+\lambda v^2},$$

giving the antiderivative

$$\log|v| - \frac{1}{2}\log(1+\lambda v^2).$$

An equivalent substitution method was accepted if it reached the same antiderivative.

Third mark
Awarded for applying the limits or initial condition consistently and simplifying to

$$\frac{1}{2}\log_e\left(\frac{Q^2(1+\lambda v^2)}{v^2(1+\lambda Q^2)}\right).$$

Issues observed

- Some students omitted the factor of v in the time integrand to give $1/(1+\lambda v^2)$ instead of $1/[v(1+\lambda v^2)]$.
- Many partial-fraction decompositions were of the form

  $$\frac{1}{v} - \frac{\lambda}{1+\lambda v^2} \neq \frac{1}{v(1+\lambda v^2)}.$$

  Forms with a constant numerator over $1+\lambda v^2$ were incorrect.
- Reversing the limits, or mishandling the leading negative sign, produced the reciprocal of the required logarithm argument.
- Not everyone used absolute values in the intermediate logarithms. This was not penalised since $Q > 0$, $v > 0$, and $\lambda > 0$. But students must at least state these conditions if avoiding absolute value in the argument of the logarithm.
- Many students wrote the required final expression after an incorrect decomposition or integration. This did not earn the final mark.

Question 14

(a) **9**

(i) Terminal speed **1**

At terminal speed, $\frac{dv}{dt} = 0$:

$$0 = \frac{g}{5} - kv \quad \Longrightarrow \quad v = \frac{g}{5k}.$$

$$\therefore \text{ terminal speed} = \frac{g}{5k} \text{ m s}^{-1}. \textbf{1 mark}$$

(ii) Show that $v = \frac{g}{5k} + \left(u - \frac{g}{5k}\right) e^{-kt}$ **2**

Separating variables in $\frac{dv}{dt} = \frac{g}{5} - kv$:

$$\int_u^v \frac{dv'}{\frac{g}{5} - kv'} = \int_0^t dt'.$$

Integrating the left-hand side:

$$\left[-\frac{1}{k} \ln\left(\frac{g}{5} - kv'\right)\right]_u^v = t.$$

$$-\frac{1}{k}\left[\ln\left(\frac{g}{5} - kv\right) - \ln\left(\frac{g}{5} - ku\right)\right] = t \ \textbf{1 mark}$$

$$\ln\left(\frac{\frac{g}{5} - kv}{\frac{g}{5} - ku}\right) = -kt$$

$$\frac{\frac{g}{5} - kv}{\frac{g}{5} - ku} = e^{-kt}$$

$$\frac{g}{5} - kv = \left(\frac{g}{5} - ku\right) e^{-kt}$$

$$kv = \frac{g}{5} - \left(\frac{g}{5} - ku\right) e^{-kt}$$

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$$v = \frac{g}{5k} - \left(\frac{g}{5k} - u\right) e^{-kt} = \frac{g}{5k} + \left(u - \frac{g}{5k}\right) e^{-kt},$$

as required. **1 mark**

(iii) Show that $x = \frac{g}{5k}t + \frac{1}{k}\left(u - \frac{g}{5k}\right)\left(1 - e^{-kt}\right)$ **2**

Since x is the distance descended, $x = \int_0^t v\, dt'$:

$$x = \int_0^t \left[\frac{g}{5k} + \left(u - \frac{g}{5k}\right) e^{-kt'}\right] dt'$$

$$= \left[\frac{g}{5k}t' - \frac{1}{k}\left(u - \frac{g}{5k}\right) e^{-kt'}\right]_0^t \quad \textbf{1 mark}$$

$$= \frac{g}{5k}t - \frac{1}{k}\left(u - \frac{g}{5k}\right) e^{-kt} - \left(0 - \frac{1}{k}\left(u - \frac{g}{5k}\right)\right)$$

$$= \frac{g}{5k}t + \frac{1}{k}\left(u - \frac{g}{5k}\right)\left(1 - e^{-kt}\right),$$

as required. **1 mark**

(iv) **4**

(a) Time at which the thrusters switch on **2**

The thrusters switch on when v first reaches the value halfway between the initial speed u and the terminal speed $\frac{g}{5k}$, i.e. when

$$v = \frac{1}{2}\left(u + \frac{g}{5k}\right).$$

Using the result from (ii):

$$\frac{g}{5k} + \left(u - \frac{g}{5k}\right) e^{-kt} = \frac{1}{2}\left(u + \frac{g}{5k}\right)$$

$$\left(u - \frac{g}{5k}\right) e^{-kt} = \frac{1}{2}\left(u + \frac{g}{5k}\right) - \frac{g}{5k} = \frac{1}{2}\left(u - \frac{g}{5k}\right) \quad \textbf{1 mark}$$

