NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2026 Year 12 Course Assessment Task 4 (Trial Examination)
Friday, 31 July 2026

General instructions

- Working time – 3 hours.
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

SECTION I

- Attempt Questions 1- 10
- Allow about 15 minutes for this section

SECTION II

- Attempt Questions 11 - 16
- Allow about 2 hour and 45 minutes for this section

NESA STUDENT #: ..... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12MXX.1 – Ms Ham ◯ 12MXX.2 – Mr Ho

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	%
MARKS	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{17}$	$\overline{14}$	$\overline{16}$	$\overline{13}$	$\overline{100}$

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 17).

Questions **Marks**

1. If $z = 2 + i$, which of the following is the value of $i(z - 2\overline{z})$? **1**

(A) $3 + 2i$

(B) $3 - 2i$

(C) $-3 + 2i$

(D) $-3 - 2i$

2. Which of the following is the converse of the statement below? **1**

If $a = 2$ and $b = 0$ then $ab = 0$

(A) If $a \neq 2$ or $b \neq 0$ then $ab = 0$

(B) If $a \neq 2$ or $b \neq 0$ then $ab \neq 0$

(C) If $ab = 0$ then $a = 2$ and $b = 0$

(D) If $ab = 0$ then $a = 2$ or $b = 0$

3. It is given that **1**

$$\underset{\sim}{a} = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$$

Consider the following two statements.

(i) When $t = 0$, the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is obtuse.

(ii) When $t = -3$, $\underset{\sim}{a} \parallel \underset{\sim}{b}$.

Which of the following is true?

(A) Neither statement is correct

(B) Only statement (i) is correct

(C) Only statement (ii) is correct

(D) Both statments are correct

4. Let ω be a non-real cube root of unity. **1**

Which of the following is equal to $\omega^{2026} + \omega^{2027} + \omega^{2028}$?

(A) 0 (B) 1 (C) ω (D) ω^2

5. It is given that $1 - i = e^{a+ib}$, where $a, b \in \mathbb{R}$ and $-\dfrac{\pi}{2} < b < \dfrac{\pi}{2}$. **1**

Which of the following gives the exact values of a and b?

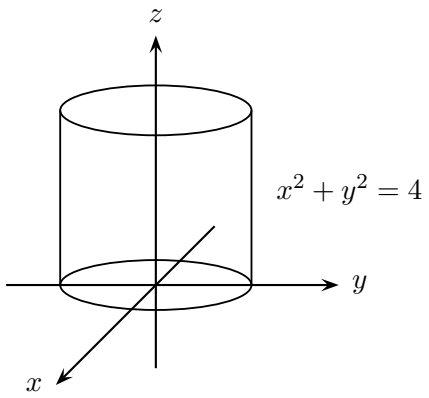
(A) $a = \dfrac{1}{2}\ln 2, \quad b = -\dfrac{\pi}{4}$

(B) $a = \dfrac{1}{2}\ln 2, \quad b = \dfrac{\pi}{4}$

(C) $a = \ln 2, \quad b = -\dfrac{\pi}{4}$

(D) $a = -\dfrac{1}{2}\ln 2, \quad b = -\dfrac{\pi}{4}$

6. A particle moves on the surface of the cylinder $x^2 + y^2 = 4$, shown below. **1**



For $t \geq 0$, the particle starts at $(2, 0, 0)$ and travels anticlockwise around the cylinder (as viewed along the positive z-axis) while rising uniformly.

Which of the following could represent the motion?

(A) $x = 2\cos t,\ y = 2\sin t,\ z = t$

(B) $x = 2\cos t,\ y = 2\sin 2t,\ z = t$

(C) $x = 2\sin t,\ y = 2\cos t,\ z = t$

(D) $x = 2\cos t,\ y = -2\sin t,\ z = t$

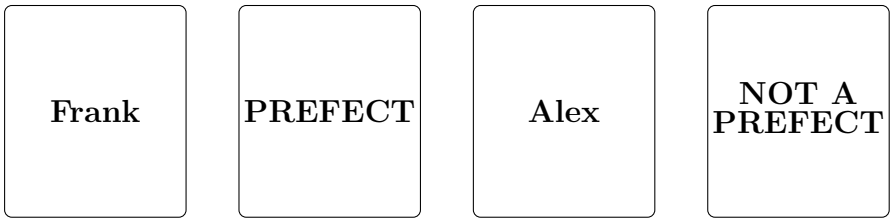
Examination continues overleaf...

7. A unit vector lies in the first octant, where $x, y, z > 0$. It makes equal angles with the positive directions of the y-axis and the z-axis, and makes an angle of $\frac{\pi}{4}$ with the positive direction of the x-axis. **1**

What angle does the vector make with the positive direction of the z-axis?

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

8. Every student ID card has the student's name on one side and whether the student is a prefect on the other side. The four cards are placed on a table, as shown below. **1**



Card 1 Card 2 Card 3 Card 4

The following statement is made:

If a student is Frank, then the student is a prefect.

Mr Normo would like to determine whether the statement is true. Which cards should he turn over?

(A) Cards 1 and 2

(B) Cards 1 and 4

(C) Cards 2 and 3

(D) Cards 3 and 4

9. A particle falls from rest through a medium where the resistance is proportional to its speed. **1**

As the particle approaches terminal velocity, which of the following is correct?

(A) Both the particle's speed and the magnitude of its acceleration decrease.

(B) Both the particle's speed and the magnitude of its acceleration increase.

(C) The particle's speed decreases while the magnitude of its acceleration increases.

(D) The particle's speed increases while the magnitude of its acceleration decreases.

10. A curve $y = f(x)$ satisfies **1**

$$y\frac{dy}{dx} = \frac{1}{2x} \qquad \text{and} \qquad y(1) = 1.$$

The region bounded by the curve, the x-axis, and the lines $x = 1$ and $x = e$ is rotated about the x-axis.

Which of the following is the volume of the resulting solid?

(A) $\pi(e-1)$

(B) πe

(C) $2\pi(e-1)$

(D) $2\pi e$

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Answer the questions in the Section II Writing Booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) Solve the quadratic equation **2**

$$z^2 - 5z + 8 = 0$$

where z is a complex number. Give your answers in Cartesian form.

(b) A sphere S has centre $(3, -7, -2)$ and radius 9.

i. Find the Cartesian equation for the sphere S. **1**

ii. Find the vector equation for the sphere S. **1**

(c) It is given that **3**

$$\underset{\sim}{a} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k}$$

$$\underset{\sim}{b} = -2\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}.$$

Find the values of λ_1 and λ_2 such that $\lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{b} = 14\underset{\sim}{i} + 26\underset{\sim}{j} - 18\underset{\sim}{k}$, where $\lambda_1, \lambda_2 \in \mathbb{R}$.

(d) A particle moves in simple harmonic motion described by the equation $\ddot{x} = -4(x - 3)$. **2**

The particle is initially at $x = 3$ with velocity -6 cm/s. Find the equation of the displacement-time function for the particle.