Since $u \neq \frac{g}{5k}$, we may divide both sides by $\left(u - \frac{g}{5k}\right)$:

$$e^{-kt} = \frac{1}{2}$$

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$$-kt = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\therefore\ t = \frac{\ln 2}{k}.\ \textbf{1 mark}$$

(a) Show that the speed approaches zero as $t \to \infty$ **2**

Once the thrusters switch on, the forces acting on the AUV are:

- weight mg acting downwards;
- buoyant force $\frac{4}{5}mg$ acting upwards;
- thruster force $\frac{1}{5}mg$ acting upwards;
- resistive force mkv acting upwards (opposing the downward motion).

By Newton's second law (taking downwards as positive):

$$m\frac{dv}{dt} = mg - \frac{4}{5}mg - \frac{1}{5}mg - mkv = -mkv$$

$$\therefore\ \frac{dv}{dt} = -kv.\ \textbf{1 mark}$$

Let $t_1 = \dfrac{\ln 2}{k}$ be the time the thrusters switch on, with speed $v_1 = \dfrac{1}{2}\left(u + \dfrac{g}{5k}\right)$ at that instant. Separating variables:

$$\int_{v_1}^{v} \frac{dv'}{v'} = \int_{t_1}^{t} -k\,dt'$$

$$\ln\left(\frac{v}{v_1}\right) = -k(t - t_1)$$

$$v = v_1 e^{-k(t-t_1)}.$$

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Since $k > 0$, as $t \to \infty$, $e^{-k(t-t_1)} \to 0$, and since v_1 is a fixed finite constant,

$$v \to 0 \quad \text{as } t \to \infty$$

$\therefore$ the speed of the AUV approaches zero as $t \to \infty$. **1 mark**

(b) Given Area of $\triangle ABC = M$, so **3**

$$h_a = \frac{2M}{a}, \qquad h_b = \frac{2M}{b}, \qquad h_c = \frac{2M}{c}.$$

Show that

$$\left(\frac{a^2+b^2}{c}\right)h_c + \left(\frac{b^2+c^2}{a}\right)h_a + \left(\frac{c^2+a^2}{b}\right)h_b \geq 12M.$$

Method 1

Substitute h_a, h_b, h_c into the LHS:

$$\text{LHS} = \left(\frac{a^2+b^2}{c}\right)\left(\frac{2M}{c}\right) + \left(\frac{b^2+c^2}{a}\right)\left(\frac{2M}{a}\right) + \left(\frac{c^2+a^2}{b}\right)\left(\frac{2M}{b}\right) \textbf{ 1 mark}$$

$$= 2M\left[\frac{a^2+b^2}{c^2} + \frac{b^2+c^2}{a^2} + \frac{c^2+a^2}{b^2}\right]$$

$$= 2M\left[\frac{a^2}{c^2} + \frac{b^2}{c^2} + \frac{b^2}{a^2} + \frac{c^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{b^2}\right]$$

$$= 2M\left[\left(\frac{a^2}{c^2} + \frac{c^2}{a^2}\right) + \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) + \left(\frac{b^2}{c^2} + \frac{c^2}{b^2}\right)\right] \textbf{ 1 mark}$$

Applying the inequality $p^2 + q^2 \geq 2pq$ to each bracket (with, e.g., $p = \frac{a}{c},\ q = \frac{c}{a}$, so $2pq = 2$):

$$\geq 2M\left[\, 2\sqrt{\tfrac{a^2}{c^2}\cdot\tfrac{c^2}{a^2}} + 2\sqrt{\tfrac{a^2}{b^2}\cdot\tfrac{b^2}{a^2}} + 2\sqrt{\tfrac{b^2}{c^2}\cdot\tfrac{c^2}{b^2}}\,\right]$$

$$= 2M\left[\, 2 + 2 + 2 \,\right] = 2M(6) = 12M \textbf{ 1 mark}$$

$\therefore$ LHS $\geq 12M$, as required.

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Method 2 (AM–GM)

Using $a^2 + b^2 \geq 2ab$, $b^2 + c^2 \geq 2bc$, $c^2 + a^2 \geq 2ac$:

$$\left(\frac{a^2+b^2}{c}\right)h_c + \left(\frac{b^2+c^2}{a}\right)h_a + \left(\frac{c^2+a^2}{b}\right)h_b \geq \frac{2ab}{c}h_c + \frac{2bc}{a}h_a + \frac{2ac}{b}h_b$$ **1 mark**

Applying the AM–GM inequality $\frac{x_1+x_2+x_3}{3} \geq \sqrt[3]{x_1x_2x_3}$ to the three terms on the right:

$$\geq 3\sqrt[3]{\frac{2ab}{c}h_c \cdot \frac{2bc}{a}h_a \cdot \frac{2ac}{b}h_b} = 3\sqrt[3]{\frac{8a^2b^2c^2}{abc}h_ah_bh_c} = 3\sqrt[3]{8abc \cdot h_ah_bh_c}$$ **1 mark**

Substituting $h_a = \frac{2M}{a}$, $h_b = \frac{2M}{b}$, $h_c = \frac{2M}{c}$, so that $abc \cdot h_ah_bh_c = 8M^3$:

$$= 3\sqrt[3]{8 \cdot 8M^3} = 3\sqrt[3]{8} \cdot \sqrt[3]{(2M)^3} = 6(2M) = 12M$$ **1 mark**

$\therefore$ LHS $\geq 12M$, as required.

Marking Criteria

Criteria	Marks
Correct substitution of h_a, h_b, h_c into the LHS	1
Correct application of a valid inequality (e.g. $a^2 + b^2 \geq 2ab$ together with AM–GM)	1
Correct simplification to $12M$, with conclusion stated	1
Total	**3**

(c) **5**

(i) Let x, y and k be positive real numbers. Show that **2**

$$x + y\sqrt{k} \leq \sqrt{x^2+y^2}\,\sqrt{k+1}.$$

Solution 1 — Squaring and comparing (RHS − LHS)

Since $x, y, k > 0$, both sides of the required inequality are positive, so it suffices to compare the squares.

$$(\text{LHS})^2 = \left(x + y\sqrt{k}\right)^2 = x^2 + 2xy\sqrt{k} + ky^2,$$

$$(\text{RHS})^2 = \left(x^2+y^2\right)(k+1) = x^2 + k\left(x^2+y^2\right) + y^2.$$

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Consider

$$\begin{aligned}(\text{RHS})^2 - (\text{LHS})^2 &= x^2 + k\left(x^2 + y^2\right) + y^2 - \left(x^2 + 2xy\sqrt{k} + ky^2\right) \\ &= kx^2 + y^2 - 2xy\sqrt{k} \\ &= \left(\sqrt{k}\,x - y\right)^2 \;\geq\; 0.\end{aligned}$$

1 mark: correctly expands and reduces to a perfect square

Hence

$$\left(x + y\sqrt{k}\right)^2 \;\leq\; \left(x^2 + y^2\right)(k+1),$$

and taking square roots of both sides (valid since both sides are non-negative) gives

$$x + y\sqrt{k} \;\leq\; \sqrt{x^2 + y^2}\,\sqrt{k+1},$$

1 mark: correct conclusion, taking square roots validly

with equality if and only if $y = \sqrt{k}\,x$.

No marks were awarded if a student assumed the inequality and worked backwards to $0 \geq 0$ without justifying the reversibility of each step; students were required to *prove* the inequality.

Solution 2 — Starting from a perfect square (reverse direction)

Since x, y, k are real, $\left(\sqrt{k}\,x - y\right)^2 \geq 0$.