(e) Find $\displaystyle\int \tan^{-1} x \, dx$. **2**

(f) Find $\displaystyle\int_2^6 \frac{x+1}{(x-1)(x+3)} \, dx$. **4**

Question 12 (15 marks) **Marks**

(a) It is given that $P(x) = x^3 - x^2 + mx + n$, where both m and n are integers.

i. Show that $P(-i) = (1 + n) + i(1 - m)$. **1**

ii. When $P(x)$ is divided by $x^2 + 1$, the remainder is $6x - 3$. Find the values of m and n. **2**

(b) Sketch the region defined by $|z| < 3$ and $0 \leq \arg(z + i) \leq \dfrac{\pi}{2}$. **3**

(c) i. Let $z = \cos\theta + i\sin\theta$. Show that $z^n + z^{-n} = 2\cos n\theta$. **1**

ii. Use the expansion of $(z + \bar{z})^3$ to show that $\cos^3\theta = \dfrac{1}{4}\cos 3\theta + \dfrac{3}{4}\cos\theta$ **2**

(d) Suppose that $a, b \in \mathbb{Z}^+$. **3**

By proving the contrapositive, prove that if ab is even, then a is even or b is even.

(e) i. Prove that $x + y \geq 2\sqrt{xy}$ for $x, y > 0$. **1**

ii. Hence, or otherwise, prove that $(a + b)(1 + ab) \geq 4ab$ for $a, b > 0$. **2**

Examination continues overleaf...

Question 13 (17 marks) **Marks**

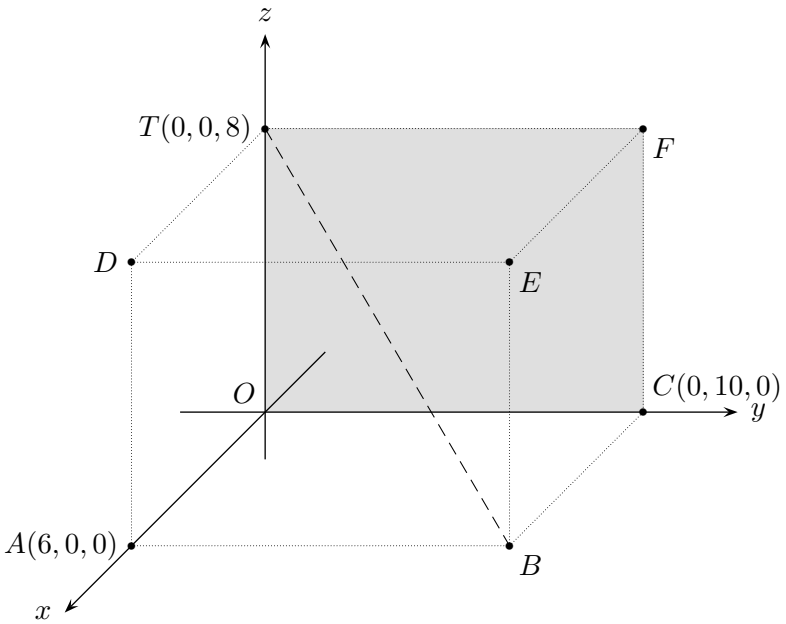
(a) Find $\int x\sqrt{x+1}\,dx$. **2**

(b) It is given that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

Using this identity, evaluate $\int_0^1 \dfrac{e^x}{e^x + e^{1-x}}\,dx$.

(c) Find $\int \dfrac{1}{5+3\sin x}\,dx$. **4**

(d) A rectangular prism is defined with vertices $A(6,0,0)$, $C(0,10,0)$ and $T(0,0,8)$.



i. Find a vector equation of the line through B and T, where BT is a diagonal of the prism. **2**

ii. Using (i), show that the shortest distance from A to the line BT is $5\sqrt{2}$. **3**

iii. Hence, or otherwise, find the area of triangle ABT. **2**

(e) For $\underset{\sim}{u} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, find a non-zero vector that is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

Question 14 (14 marks) **Marks**

(a) i. Use the seventh roots of -1 to show that **3**

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$$

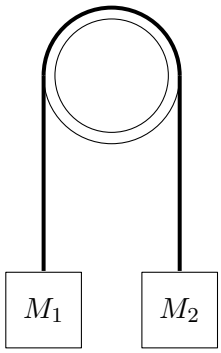
ii. Hence find the exact value of **1**

$$\cos^2\frac{\pi}{7}+\cos^2\frac{3\pi}{7}+\cos^2\frac{5\pi}{7}$$

(b) A particle moving in simple harmonic motion about the origin starts at the origin with velocity V_0. **2**

Show that the particle first comes to rest after traveling a distance of $\dfrac{|V_0|}{n}$.

(c) Two objects M_1 and M_2 of respective masses 5 kg and 9 kg are each attached to the two ends of a light inextensible string which passes over a smooth pulley.



The two particles are initially held at rest, each 1.75 m above a horizontal floor. The system is then released from rest.

Each particle experiences a constant air resistance of 3.5 N while in motion. M_2 reaches the floor before M_1 reaches the pulley.

Let the acceleration g due to gravity be 9.8 m/s^2.

i. Find the acceleration of the particles. **2**

ii. Find the tension in the string. **1**

iii. Find the speed of M_2 as it strikes the floor, giving your answer correct to two decimal places. **2**

Examination continues overleaf...

(d) An object is dropped from rest from an altitude $x = R$ above the Earth's surface. Its speed v satisfies **3**

$$v^2 = 2k\left(\frac{1}{x+R} - \frac{1}{2R}\right)$$

where $k > 0$, R is the radius of the Earth and x is the altitude above the Earth's surface.

Let g be the magnitude of acceleration due to gravity at the Earth's surface.

Find the altitude at which the object's acceleration has magnitude $\frac{g}{2}$, in terms of R only.

Question 15 (16 marks) **Marks**

(a) Prove that $\log_3 15$ is irrational. **3**

(b) Use mathematical induction to prove that the nth derivative of $y = xe^{-x}$ is **3**

$$\frac{d^n y}{dx^n} = (-1)^n (x - n)e^{-x}$$

for all integers $n \geq 1$.

(c) Let $I_n = \displaystyle\int_0^1 \frac{x^{n-1}}{(x+1)^n}\, dx$, for $n \in \mathbb{Z}^+$.

i. By using integration by parts, or otherwise, show that $I_{n+1} = I_n - \dfrac{1}{n2^n}$. **3**

ii. Show that the maximum value of $\dfrac{x}{x+1}$ for $0 \leq x \leq 1$ is $\dfrac{1}{2}$. **1**

iii. Hence, or otherwise, show that $I_{n+1} < \dfrac{1}{2} I_n$. **1**

iv. Deduce that $I_n < \dfrac{1}{n2^{n-1}}$. **2**

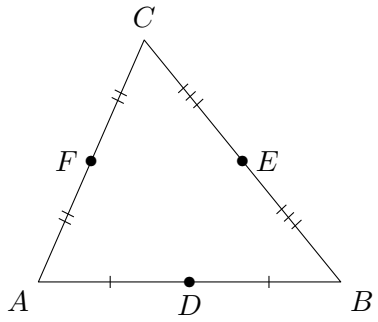
v. Hence show that $\dfrac{5}{8} < \ln 2 < \dfrac{17}{24}$. **3**

(Hint: You may consider I_1)

Examination continues overleaf...

Question 16 (13 marks) **Marks**

(a) In triangle ABC shown below, D, E and F are the midpoints of AB, BC and AC, respectively. **3**



Prove that the perpendicular bisectors of AB, BC and AC are concurrent.

(b) A ball of unit mass is projected vertically upwards from the ground with initial speed U m/s. The ball moves through the air with speed v and experiences air resistance of magnitude kv newtons, where $k > 0$.

After reaching its maximum height, the ball falls back to the ground.

During the downward motion,

$$\ddot{y} = g - kv \qquad \text{and} \qquad v = \frac{g}{k}(1 - e^{-kt}),$$

where y is the displacement of the ball measured downwards from its maximum height, t is the time elapsed since the ball reached its maximum height, and g is the acceleration due to gravity.

i. By obtaining a relationship between y and v, show that **3**

$$t = \frac{v + ky}{g}$$

ii. Hence show that the total time of flight is $T = \dfrac{U + V_f}{g}$, where V_f is the speed of the ball immediately before it hits the ground. **3**

(c) Three drones are positioned over a flat region, with their locations represented on an Argand diagram. The first drone is located at $z_1 = 4 + 2i$. **4**

The second drone lies on $\text{Re}(z) = -4$, while the third drone lies on $\text{Im}(z) = -4$.

It is given that

$$\left|\frac{z_3 - z_1}{z_2 - z_1}\right| = 1 \qquad \text{and} \qquad \text{Re}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{1}{2}.$$

Find all possible locations of the second drone in Cartesian form.

End of paper.

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Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

NESA STUDENT #: .....

Class (please ✔)

◯ 12MXX.1 – Ms Ham

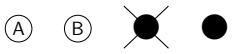
◯ 12MXX.2 – Mr Ho

Directions for multiple choice answers

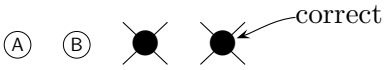
- Read each question and its suggested answers.
- Select the alternative Ⓐ, Ⓑ, Ⓒ, or Ⓓ that best answers the question.
- Mark only one circle per question. There is only *one* correct choice per question.
- Fill in the response circle completely, using blue or black pen, e.g.



- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.



- If you continue to change your mind, write the word `correct` and clearly indicate your final choice with an arrow as shown below:



1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ Ⓓ

6 – Ⓐ Ⓑ Ⓒ Ⓓ

7 – Ⓐ Ⓑ Ⓒ Ⓓ

8 – Ⓐ Ⓑ Ⓒ Ⓓ

9 – Ⓐ Ⓑ Ⓒ Ⓓ

10 – Ⓐ Ⓑ Ⓒ Ⓓ

Sample Band E4 Responses

1. (D) **2.** (C) **3.** (B) **4.** (A) **5.** (A)
6. (A) **7.** (C) **8.** (B) **9.** (D) **10.** (B)

Question 11 (Ham)

(a) (2 marks)

✓ [1] for use of quadratic formula, or equivalent merit

✓ [2] for correct solution

$$z = \frac{5 \pm \sqrt{25 - 4(8)}}{2} = \frac{5 \pm i\sqrt{7}}{2}$$

$$\therefore z = \frac{5}{2} \pm \frac{\sqrt{7}}{2}i$$

(b) i. (1 mark)

$$(x-3)^2 + (y+7)^2 + (z+2)^2 = 81$$

ii. (1 mark)

$$\left| \underset{\sim}{\mathrm{r}} - \begin{pmatrix} 3 \\ -7 \\ -2 \end{pmatrix} \right| = 9$$

(c) (3 marks)

✓ [1] for parametric equations

✓ [2] for obtaining correct values of λ_1 and λ_2

✓ [3] for correct solution

$$\lambda_1 \underset{\sim}{\mathrm{a}} + \lambda_2 \underset{\sim}{\mathrm{b}} = 14\underset{\sim}{\mathrm{i}} + 26\underset{\sim}{\mathrm{j}} - 18\underset{\sim}{\mathrm{k}}$$

$$\lambda_1 \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 26 \\ -18 \end{pmatrix}$$

$$\begin{cases} 3\lambda_1 - 2\lambda_2 = 14 & \dots(1) \\ 5\lambda_1 - 4\lambda_2 = 26 & \dots(2) \\ -\lambda_1 + 4\lambda_2 = -18 & \dots(3) \end{cases}$$

(2)+(3),

$$4\lambda_1 = 8$$
$$\lambda_1 = 2, \quad \lambda_2 = -4$$

These values satisfy (1) because

$$3\lambda_1 - 2\lambda_2 = 3(2) - 2(-4) = 14$$

(d) (2 marks)

✓ [1] for value of n and centre of motion

✓ [1] for correct solution

$n = 2$ and the centre of motion is at $x = 3$.
Let $x = -a\sin(2t) + 3$.

$$v = -2a\cos(2t)$$

$\because$ When $t = 0, v = -6$,

$$-6 = -2a, \quad a = 3$$
$$\therefore x = -3\sin(2t) + 3$$

(e) (2 marks)

✓ [1] for integration by parts

✓ [1] for final answer

$$u = \tan^{-1} x \qquad v = x$$
$$u' = \frac{1}{1+x^2} \qquad v' = 1$$

$$\begin{aligned}\int \tan^{-1} x\, dx &= x\tan^{-1} x - \int \frac{x}{1+x^2}\, dx \\ &= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C\end{aligned}$$

(f) (4 marks)

✓ [1] for attempt to use partial fractions

✓ [1] for $\frac{x+1}{(x-1)(x+3)} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+3}\right)$

✓ [1] for integrating, or equivalent merit

✓ [1] for correct solution

$$\frac{x+1}{(x-1)(x+3)} \equiv \frac{A}{x-1} + \frac{B}{x+3}$$
$$x + 1 \equiv A(x+3) + B(x-1)$$

Let $x = 1$,

$$A = \frac{1}{2}$$

Let $x = -3$,

$$B = -\frac{1}{2}$$

$$\begin{aligned}\int_2^6 \frac{x+1}{(x-1)(x+3)}\,dx &= \frac{1}{2}\int_2^6 \left(\frac{1}{x-1}-\frac{1}{x+3}\right)\,dx \\ &= \frac{1}{2}\Big[\ln|x-1|-\ln|x+3|\Big]_2^6 \\ &= \frac{1}{2}\Big(\ln 5+\ln 9-(0-\ln 5)\Big) \\ &= \frac{1}{2}\ln 9 \\ &= \ln 3\end{aligned}$$

Alternatively

$$\begin{aligned}\int_2^6 \frac{x+1}{(x-1)(x+3)}\,dx &= \int_2^6 \frac{x+1}{x^2+2x-3}\,dx \\ &= \frac{1}{2}\Big[\ln(x^2+2x-3)\Big]_2^6 \\ &= \frac{1}{2}(\ln 45-\ln 5) \\ &= \frac{1}{2}\ln 9 \\ &= \ln 3\end{aligned}$$