Expanding,

$$kx^2 + y^2 - 2xy\sqrt{k} \;\geq\; 0 \quad\Longrightarrow\quad kx^2 + y^2 \;\geq\; 2xy\sqrt{k}.$$

1 mark: correct expansion/rearrangement

Adding $x^2 + ky^2$ to both sides:

$$(k+1)x^2 + (k+1)y^2 \;\geq\; x^2 + ky^2 + 2xy\sqrt{k},$$

so

$$(k+1)\left(x^2 + y^2\right) \;\geq\; \left(x + y\sqrt{k}\right)^2.$$

Since $x, y, k > 0$, both sides are non-negative, so taking square roots gives

$$\sqrt{(k+1)\left(x^2 + y^2\right)} \;\geq\; x + y\sqrt{k},$$

i.e.

$$x + y\sqrt{k} \;\leq\; \sqrt{x^2 + y^2}\,\sqrt{k+1}.$$

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1 mark: correct final step, respecting positivity

Solution 3 — Using the AM–GM inequality

We use the standard result that for real a, b,

$$\frac{a^2 + b^2}{2} \;\geq\; ab.$$

Squaring the required inequality, it suffices to show

$$x^2 + 2xy\sqrt{k} + y^2k \;\leq\; \left(x^2 + y^2\right)(k+1) = x^2k + x^2 + y^2k + y^2,$$

which after cancelling common terms reduces to

$$2xy\sqrt{k} \;\leq\; x^2k + y^2.$$

1: correctly reduces the squared inequality to this key statement

Now apply the AM–GM result above with $a = x\sqrt{k}$ and $b = y$:

$$\frac{\left(x\sqrt{k}\right)^2 + y^2}{2} \;\geq\; \left(x\sqrt{k}\right)y \quad\Longrightarrow\quad x^2k + y^2 \;\geq\; 2xy\sqrt{k},$$

which is exactly the statement required above.

Reversing the steps (all reversible since every quantity involved is non-negative),

$$\left(x^2 + y^2\right)(k+1) \;\geq\; \left(x + y\sqrt{k}\right)^2,$$

and since x, y, k are all positive, taking square roots gives

$$\sqrt{x^2 + y^2}\,\sqrt{k+1} \;\geq\; x + y\sqrt{k}.$$

1 mark: correctly reverses squaring step using positivity

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Solution 4 — Cauchy–Schwarz (vector) approach

Consider the vectors

$$A = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ \sqrt{k} \end{pmatrix}.$$

1 mark: sets up an appropriate pair of vectors

Then

$$A \cdot B = x + y\sqrt{k},$$

and by the Cauchy–Schwarz inequality (equivalently, since $A \cdot B = |A||B| \cos\theta \leq |A||B|$),

$$A \cdot B \ \leq \ |A|\,|B|.$$

Since

$$|A| = \sqrt{x^2 + y^2}, \qquad |B| = \sqrt{1 + k},$$

we obtain

$$x + y\sqrt{k} \ \leq \ \sqrt{x^2 + y^2}\ \sqrt{k + 1},$$

1 mark: correctly identifies $|A|$, $|B|$ and concludesas required, with equality when A and B are parallel, i.e. when $y = \sqrt{k}\,x$.

$x + y\sqrt{k} \leq \sqrt{x^2 + y^2}\sqrt{k + 1}$, with equality iff $y = \sqrt{k}\,x$.

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(ii) **3**

Prove for $n = 1$:

$$H = r \leq r\sqrt{1}$$

$\therefore$ Max H is $r\sqrt{1}$ cm.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true, i.e. for a tower with k hemispheres,

$$\text{Max } H = r\sqrt{k}.$$

Prove for $P(k + 1)$:

Consider the top k hemispheres. By assumption,

$$\text{Max } H = R\sqrt{k}.$$



$\therefore$ Max H of $(k + 1)$ hemispheres

$$\begin{aligned} &= x + R\sqrt{k} \\ &\leq \sqrt{x^2 + R^2}\,\sqrt{k + 1} \qquad \text{by (i)} \\ &= r\sqrt{k + 1} \end{aligned}$$

$\therefore$ true for $k + 1$.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Criteria	**Marks**
Correct base case	1
Correct proof by mathematical induction	2
Total	**3**

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Why the inductive step cannot add a hemisphere to the *bottom* of the tower

By construction, each hemisphere's base is a horizontal cross-section of the hemisphere immediately below it, so every hemisphere in the tower has radius *no greater than* the one beneath it. In particular, r is defined as the radius of the *lowest* hemisphere of the tower — it is the largest radius in the tower and is fixed by the problem.