Question 12 (Ho)

(a) i. (1 mark)

✓ [1] for correct proof showing substitution

$P(-i) = (-i)^3 - (-i)^2 + m(-i) + n = i + 1 - mi + n = (1+n) + i(1-m)$

ii. (2 marks)

✓ [1] for $P(-i) = -6i - 3$, or equivalent merit

✓ [1] for correct solution

$P(x) = (x^2+1)Q(x) + (6x-3)$ and $P(-i) = -6i - 3$

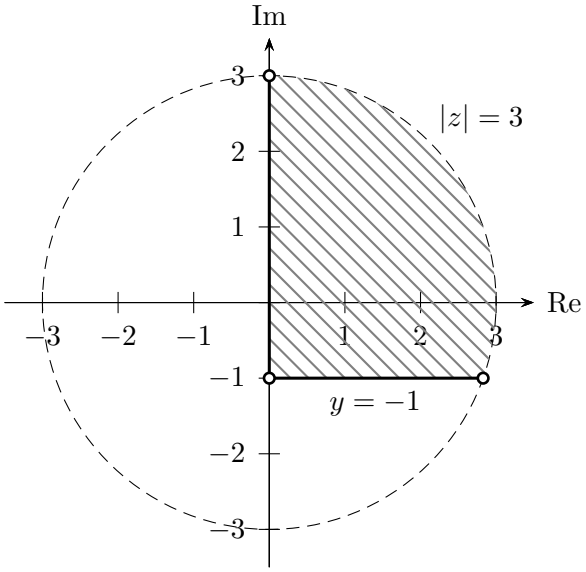
$$1 + n = -3 \text{ and } 1 - m = -6$$
$$\therefore n = -4, \quad m = 7$$

(b) (3 marks)

✓ [1] for correct sketch of interior of either $|z| < 3$ or $0 \leq \arg(z+i) \leq \dfrac{\pi}{2}$, with a correct scale and labels

✓ [1] for correct shading of the interior of the region, or equivalent merit

✓ [1] for correct sketch, with correction boundary inclusions/exclusions



(c) i. (1 mark)

✓ [1] for correct proof showing sufficient working out

$$\begin{aligned} z^n + z^{-n} &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \\ &= 2\cos n\theta \end{aligned}$$

ii. (2 marks)

✓ [1] for correct binomial expansion

✓ [1] for correct proof

$$(z+\bar{z})^3 = (z+z^{-1})^3 = (2\cos\theta)^3 = 8\cos^3\theta$$

$$\begin{aligned} (z+\bar{z})^3 &= (z+z^{-1})^3 \\ &= z^3 + 3z^2z^{-1} + 3z(z^{-1})^2 + (z^{-1})^3 \\ &= z^3 + z^{-3} + 3\left(z+z^{-1}\right) \\ &= 2\cos 3\theta + 3\cos\theta \end{aligned}$$

$$\begin{aligned} \therefore 8\cos^3\theta &= 2\cos 3\theta + 3\cos\theta \\ \cos^3\theta &= \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta \end{aligned}$$

(d) (3 marks)

✓ [1] for correct contrapositive

✓ [1] for expressing ab as a product of two odd integer expressions

✓ [1] for correct proof

Contrapositive: If a is odd and b is odd, then ab is odd.

Let $a = 2p + 1$ and $b = 2q + 1$, where $p, q \in \mathbb{Z}$.

$$\begin{aligned} ab &= (2p+1)(2q+1) \\ &= 4pq + 2p + 2q + 1 \\ &= 2(pq + p + q) + 1 \qquad \text{where } pq + p + q \in \mathbb{Z} \end{aligned}$$

$\therefore$ The contrapositive is true.

Hence, if ab is even, then a is even or b is even.

(e) i. (1 mark)

✓ [1] for correct proof showing sufficient working out

$$\begin{aligned} (\sqrt{x} - \sqrt{y})^2 &\geq 0 \\ x + y - 2\sqrt{x}\sqrt{y} &\geq 0 \\ x + y &\geq 2\sqrt{xy} \end{aligned}$$

ii. (2 marks)

✓ [1] for using (i), or equivalent merit

✓ [2] for correct proof

$$\begin{aligned} (a+b)(1+ab) &\geq 2\sqrt{ab} \times 2\sqrt{1 \cdot ab} \\ &= 4\sqrt{a^2b^2} \\ &= 4ab \qquad (\because a, b > 0) \end{aligned}$$

Question 13 (Ho)

(a) (2 marks)

✓ [1] for manipulation of the integral, or equivalent merit

✓ [1] correct solution, or equivalent merit

$$\begin{aligned} \int x\sqrt{x+1}\,dx &= \int \Big((x+1) - 1\Big)\sqrt{x+1}\,dx \\ &= \int (x+1)^{\frac{3}{2}} - (x+1)^{\frac{1}{2}}\,dx \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C \end{aligned}$$

Alternate solution: $\dfrac{2}{3}x(x+1)^{\frac{3}{2}} - \dfrac{4}{15}(x+1)^{\frac{5}{2}} + C$

(b) (2 marks)

✓ [1] for using the given identity

✓ [1] for correct solution showing sufficient working out

$$I = \int_0^1 \frac{e^x}{e^x + e^{1-x}}\, dx = \int_0^1 \frac{e^{1-x}}{e^{1-x} + e^x}\, dx$$

$$\begin{aligned} \therefore 2I &= \int_0^1 \frac{e^x + e^{1-x}}{e^x + e^{1-x}}\, dx \\ &= \int_0^1 1\, dx \\ &= \Big[x\Big]_0^1 \\ &= 1 \end{aligned}$$

$$\therefore I = \frac{1}{2}$$

(c) (4 marks)

✓ [1] for converting differential in x into differential in t

✓ [1] for completing square in denominator

✓ [1] for correct primitive as inverse tan function

✓ [1] for correct solution in terms of x

Using t formulae: let $t = \tan\frac{x}{2}$

$$\begin{aligned} t &= \tan\frac{x}{2} \\ x &= 2\tan^{-1} t \\ \therefore dx &= \frac{2}{1+t^2}\, dt \end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{5+3\sin x} &= \int \frac{\frac{2}{1+t^2}}{5+3\left(\frac{2t}{1+t^2}\right)}\,dt \\
&= \int \frac{2}{5(1+t^2)+6t}\,dt \\
&= \int \frac{2}{5t^2+6t+5}\,dt \\
&= \int \frac{2}{5\left(t^2+\frac{6}{5}t+1\right)}\,dt \\
&= \frac{2}{5}\int \frac{1}{\left(t+\frac{3}{5}\right)^2+\frac{16}{25}}\,dt \\
&= \frac{2}{5}\int \frac{1}{\left(t+\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}\,dt \\
&= \frac{2}{5}\left(\frac{5}{4}\right)\tan^{-1}\left(\frac{t+\frac{3}{5}}{\frac{4}{5}}\right)+C \\
&= \frac{1}{2}\tan^{-1}\left(\frac{5t+3}{4}\right)+C \\
&= \frac{1}{2}\tan^{-1}\left(\frac{1}{4}\left(5\tan\frac{x}{2}+3\right)\right)+C
\end{aligned}$$

(d) i. (2 marks)

✓ [1] for $\overrightarrow{TB}$, or equivalent merit

✓ [2] for a correct solution

Since $B=(6,10,0)$ and $T=(0,0,8)$,

$$\overrightarrow{TB}=\begin{pmatrix}6\\10\\-8\end{pmatrix}=2\begin{pmatrix}3\\5\\-4\end{pmatrix}.$$

Therefore, a vector equation of the line BT is

$$\underset{\sim}{\mathrm{r}}=\begin{pmatrix}0\\0\\8\end{pmatrix}+\lambda\begin{pmatrix}3\\5\\-4\end{pmatrix},\qquad \lambda\in\mathbb{R}.$$

ii. (3 marks)

✓ [1] for a general point on the line BT, or equivalent merit

✓ [2] for AP^2, or equivalent merit

✓ [3] for correct result, with sufficient working out

Let P be a general point on the line BT. Then

$$\overrightarrow{OP} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 3\lambda \\ 5\lambda \\ 8-4\lambda \end{pmatrix}$$

$$\overrightarrow{AP} = \begin{pmatrix} 3\lambda \\ 5\lambda \\ 8-4\lambda \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3\lambda - 6 \\ 5\lambda \\ 8-4\lambda \end{pmatrix}.$$

$$\begin{aligned} AP^2 &= (3\lambda-6)^2 + (5\lambda)^2 + (8-4\lambda)^2 \\ &= 9\lambda^2 - 36\lambda + 36 + 25\lambda^2 + 16\lambda^2 - 64\lambda + 64 \\ &= 50\lambda^2 - 100\lambda + 100 \\ &= 50\left(\lambda^2 - 2\lambda + 2\right) \\ &= 50\left((\lambda-1)^2 + 1\right) \end{aligned}$$

The minimum value of AP^2 occurs when $\lambda = 1$. ($\because$ concave down parabola)

$$\begin{aligned} AP^2_{min} &= 50 \\ \therefore AP_{min} &= \sqrt{50} = 5\sqrt{2}. \end{aligned}$$

Hence, the shortest distance from A to the line BT is $5\sqrt{2}$.

iii. (2 marks)

✓ [1] for length of BT

✓ [2] for correct solution

The length of BT is

$$\begin{aligned} BT &= \sqrt{(6-0)^2 + (10-0)^2 + (0-8)^2} \\ &= \sqrt{36 + 100 + 64} \\ &= \sqrt{200} \\ &= 10\sqrt{2}. \end{aligned}$$

Therefore, using BT as the base of $\triangle ABT$,

$$\begin{aligned} \text{Area of } \triangle ABT &= \frac{1}{2} \times 10\sqrt{2} \times 5\sqrt{2} \\ &= 50 \end{aligned}$$

(e) (2 marks)

✓ [1] for converting differential in x into differential in t

✓ [1] for completing square in denominator

Let $\underset{\sim}{\mathrm{w}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a non-zero vector perpendicular to $\underset{\sim}{\mathrm{u}}$ and $\underset{\sim}{\mathrm{v}}$.

$$\underset{\sim}{\mathrm{u}} \cdot \underset{\sim}{\mathrm{w}} = 0$$
$$\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$
$$5x - y + 2z = 0 \qquad \ldots (1)$$

$$\underset{\sim}{\mathrm{v}} \cdot \underset{\sim}{\mathrm{w}} = 0$$
$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$
$$x + 2y + 2z = 0 \qquad \ldots (2)$$

Equation (1) − (2) : $\quad 4x = 3y$

Let $x = 3$: $\quad y = 4$

From equation (2) :

$$3 + 8 + 2z = 0$$
$$z = -\frac{11}{2}$$

Hence $\underset{\sim}{\mathrm{w}} = \begin{pmatrix} 3 \\ 4 \\ -\frac{11}{2} \end{pmatrix}$ is one such vector.

Alternately: $\quad \underset{\sim}{\mathrm{w}} = \begin{pmatrix} 6 \\ 8 \\ -11 \end{pmatrix}$

Question 14 (Ham)

(a) i. (3 marks)

✓ [1] for seventh roots of -1

✓ [2] for sum of roots, or equivalent merit

✓ [3] for correct solution

The seventh roots of -1 satisfy

$$z^7 = -1 = e^{(2k+1)\pi i} \qquad k \in \mathbb{Z}$$

$$\therefore z = e^{\frac{(2k+1)\pi}{7}}$$

Therefore, the seventh roots of -1 are

$$z = e^{\frac{\pi}{7}}, \quad e^{\frac{3\pi}{7}}, \quad e^{\frac{5\pi}{7}}, \quad -1, \quad e^{-\frac{5\pi}{7}}, \quad e^{-\frac{3\pi}{7}}, \quad e^{-\frac{\pi}{7}}$$

The sum of the roots is zero and since $|z| = 1$, $z + \overline{z} = 2\cos\theta$,

$$e^{\frac{\pi}{7}} + e^{-\frac{\pi}{7}} + e^{\frac{3\pi}{7}} + e^{-\frac{3\pi}{7}} + e^{\frac{5\pi}{7}} + e^{-\frac{5\pi}{7}} - 1 = 0$$

$$2\cos\frac{\pi}{7} + 2\cos\frac{3\pi}{7} + 2\cos\frac{5\pi}{7} - 1 = 0$$

$$\boxed{\therefore \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}}$$

ii. (1 mark)

Using $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$,

$$\begin{aligned}
&\cos^2\frac{\pi}{7} + \cos^2\frac{3\pi}{7} + \cos^2\frac{5\pi}{7} \\
&= \frac{1}{2}\left(1 + \cos\frac{2\pi}{7}\right) + \frac{1}{2}\left(1 + \cos\frac{6\pi}{7}\right) + \frac{1}{2}\left(1 + \cos\frac{10\pi}{7}\right) \\
&= \frac{1}{2}\left(1 - \cos\frac{5\pi}{7}\right) + \frac{1}{2}\left(1 - \cos\frac{\pi}{7}\right) + \frac{1}{2}\left(1 - \cos\frac{3\pi}{7}\right)
\end{aligned}$$

it follows that

$$\begin{aligned}
\cos^2\frac{\pi}{7} + \cos^2\frac{3\pi}{7} + \cos^2\frac{5\pi}{7} &= \frac{3}{2} - \frac{1}{2}\left(\frac{1}{2}\right) \quad \ldots \text{using (i)} \\
&= \frac{5}{4}
\end{aligned}$$

(b) (2 marks)

✓ [1] for equation of v^2

✓ [2] for correct solution

$$\ddot{x} = -n^2 x$$

We have

$$\begin{aligned} v\frac{dv}{dx} &= -n^2 x \\ \int v\, dv &= -n^2 \int x\, dx \\ \frac{1}{2}v^2 &= -\frac{1}{2}n^2 x^2 + C \end{aligned}$$

Initially $x = 0, v = V_0$,

$$\begin{aligned} C &= \frac{1}{2}V_0^2 \\ \therefore v^2 &= V_0^2 - n^2 x^2 \end{aligned}$$

When $v = 0$,

$$\begin{aligned} 0 &= V_0^2 - n^2 x^2 \\ n^2 x^2 &= V_0^2 \\ |x| &= \frac{|V_0|}{n} \end{aligned}$$

Hence, the particle first comes to rest after travelling a distance of $\frac{|V_0|}{n}$ as it starts at the origin.