Attempting to build $P(k+1)$ from $P(k)$ by adding underneath:

Suppose a valid tower of k hemispheres has base radius ρ, so that (by the inductive hypothesis) its maximum height is $\rho\sqrt{k}$. To turn this into a tower of $k+1$ hemispheres by placing a new hemisphere *underneath* it, that new hemisphere would need some radius r' such that ρ is a cross-section of it, i.e.

$$\rho \leq r'.$$

But then r', not ρ, is the radius of the lowest chamber of the resulting $(k+1)$-tower. This contradicts the setup of the problem: the radius r given in the question is meant to be the lowest hemisphere of the tower being analysed, not a value that only appears partway up the tower as a cross-section. In other words, adding to the bottom changes *which* hemisphere plays the role of "the tower with base radius r", so the inductive hypothesis $P(k)$ (stated for base radius r) can no longer be applied consistently.

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Question 15:

(a)

(i) $\underset{\sim}{v} = 0.4\underset{\sim}{i} - 0.4\underset{\sim}{j} - 0.1\underset{\sim}{k}$

$\underset{\sim}{r}(t) = 0.4t\,\underset{\sim}{i} - 0.4t\,\underset{\sim}{j} - 0.1t\,\underset{\sim}{k} + \underset{\sim}{a}$ ✓

$\underset{\sim}{r}(60) = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k} = 24\underset{\sim}{i} - 24\underset{\sim}{j} - 6\underset{\sim}{k} + \underset{\sim}{a}$ ✓

$\underset{\sim}{a} = -23\underset{\sim}{i} + 26\underset{\sim}{j} + 8\underset{\sim}{k}$

$\underset{\sim}{r}(t) = (0.4t - 23)\underset{\sim}{i} + (26 - 0.4t)\underset{\sim}{j} + (8 - 0.1t)\underset{\sim}{k}$

1 mark to use the velocity to find $\underset{\sim}{r}(t)$

1 mark to use the position vector at $t = 60$s

(ii) $8 - 0.1t = 0 \therefore t = 80$ sec ✓

$\underset{\sim}{r}(80) = (32 - 23)\underset{\sim}{i} + (26 - 32)\underset{\sim}{j} + (8 - 8)\underset{\sim}{k}$ ✓

$= 9\underset{\sim}{i} - 6\underset{\sim}{j}$

1 mark to consider the z-component of $\underset{\sim}{r}(t)$ is zero and find the time value.

1 mark to find the position vector for this time

(iii) $\underset{\sim}{a} = \frac{1}{225}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k})$

$\underset{\sim}{v} = \frac{1}{225}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k})t + \underset{\sim}{b}$

$\frac{1}{225}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k}) \times 70 + \underset{\sim}{b} = \underset{\sim}{u}$

$$\underset{\sim}{v} = \frac{1}{225}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k})t + \underset{\sim}{u} - \frac{70}{225}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k})$$

$$= \underset{\sim}{u} + \frac{t-70}{225}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k})$$

✓

$$\underset{\sim}{r}(t) = \underset{\sim}{u}t + \frac{(t-70)^2}{450}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k}) + \underset{\sim}{d}$$

$$\underset{\sim}{r}(70) = 3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$

$$3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k} = 70\underset{\sim}{u} + \underset{\sim}{d}$$

$$\underset{\sim}{r}(t) = (t-70)\underset{\sim}{u} + \frac{(t-70)^2}{450}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k}) + 3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$

✓

$$\underset{\sim}{r}(85) = 9\underset{\sim}{i} - 6\underset{\sim}{j}$$

$$15\underset{\sim}{u} + \frac{15^2}{450}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k}) + 3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k} = 9\underset{\sim}{i} - 6\underset{\sim}{j}$$