(c) i. (2 marks)

✓ [1] for correct force equations for both particles

✓ [2] for correct solution

Let the magnitude of the acceleration be $\ddot{x}\,\text{m/s}^2$ and the tension in the string be T. Considering M_1, taking upwards as positive,

$$T - 5g - 3.5 = 5\ddot{x} \quad \ldots (1)$$

Considering M_2, taking downwards as positive,

$$9g - T - 3.5 = 9\ddot{x} \quad \ldots (2)$$

Adding (1) and (2),

$$\begin{aligned} 4g - 7 &= 14\ddot{x} \\ 4(9.8) - 7 &= 14\ddot{x} \\ 32.2 &= 14\ddot{x} \\ \therefore \ddot{x} &= 2.3\,\text{m/s}^2 \end{aligned}$$

ii. (1 mark)
Substituting $\ddot{x} = 2.3$ into (1),

$$\begin{aligned} T - 5(9.8) - 3.5 &= 5(2.3) \\ T - 52.5 &= 11.5 \\ T &= 64 \text{ N} \end{aligned}$$

iii. (2 marks)

✓ [1] for use of $v\dfrac{dv}{dx}$ to integrate from $x = 0$ to $x = 1.75$

✓ [2] for correct solution

$$\ddot{x} = 2.3$$

$$v\frac{dv}{dx} = 2.3$$

Since M_2 starts from rest and travels 1.75 m before reaching the floor,

$$\begin{aligned} \int_0^v v\,dv &= \int_0^{1.75} 2.3\,dx \\ \left[\frac{1}{2}v^2\right]_0^v &= 2.3\Big[x\Big]_0^{1.75} \\ \frac{1}{2}v^2 &= 2.3(1.75) \\ v^2 &= 8.05 \\ v &= \sqrt{8.05} \qquad (v > 0) \\ &\approx 2.84 \text{ m s}^{-1} \end{aligned}$$

(d) (3 marks)

✓ [1] for $\ddot{x} = -\frac{k}{(x+R)^2}$

✓ [2] for $k = gR^2$

✓ [3] for correct solution

$$\begin{aligned} \ddot{x} &= \frac{1}{2}\frac{d}{dx}\left(v^2\right) \\ &= \frac{1}{2}\frac{d}{dx}\left[2k\left(\frac{1}{x+R} - \frac{1}{2R}\right)\right] \\ &= k\frac{d}{dx}\left(\frac{1}{x+R}\right) \\ &= -\frac{k}{(x+R)^2} \end{aligned}$$

At the Earth's surface, $x = 0$ and $|\ddot{x}| = g$,

$$\begin{aligned} g &= \frac{k}{R^2} \\ \therefore k &= gR^2 \end{aligned}$$

When $|\ddot{x}| = \dfrac{g}{2}$,

$$\begin{aligned}\frac{k}{(x+R)^2} &= \frac{g}{2}\\ \frac{gR^2}{(x+R)^2} &= \frac{g}{2}\\ (x+R)^2 &= 2R^2\end{aligned}$$

Since $x + R > 0$,

$$\begin{aligned}x + R &= \sqrt{2}R\\ x &= (\sqrt{2}-1)R\end{aligned}$$

Question 15 (Ham)

(a) (3 marks)

✓ [1] for assuming $\log_3 15 = \dfrac{p}{q}$, where $\gcd(p, q) = 1$, or equivalent merit

✓ [2] for obtaining $3^p = 15^q$, or equivalent merit

✓ [3] for correct proof

Assume $\log_3 15$ is rational. Then,

$$\exists p, q \in \mathbb{Z}^+ \text{ s.t. } \log_3 15 = \frac{p}{q}$$

with $\gcd(p, q) = 1$. Also as $\log_3 15 > 1$, $p > q$.

$$\begin{aligned}3^{\frac{p}{q}} &= 15\\ 3^p &= 15^q\\ 3^p &= 3^q 5^q\end{aligned}$$

Therefore,

$$3^{p-q} = 5^q$$

The left-hand side has only 3 as a prime factor, while the right-hand side has only 5 as a prime factor. This contradicts the uniqueness of prime factorisation.

$$\therefore \boxed{\log_3 15 \text{ is irrational.}}$$

(b) (3 marks)

✓ [1] for proving the base case.

✓ [2] Substantial progress to correctly incorporate $P(k)$ into $P(k+1)$.

✓ [3] for correct proof

Let $P(n)$ be the proposition

$$\frac{d^n y}{dx^n} = (-1)^n (x-n)e^{-x}, \qquad \text{where } y = xe^{-x}.$$

- Prove $P(1)$ is true

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(xe^{-x}\right)\\ &= e^{-x} - xe^{-x}\\ &= (1-x)e^{-x}\\ &= (-1)^1(x-1)e^{-x}\end{aligned}$$

Hence $P(1)$ is true.

- Assume $P(k)$ is true

$$\frac{d^k y}{dx^k} = (-1)^k(x-k)e^{-x}$$

- Prove $P(k+1)$ is true

$$\text{RTP:}\quad \frac{d^{k+1}y}{dx^{k+1}} = (-1)^{k+1}\left(x-(k+1)\right)e^{-x}$$

$$\begin{aligned}\frac{d^{k+1}y}{dx^{k+1}} &= \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)\\ &= \frac{d}{dx}\left[(-1)^k(x-k)e^{-x}\right]\\ &= (-1)^k\left[e^{-x} - (x-k)e^{-x}\right]\\ &= (-1)^k(k+1-x)e^{-x}\\ &= (-1)^k\left[-(x-(k+1))\right]e^{-x}\\ &= (-1)^{k+1}\left(x-(k+1)\right)e^{-x}.\end{aligned}$$

Hence $P(k+1)$ is also true, and by mathematical induction $P(n)$ is true for all integers $n \geq 1$

(c) i. (3 marks)

✓ [1] for I_{n+1}

✓ [2] for use of integration by parts

✓ [3] for correct solution

$$I_{n+1} = \int_0^1 \frac{x^n}{(x+1)^{n+1}}\,dx$$

$$u = x^n \qquad v = -\frac{1}{n(x+1)^n}$$

$$u' = nx^{n-1} \qquad v' = \frac{1}{(x+1)^{n+1}}$$

$$\begin{aligned}I_{n+1} &= \left[-\frac{x^n}{n(x+1)^n}\right]_0^1 + \int_0^1 \frac{x^{n-1}}{(x+1)^n}\,dx\\ &= -\frac{1}{n2^n} + I_n.\end{aligned}$$

Therefore,

$$\boxed{I_{n+1} = I_n - \frac{1}{n2^n}}.$$

ii. (1 mark)
Let

$$f(x) = \frac{x}{x+1} = 1 - \frac{1}{1+x}$$

Then

$$\begin{aligned} f'(x) &= \frac{1}{(x+1)^2} \\ &> 0 \end{aligned}$$

for $0 \leq x \leq 1$.
Therefore, $f(x)$ is increasing on $[0,1]$ so its maximum occurs at $x = 1$, and its maximum value is $\frac{1}{2}$.

iii. (1 mark)
✓ [1] for correct proof with appropriate justification for the strict inequality
Since for $0 \leq x \leq 1$,

$$0 \leq \frac{x}{x+1} \leq \frac{1}{2},$$

$$\begin{aligned} I_{n+1} &= \int_0^1 \frac{x^n}{(x+1)^{n+1}}\,dx \\ &= \int_0^1 \frac{x}{x+1}\frac{x^{n-1}}{(x+1)^n}\,dx \\ &< \frac{1}{2}\int_0^1 \frac{x^{n-1}}{(x+1)^n}\,dx \quad \text{(since equality occurs only at the single point } x=1) \\ &= \frac{1}{2}I_n, \end{aligned}$$

$$\therefore \boxed{I_{n+1} < \frac{1}{2}I_n}.$$

iv. (2 marks)
✓ [1] for use of (i) and (iii)
✓ [2] for correct solution
From part (i),

$$I_n - I_{n+1} = \frac{1}{n2^n}$$

Also, from part (iii),

$$I_{n+1} < \frac{1}{2}I_n$$

$$\begin{aligned} \therefore I_n - I_{n+1} &> I_n - \frac{1}{2}I_n \\ &= \frac{1}{2}I_n \end{aligned}$$

Hence,

$$\frac{1}{n2^n} > \frac{1}{2}I_n$$

$$\boxed{\therefore I_n < \frac{1}{n2^{n-1}}}$$

v. (3 marks)

✓ [1] for $I_3 = I_2 - \frac{1}{8}$, or equivalent merit

✓ [2] for showing $I_1 < \dfrac{17}{24}$, or equivalent merit

✓ [1] for correct solution

$$\begin{aligned} I_1 &= \int_0^1 \frac{1}{x+1}\,dx \\ &= \Big[\ln(x+1)\Big]_0^1 \\ &= \ln 2. \end{aligned}$$

Using part (i),

$$\begin{aligned} I_3 &= I_2 - \frac{1}{8} \\ &= \left(I_1 - \frac{1}{2}\right) - \frac{1}{8} \\ &= I_1 - \frac{5}{8} \end{aligned}$$

$$\therefore I_1 = I_3 + \frac{5}{8}$$

Since $I_3 > 0$,

$$\ln 2 = I_1 > \frac{5}{8}$$

Also from part (iv),

$$\begin{aligned} I_3 &< \frac{1}{3 \cdot 2^2} \\ &= \frac{1}{12} \end{aligned}$$

$$\begin{aligned} \therefore I_1 = \ln 2 &= \frac{5}{8} + I_3 \\ &< \frac{5}{8} + \frac{1}{12} \\ &= \frac{17}{24} \end{aligned}$$

$$\boxed{\therefore \frac{5}{8} < \ln 2 < \frac{17}{24}}$$

Question 16 (Ho)

(a) (3 marks)

✓ [1] for applying $\overrightarrow{OD} \cdot \overrightarrow{AB} = 0$ to show $|\underset{\sim}{a}| = |\underset{\sim}{b}|$, or equivalent merit

✓ [1] for applying $\overrightarrow{OE} \cdot \overrightarrow{BC} = 0$ to show $|\underset{\sim}{b}| = |\underset{\sim}{c}|$, or equivalent merit

✓ [1] for correct proof showing sufficient working out

Let O be the point of intersection of the perpendicular bisectors of AB and BC.

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$, and the midpoint of AC be F.

RTP: $OF \perp AC$

$$\because \overrightarrow{OD} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) \text{ and } \overrightarrow{OD} \cdot \overrightarrow{AB} = 0,$$

$$\begin{aligned} \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{a}) &= 0 \\ \frac{1}{2}\left(|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2\right) &= 0 \\ |\underset{\sim}{a}| &= |\underset{\sim}{b}| \end{aligned}$$

$$\because \overrightarrow{OE} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) \text{ and } \overrightarrow{OE} \cdot \overrightarrow{BC} = 0,$$

$$\begin{aligned} \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{c} - \underset{\sim}{b}) &= 0 \\ \frac{1}{2}\left(|\underset{\sim}{c}|^2 - |\underset{\sim}{b}|^2\right) &= 0 \\ |\underset{\sim}{b}| &= |\underset{\sim}{c}| \end{aligned}$$

Hence,

$$|\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{c}|$$

Now,

$$\overrightarrow{OF} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c})$$

$$\begin{aligned} \overrightarrow{OF} \cdot \overrightarrow{AC} &= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{c} - \underset{\sim}{a}) \\ &= \frac{1}{2}\left(|\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2\right) \\ &= 0 \end{aligned}$$

$$\therefore OF \perp AC$$

Since O also lies on the perpendicular bisector of AC, the perpendicular bisectors of AB, BC and AC are concurrent.

(b) i. (3 marks)

✓ [1] for recognising $\ddot{y} = v\dfrac{dv}{dy}$ and correctly splitting fraction, or equivalent merit

✓ [1] for substituting given $v = \dfrac{g}{k}\left(1 - e^{-kt}\right)$

✓ [1] for correct proof with sufficient working out

Method 1:

During the downward motion,

$$\begin{aligned} \ddot{y} &= g - kv \\ v\frac{dv}{dy} &= g - kv \\ \frac{dy}{dv} &= \frac{v}{g - kv} \end{aligned}$$

At the maximum height, $y = 0$ and $v = 0$. Therefore,

$$\begin{aligned} y &= \int_0^v \frac{v}{g - kv}\, dv \\ &= -\frac{1}{k}\int_0^v 1 - \frac{g}{g - kv}\, dv \\ &= -\frac{1}{k}\left[v + \frac{g}{k}\ln|g - kv|\right]_0^v \\ &= -\frac{1}{k}\left(v + \frac{g}{k}\ln\left(\frac{g - kv}{g}\right)\right) \quad \ldots (\because g - kv > 0) \end{aligned}$$

$$\therefore v + ky = -\frac{g}{k}\ln\left(\frac{g - kv}{g}\right)$$

From given,

$$\begin{aligned} v &= \frac{g}{k}\left(1 - e^{-kt}\right) \\ e^{-kt} &= \frac{g - kv}{g} \\ t &= -\frac{1}{k}\ln\left(\frac{g - kv}{g}\right) \end{aligned}$$

Therefore,

$$v + ky = gt$$

$$\boxed{\therefore t = \frac{v + ky}{g}}$$

Method 2:

During the downward motion,

$$\begin{aligned}
\ddot{y} &= g - kv \\
\frac{dv}{dt} &= g - k\frac{dy}{dt} \\
\int_0^t \frac{dv}{dt}\, dt &= \int_0^t \left(g - k\frac{dy}{dt}\right)\, dt \\
v(t) - v(0) &= \Big[gt\Big]_0^t - k\Big[y(t) - y(0)\Big] \\
v &= gt - ky
\end{aligned}$$

$$\boxed{\therefore t = \frac{v + ky}{g}}$$

Method 3:

During the downward motion,

$$\begin{aligned}
v &= \frac{g}{k}\left(1 - e^{-kt}\right) \\
\frac{dy}{dt} &= \frac{g}{k}\left(1 - e^{-kt}\right) \\
\int_0^y dy &= \frac{g}{k}\int_0^t 1 - e^{-kt}\, dy \\
y &= \frac{g}{k}\Big[t + \frac{1}{k}e^{-kt}\Big]_0^t \\
ky &= g\left(t + \frac{1}{k}e^{-kt} - \frac{1}{k}\right) \\
&= gt - \frac{g}{k}\left(1 - e^{-kt}\right) \\
&= gt - v \qquad \ldots \text{(from given)}
\end{aligned}$$

$$\boxed{\therefore t = \frac{v + ky}{g}}$$

ii. (3 marks)

✓ [1] for correct and relevant integral of $\ddot{y} = -g - kv$ using correct conditions, or equivalent merit

✓ [1] for correct proof that $t_1 = \dfrac{U - kh}{g}$ for upwards motion

✓ [1] for correct proof showing sufficient working out

Method 1:

Let t_1 be the time taken for the ball to reach its maximum height, and let the maximum height be h.

During the upward motion, let v be the upward speed of the ball. Then

$$\frac{dv}{dt} = -g - kv$$

Integrating from the time of projection until the ball reaches its maximum height,

$$\begin{aligned}
\int_U^0 dv &= -\int_0^{t_1} (g + kv)\, dt \\
-U &= -\left(g\Big[t\Big]_0^{t_1} + k\int_0^{t_1} v\, dt \right) \\
U &= gt_1 + k\int_0^{t_1} \frac{dy}{dt}\, dt
\end{aligned}$$

When $t = 0, y = 0$ and $t = t_1, y = h$.

$$\begin{aligned}
U &= gt_1 + k\Big[y(t_1) - y(0)\Big] \\
&= gt_1 + k(h - 0)
\end{aligned}$$

$$\therefore t_1 = \frac{U - kh}{g}$$

Let t_2 be the time taken to fall from the maximum height to the ground.
From part (i), using $y = h$ and $v = V_f$,

$$t_2 = \frac{V_f + kh}{g}$$

$\therefore$ The total time of flight is

$$\begin{aligned}
T &= t_1 + t_2 \\
&= \frac{U - kh}{g} + \frac{V_f + kh}{g} \\
&= \boxed{\frac{U + V_f}{g}}
\end{aligned}$$

Method 2:

Let t_1 be the time taken for the ball to reach its maximum height, and let the maximum height be h.

During the upward motion, let v be the upward speed of the ball.
When $t = 0$, $y = 0$, $v = U$ and $t = t_1$, $y = h$, $v = 0$,

$$\begin{aligned}
\ddot{y} &= -g - kv \\
v\frac{dv}{dy} &= -(g + kv) \\
\int_U^0 \frac{v}{g + kv}\, dv &= -\int_0^h dy \\
\frac{1}{k}\int_U^0 1 - \frac{g}{g + kv}\, dv &= -\Big[y\Big]_0^h \\
\frac{1}{k}\Big[v - \frac{g}{k}\ln|g + kv|\Big]_U^0 &= -h \\
-\frac{g}{k}\ln(g) - \Big(U - \frac{g}{k}ln(g + kU)\Big) &= -kh \qquad \dots (\because g - kv > 0) \\
-U + \frac{g}{k}\ln\left(\frac{g + kU}{g}\right) &= -kh \qquad \dots (1)
\end{aligned}$$

Also,

$$\begin{aligned}
\ddot{y} &= -g - kv \\
\frac{dv}{dt} &= -(g + kv) \\
\int_U^0 \frac{1}{g + kv}\, dv &= -\int_0^{t_1} dt \\
\frac{1}{k}\Big[\ln|g + kv|\Big]_U^0 &= -\Big[t\Big]_0^{t_1} \\
\frac{1}{k}\Big(\ln(g) - \ln(g + kU)\Big) &= -t_1 \\
\frac{g}{k}\ln\left(\frac{g + kU}{g}\right) &= gt_1 \qquad \dots (2)
\end{aligned}$$

Substitute (2) into (1):

$$\begin{aligned}
-U + gt_1 &= -kh \\
\therefore\ t_1 &= \frac{U - kh}{g}
\end{aligned}$$

Let t_2 be the time taken to fall from the maximum height to the ground.
From part (i), using $y = h$ and $v = V_f$,

$$t_2 = \frac{V_f + kh}{g}$$

$\therefore$ The total time of flight is

$$\begin{aligned}
T &= t_1 + t_2 \\
&= \frac{U - kh}{g} + \frac{V_f + kh}{g} \\
&= \boxed{\frac{U + V_f}{g}}
\end{aligned}$$

(c) (4 marks)

✓ [1] for a correct value for $\text{Im}(z)$

✓ [1] for correct equation for $z_3 - z_1 = w(z_2 - z_1)$ in terms of a and b

✓ [1] for correct equation for a by equating imaginary parts

✓ [1] for correct solutions

Let

$$z_2 = -4 + ai \qquad \text{and} \qquad z_3 = b - 4i$$

where $a, b \in \mathbb{R}$.

Let

$$w = \frac{z_3 - z_1}{z_2 - z_1}$$

$$\because |w| = 1 \qquad \text{and} \qquad \text{Re}(w) = \frac{1}{2},$$

$$w = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Now,

$$\begin{aligned} z_2 - z_1 &= (-4 + ai) - (4 + 2i) \\ &= -8 + (a - 2)i \end{aligned}$$

and

$$\begin{aligned} z_3 - z_1 &= (b - 4i) - (4 + 2i) \\ &= (b - 4) - 6i \end{aligned}$$

As $z_3 - z_1 = w(z_2 - z_1)$,

$$\begin{aligned} (b - 4) - 6i &= \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)(-8 + (a - 2)i) \\ &= -4 - \frac{\sqrt{3}}{2}(a - 2) + \left(\frac{a - 2}{2} - 4\sqrt{3}\right)i \quad \text{or} \\ &\quad -4 + \frac{\sqrt{3}}{2}(a - 2) + \left(\frac{a - 2}{2} + 4\sqrt{3}\right)i \end{aligned}$$

Equating imaginary parts,

$$\begin{aligned} -6 &= \frac{a - 2}{2} \mp 4\sqrt{3} \\ a - 2 &= -12 \pm 8\sqrt{3} \\ a &= -10 \pm 8\sqrt{3} \end{aligned}$$

Thus,

$$\boxed{z_2 = -4 + \left(-10 + 8\sqrt{3}\right)i}$$

or

$$\boxed{z_2 = -4 + \left(-10 - 8\sqrt{3}\right)i}$$