✓

$$15\underset{\sim}{u} + \frac{1}{2}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 21\underset{\sim}{k}) = 6\underset{\sim}{i} - 5\underset{\sim}{j} - 3\underset{\sim}{k}$$

$$15\underset{\sim}{u} = \frac{1}{2}(9\underset{\sim}{i} - 12\underset{\sim}{j} + 15\underset{\sim}{k})$$

$$\frac{3}{\sqrt{2}}\cos\theta(3\underset{\sim}{i} - 4\underset{\sim}{j}) + \frac{15}{\sqrt{2}}\sin\theta\,\underset{\sim}{k} = \frac{1}{2}(9\underset{\sim}{i} - 12\underset{\sim}{j} + 15\underset{\sim}{k})$$

$$\frac{9}{\sqrt{2}}\cos\theta = \frac{9}{2} \quad \therefore \quad \cos\theta = \frac{1}{\sqrt{2}}$$

$$\frac{-12}{\sqrt{2}}\cos\theta = \frac{-12}{2} \quad \therefore \quad \cos\theta = \frac{1}{\sqrt{2}}$$

$$\frac{15}{\sqrt{2}}\sin\theta = \frac{15}{2} \quad \therefore \quad \sin\theta = \frac{1}{\sqrt{2}}$$

✓

$$\therefore \quad \theta = 45^\circ$$

issues:

- with 3D vector, differentiation with respect to x, the question is what x mean
- You can not have a square of a vector
- using Column vector as a bound in the definite integral

$$\int_{\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}}^{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} dV$$

such notation is not acceptable.

- Notation with the integral of Column vector

$$\int \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} dt = \begin{pmatrix} \int v_1 dt \\ \int v_2 dt \\ \int v_3 dt \end{pmatrix}$$ is not acceptable.

(b) $$I_n = \int_0^{\pi} e^{-x}(\pi - x)^n \, dx$$

(i) $$I_{n+1} = \int_0^{\pi} e^{-x}(\pi - x)^{n+1} \, dx$$

let $u = (\pi - x)^{n+1}$ and $e^{-x}dx = dv$ ✓

$du = -(n+1)(\pi - x)^n dx$ $\quad v = -e^{-x}$

$$I_{n+1} = \left[-(\pi - x)^{n+1} e^{-x}\right]_0^{\pi} - \int_0^{\pi} e^{-x}(n+1)(\pi - x)^n dx$$ ✓

$$I_{n+1} = \pi^{n+1} - (n+1) I_n$$

(ii) $$I_n = \pi^n - n I_{n-1}$$

$$\frac{I_n}{n!} = \frac{\pi^n}{n!} - \frac{1}{(n-1)!} I_{n-1}$$

$$= \frac{\pi^n}{n!} - \frac{1}{(n-1)!}\left[\pi^{n-1} - (n-1) I_{n-2}\right]$$ ✓

$$= \frac{\pi^n}{n!} - \frac{\pi^{n-1}}{(n-1)!} + \frac{1}{(n-2)!} I_{n-2}$$

$$= \frac{\pi^n}{n!} - \frac{\pi^{n-1}}{(n-1)!} + \frac{\pi^{n-2}}{(n-2)!} - \frac{1}{(n-3)!} I_{n-3}$$ ✓

$$\therefore (-1)^n \frac{I_n}{n!} = (-1)^n \frac{\pi^n}{n!} + (-1)^{n-1} \frac{\pi^{n-1}}{(n-1)!} + \cdots + \frac{\pi^4}{4!} - \frac{\pi^3}{3!} + \frac{\pi^2}{2!} - I_1$$ ✓

$$(-1)^n \frac{I_n}{n!} + \frac{1}{e^{\pi}} = 1 - \pi + \frac{\pi^2}{2!} - \frac{\pi^3}{3!} + \frac{\pi^4}{4!} - \cdots + (-1)^n \frac{\pi^n}{n!}$$

$$(-1)^n \frac{I_n}{n!} + \frac{1}{e^\pi} = \sum_{K=0}^{n} (-1)^K \frac{\pi^K}{K!}$$

(iii) $$I_n = \pi^n - n I_{n-1}$$

$$\frac{I_n}{n!} = \frac{\pi^n}{n!} - \frac{I_{n-1}}{(n-1)!}$$

$$\lim_{n\to\infty} \frac{I_n}{n!} = \lim_{n\to\infty} \frac{\pi^n}{n!} - \lim_{n\to\infty} \frac{I_{n-1}}{(n-1)!}$$

Since $\lim_{n\to\infty} \frac{\pi^n}{n!} = 0$ $\therefore$ $\lim_{n\to\infty} \frac{I_n}{n!} = -\lim_{n\to\infty} \frac{I_{n-1}}{(n-1)!}$

$$\therefore \lim_{n\to\infty} \frac{I_n}{n!} = 0$$

$$\therefore \frac{1}{e^\pi} = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^k}{k!}$$

Alternative method:

$$\frac{I_n}{n!} = \int_0^\pi \frac{e^{-x}(\pi - x)^n}{n!}\, dx \leq \int_0^\pi e^{-x} \frac{\pi^n}{n!}\, dx$$

$$= \frac{\pi^n}{n!} \int_0^\pi e^{-x}\, dx$$

$e^{-x} > 0$ and $(\pi - x)^n \geq 0$ for $0 \leq x \leq \pi$

$$\frac{I_n}{n!} \geq 0 \quad \therefore \lim_{n\to\infty} \frac{I_n}{n!} = 0$$

$$\therefore \frac{1}{e^\pi} = \sum_{K=0}^{\infty} (-1)^K \frac{\pi^K}{K!}$$

1 mark for using $\lim_{n\to\infty} \frac{\Pi^n}{n!} = 0$

1 mark for proving $\lim_{n\to\infty} \frac{I_n}{n!} = 0$.

(C) $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{c}$, $\overrightarrow{OD} = \underset{\sim}{d}$

$CL : LD = 1 : 2 \therefore \overrightarrow{OL} = \frac{2}{3}\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{d}$

$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{OM} = \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$

$\overrightarrow{LM} = \underset{\sim}{a} - \frac{1}{6}\underset{\sim}{c} - \frac{1}{3}\underset{\sim}{d}$

$\overrightarrow{BD} = \underset{\sim}{d} - \underset{\sim}{a} - \underset{\sim}{c}$

let $\overrightarrow{BN} = \lambda \overrightarrow{BD}$

$\overrightarrow{OP} = \mu \overrightarrow{ON}$

$\overrightarrow{LP} = \gamma \overrightarrow{LM}$

$\overrightarrow{OP} = \mu \overrightarrow{ON}$

$= \mu(\overrightarrow{OB} + \overrightarrow{BN})$

$= \mu(\overrightarrow{OB} + \lambda \overrightarrow{BD})$

$= \mu(\underset{\sim}{a} + \underset{\sim}{c} + \lambda(\underset{\sim}{d} - \underset{\sim}{a} - \underset{\sim}{c}))$

$= \mu(1-\lambda)\underset{\sim}{a} + \mu(1-\lambda)\underset{\sim}{c} + \mu\lambda\underset{\sim}{d}$ (*)

$\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP}$

$= \frac{2}{3}\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{d} + \gamma \overrightarrow{LM}$

$$\overrightarrow{OP} = \frac{2}{3}\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{d} + \gamma\left(\underset{\sim}{a} - \frac{1}{6}\underset{\sim}{c} - \frac{1}{3}\underset{\sim}{d}\right)$$

$$= \gamma\underset{\sim}{a} + \left(\frac{2}{3} - \frac{\gamma}{6}\right)\underset{\sim}{c} + \left(\frac{1}{3} - \frac{\gamma}{3}\right)\underset{\sim}{d} \quad (**)$$

$\therefore$ $\mu(1-\lambda) = \gamma$ (1)

$\mu(1-\lambda) = \frac{2}{3} - \frac{\gamma}{6}$ (2)

$\mu\lambda = \frac{1}{3} - \frac{\gamma}{3}$ (3)

(1) and (2) $\therefore$ $\gamma = \frac{4}{7}$

(3) $\therefore$ $\mu\lambda = \frac{1}{3} - \frac{4}{21} = \frac{1}{7}$

$\mu = \gamma + \mu\lambda$

$= \frac{4}{7} + \frac{1}{7} = \frac{5}{7}$

$\therefore$ $\lambda = \frac{1}{5}$

$\therefore$ $BN : BD = 1 : 5$

$\therefore$ $BN : ND = 1 : 4$

1 mark for finding one or two expression of $\overrightarrow{OP}$.

1 mark for obtaining three equations in terms of λ, μ and γ

1 mark for finding the final answer.